Chapter I:
Introduction to Mathematical Fuzzy Logic

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This chapter provides an introduction to the field of mathematical fuzzy logic, giving an overview of its basic notions and results.

Similarly as in other branches of logic, formal systems of fuzzy logic can be stratified according to the complexity of their formal means. In this way we can distinguish propositional, first-order, higher-order, modal, etc., fuzzy logics, which closely parallel their classical counterparts. In the first section of this chapter we shall start with the simplest layer of *propositional* fuzzy logics, whose language consists of propositions compounded by propositional connectives, and focus on the paradigmatic case of logics based on continuous t-norms.

In Section 2 we indicate the directions in which the apparatus of fuzzy logic introduced in the first section can be extended. We depict a rich landscape of mathematical fuzzy logic, inhabited by dozens of different propositional logical systems and their classes, with complex interrelations and interesting metamathematical properties.

The multitude of fuzzy logics introduced in the first two sections calls for a general unifying (meta)theory. Section 3 first studies the position of fuzzy logics in the logical landscape—i.e., their relationship to well-known broader families of propositional logics (such as substructural or algebraizable) as well as to particular prominent non-classical logics. Then the section briefly surveys characteristic properties of fuzzy logics, both those shared with the aforementioned classes of logics and those particular to fuzzy logic.

Section 4 then examines basic metamathematical properties of propositional fuzzy logics. Finally, Section 5 presents an introduction to predicate fuzzy logic.

In this chapter we adopt a didactic stance: we proceed from simple concrete prominent fuzzy logics to their numerous generalizations and finally to an abstract theory of mathematical fuzzy logic. This naturally gives rise to a certain redundancy and repetitions in definitions and theorems, but we feel that this price is necessary for upholding Comenius' maxim "from the simple to the complex".[1]

Particular topics in mathematical fuzzy logic are explained in detail in subsequent chapters of this Handbook, to which this introduction frequently refers. All proofs of theorems (except one) are omitted in this Introduction; we refer the reader to the specialized chapters of this Handbook and the literature listed in the Bibliography.

[1]J.A. Comenius: *Didactica magna,* XVII:25.

1 Propositional logics of continuous t-norms

Any non-classical propositional logic has many facets, which can roughly be divided into syntactical ones (e.g., Hilbert or Gentzen axiomatic systems and their induced sets of theorems and provability relations) and semantical ones (e.g., general or intended algebraic semantics, game-theoretical semantics, relational semantics, etc., and their induced set of tautologies and semantic consequence relations). The need to formulate and prove general theorems on classes of logics compels us to choose one of these facets for a formal definition of logic as a *mathematical object.* Usually any of the facets determines all other ones uniquely (in a fixed setting), and so the selection is mainly a matter of preference. We have decided to identify logics (as mathematical objects) with consequence relations, presented either syntactically (as the provability relation of some axiomatic system) or semantically (as the consequence relation of some particular semantics).

In this section we shall first present the traditional, 'standard' semantics of a certain prominent class of fuzzy logics (namely, those based on continuous t-norms), and define a consequence relation based on this semantics; then we shall proceed to the axiomatic systems corresponding to this consequence relation. As we shall see, a straightforward definition of the semantic consequence relation would lack certain desirable metamathematical properties, including axiomatizability by means of finitary derivation rules. In order to keep the correspondence between the syntactic and semantic facets of these logics, the definition of the semantic consequence relation will therefore have to be modified to ensure its finitarity. After presenting axiomatic systems for prominent logics of continuous t-norms, we shall briefly hint at some of their metamathematical properties, treated in more detail in Section 4 as well as in further chapters of this Handbook. The subsequent sections then introduce various broader classes of fuzzy logics and describe their properties in more detail.

1.1 Standard semantics of t-norm fuzzy logics

Mathematical fuzzy logic generalizes bivalent Boolean logic to larger systems L of truth values, typically the real unit interval $[0,1]$. The—now classical—exposition of [83] starts with certain natural constraints on the semantics of conjunction and other propositional connectives, which are devised in such a way as to give rise to well-designed propositional and predicate calculi. These constraints and the $[0,1]$-valued semantics (with respect to which many propositional fuzzy logics are sound and complete) distinguishes mathematical fuzzy logic from the study of other many-valued logics.

The most fundamental assumption of (mainstream) mathematical fuzzy logic is that of *truth-functionality* of all propositional connectives. That is, each n-ary propositional connective c is semantically interpreted by a *function* $F_c\colon L^n \to L$; the truth value of a formula $c(\varphi_1,\dots,\varphi_n)$ is then defined as $F_c(x_1,\dots,x_n)$, where x_i is the truth value of the subformula φ_i, for each $i \in \{1,\dots,n\}$. In other words, the truth value of a formula only depends on the *truth values* of its subformulae (from which it can be calculated by the *truth functions* F_c of the connectives),[2] independently of the meaning, structure, or other characteristics of the subformulae.

[2]Observe that the truth functions $F_c\colon L^n \to L$ generalize *truth tables* of two-valued Boolean logic, as the latter can be regarded as functions $F_c\colon \{0,1\}^n \to \{0,1\}$.

Truth-functionality is one of the *design choices* employed in (mainstream) mathematical fuzzy logic. In other words, mathematical fuzzy logic as developed here *defines* the meaning of connectives truth-functionally, and studies mathematical properties of the resulting truth-functional systems. Though it is certainly possible to develop many-valued logics that are not truth-functional with respect to $[0,1]$ (or another intended system of truth values),[3] it turns out that the truth-functional systems described here are mathematically rich and demonstrably applicable to many real-life problems. This by no means controverts the suitability and applicability of non-truth-functional many-valued logics for many purposes and applications (including logical analysis of natural language).

Another design choice selects the real unit interval $[0,1]$ as the intended (or *standard*) system of truth values. Consequently, the primary intended application of formal fuzzy logic is to propositions that can be assigned a numerical quantity (normalized to the unit interval), taken as their truth value in a semantical model. Nevertheless, the inference laws of fuzzy logic generalize over a class of admissible assignments of truth values, and so abstract from particular truth values. The inference rules of fuzzy logic are thus applicable to gradual propositions even in cases when their particular truth values cannot be determined.[4]

The truth values 0 and 1 are meant to represent, respectively, the (analogues of) classical truth values *false* and *true* (to the full degree). The values between 0 and 1 represent intermediate grades of partial truth that can be assigned to propositions. *Partial truth* is here understood as a technical term, referring just to the graded quality assigned to propositions that is studied in formal fuzzy logic. Its philosophical interpretation is left open here, as it is irrelevant to the mathematical study of the formal systems.

The usual order $\leq$ of reals is understood as representing the logical strength of propositions, which decreases from 0 to 1: the larger the truth value of a proposition, the truer (in the technical sense) the proposition. This endows the standard system $[0,1]$ of truth values with the structure of a complete linearly ordered lattice.

Further design choices of *propositional* fuzzy logic restrict possible truth-functions of the logical connectives to those satisfying certain natural constraints. Following the account of [83], we start with conditions on the truth function $*$ of conjunction ($\&$). The following conditions are required of $*\colon [0,1]^2 \to [0,1]$, for all $x, y, z \in [0,1]$:

- *Commutativity:* $x * y = y * x$
- *Associativity:* $(x * y) * z = x * (y * z)$
- *Monotonicity:* if $x \leq x'$ and $y \leq y'$, then $x * y \leq x' * y'$
- *Unit:* $x * 1 = x$
- *Continuity:* $*$ is continuous on $[0,1]^2$.

Commutativity and associativity embody the idea that the truth value of a conjunction does not depend on the order or bracketing of the conjoined propositions. Monotonicity expresses the intuition that increasing the truth values of both conjuncts should not

[3] Such systems have been developed, e.g., in [154, Part III], [71], or [5].

[4] As is often the case with natural-language predicates: even though it is impossible to determine whether a man of height 1.80 m is tall to degree 0.7 or 0.8, the laws of fuzzy logic can still be applied to the predicate *tall*.

decrease the truth value of their conjunction. The neutrality of 1 is motivated by its interpretation as full truth (conjunction with full truth should not change the truth value of a proposition). The identity $x * 0 = 0$, corresponding to the interpretation of 0 as full falsity, already follows from the above conditions (see Theorem 1.1.5(1) below). The condition of continuity formalizes the intuitive idea that an infinitesimal change of the truth value of a conjunct should not radically change the truth value of the conjunction.

We could add further conditions required of $*$ (e.g., idempotence), but it has proved useful to stop here, as the above conditions already yield a rich and interesting theory and further conditions would be too limiting.[5] It turns out that functions satisfying the first four conditions from the above list have previously been studied in the theory of probabilistic metric spaces under the name *triangular norms* (or *t-norms* for short):

DEFINITION 1.1.1. *A binary function* $*\colon [0,1]^2 \to [0,1]$ *is a* t-norm *(or* triangular norm*) if it is commutative, associative, monotone, and* 1 *is its unit element.*

The above conditions on the interpretation of conjunction thus amount to the requirement that $*$ is a *continuous t-norm.* An extensive study of the properties of t-norms and continuous t-norms can be found in [123]. Here we only recall such properties that are needed for the development of mathematical fuzzy logic (for proofs see [123] or [83]).

DEFINITION 1.1.2. *Let* $*$ *be a t-norm and* $x \in [0,1]$. *Then we define* $x^0 = 1$ *and* $x^{n+1} = x * x^n$ *for each* $n \in \mathsf{N}$. *We say that* x *is an* idempotent element *of* $*$ *if* $x * x = x$. *We say that* x *is a* nilpotent element *of* $*$ *if there is* $n \in \mathsf{N}$ *such that* $x^n = 0$.

DEFINITION 1.1.3. *We say that a t-norm* $*$ *is* Archimedean *if for each* $x, y \in (0,1)$ *there is* $n \in \mathsf{N}$ *such that* $x^n \leq y$. *A continuous Archimedean t-norm* $*$ *is called* strict *if* 0 *is its only nilpotent element; otherwise it is called* nilpotent.

We shall say that a t-norm $*$ *is* isomorphic *to a binary operation* $\star\colon [a,b]^2 \to [a,b]$ *if there is a strictly increasing function* $f\colon [0,1] \to [a,b]$ *such that* $f(x * y) = f(x) \star f(y)$ *for all* $x, y \in [0,1]$.

DEFINITION 1.1.4. *Let* T *be a system of triples* $(*_i, a_i, b_i)$ *indexed by* $i \in I$ *such that* $*_i$ *is a t-norm for each* $i \in I$ *and* $(a_i, b_i) \subseteq [0,1]$ *are mutually disjoint open intervals. Then the t-norm* $\star$ *defined for each* $x, y \in [0,1]$ *as*

$$x \star y = \begin{cases} a_i + (b_i - a_i) \cdot \left(\frac{x - a_i}{b_i - a_i} *_i \frac{y - a_i}{b_i - a_i} \right) & \text{if } x, y \in (a_i, b_i) \text{ for some } i \in I \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

is called the ordinal sum *of the t-norms* $*_i$ *on the intervals* (a_i, b_i) *and denoted by* $\bigoplus T$.

If T *is a finite system of triples* $(*_1, a_1, b_1), \ldots, (*_n, a_n, b_n)$ *and* $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, *then the ordinal sum* $\bigoplus T$ *can also be written as* $*_1 \oplus \cdots \oplus *_n$, *provided* a_i, b_i *are known from the context or are irrelevant (as all such ordinal sums*

[5] For instance, the additional requirement of idempotence ($x * x = x$ for all $x \in [0,1]$) would already limit $*$ to a single possible function, namely the minimum: $x * y = \min\{x, y\}$. However, there are contexts in applications of mathematical fuzzy logic where interpretations of conjunction other than the minimum are appropriate.

are isomorphic). In general, T induces an order $\prec$ on I by letting $i \prec j$ iff $a_i < a_j$; in this sense we shall speak of the first, last, second, etc. component of the ordinal sum $\bigoplus T$ (if they exist in the order $\prec$).

Without loss of generality, we shall assume further on that the union $\bigcup_{i\in I}(a_i, b_i)$ of intervals from T is dense in $[0, 1]$ (this can always be achieved by adding to T all triples $(*_{\mathrm{G}}, a_j, b_j)$ such that (a_j, b_j) is a maximal open interval of idempotent elements and $*_{\mathrm{G}}$ is the minimum t-norm of Example 1.1.6 below).

THEOREM 1.1.5. *Let $*$ be a t-norm. Then:*

1. $0 * x = 0$ *for all* $x \in [0, 1]$.
2. *The set of nilpotent elements of $*$ is the interval $[0, a)$ or $[0, a]$ for some $a \in [0, 1]$.*
3. *If x is an idempotent element of $*$, then $x * a = \min\{x, a\}$ for any $a \in [0, 1]$ and $a * b = a$ for any $a, b \in [0, 1]$ such that $a \leq x \leq b$.*
4. *The t-norm $*$ is continuous iff it is continuous in one variable, i.e., iff $f_x(y) = x * y$ is continuous for each $x \in [0, 1]$; an analogous claim is valid for lower semi-continuity (also called* left-continuity*) and upper semi-continuity (or* right-continuity*).*

EXAMPLE 1.1.6. The following three are prominent examples of continuous t-norms:

- The *minimum,* also called the *Gödel t-norm:* $x *_{\mathrm{G}} y = \min\{x, y\}$
- The *product,* also called the *product t-norm:* $x *_{\Pi} y = x \cdot y$
- The *Łukasiewicz t-norm:* $x *_{\text{Ł}} y = \max\{x + y - 1, 0\}$.

In subscripts and ordinal sums, we shall often write just G, Π, and Ł instead of $*_{\mathrm{G}}$, $*_{\Pi}$, and $*_{\text{Ł}}$, respectively.

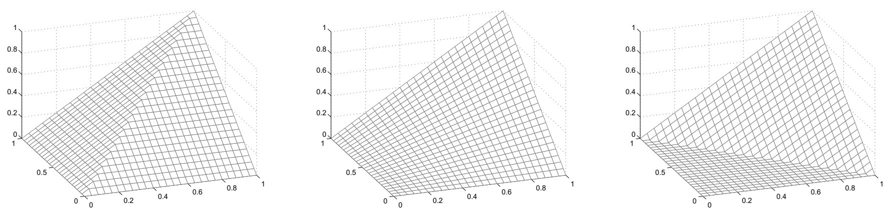

Figure 1. The graphs of the minimum, product, and Łukasiewicz t-norms

The minimum is the only idempotent t-norm. The Łukasiewicz and product t-norms are prototypical (and unique modulo isomorphism, by Theorem 1.1.7(3) below) representatives of, respectively, nilpotent and strict Archimedean t-norms. Moreover, all continuous t-norms can be decomposed into ordinal sums of isomorphic copies of these t-norms (the Mostert–Shields Theorem, see Theorem 1.1.7(5) below). The following theorem summarizes the most important properties of continuous t-norms.

THEOREM 1.1.7. *Let $*$ be a continuous t-norm. Then:*

1. *$*$ is Archimedean iff 0 and 1 are its only idempotent elements.*
2. *If $*$ is Archimedean, then it is nilpotent iff each $x \in [0,1)$ is a nilpotent element of $*$.*
3. *All continuous Archimedean t-norms are isomorphic to either the Łukasiewicz t-norm $*_{\text{Ł}}$ (nilpotent t-norms) or the product t-norm $*_{\Pi}$ (strict t-norms).*
4. *The Łukasiewicz t-norm $*_{\text{Ł}}$ is isomorphic to the product t-norm undercut at any $a \in (0,1)$, i.e., to the function $x \star y = \min\{a, x \cdot y\}$ on $[a,1]^2$.*
5. *The set of idempotent elements of $*$ is a closed subset of $[0,1]$. Its complement is therefore a countable union of disjoint open intervals (a_i, b_i), for $i \in \mathsf{N}$. Since the interval (a_i, b_i) contains no idempotent elements of $*$, the restriction of $*$ to (a_i, b_i) is isomorphic to a continuous Archimedean t-norm (either strict or nilpotent), and so is isomorphic either to $*_{\text{Ł}}$ or $*_{\Pi}$. If there is no $i \in \mathsf{N}$ such that $x, y \in (a_i, b_i)$, then $x * y = \min\{x, y\}$ by Theorem 1.1.5(3). Consequently, every continuous t-norm is isomorphic to an ordinal sum of the three t-norms from Example 1.1.6. Since also any ordinal sum of continuous t-norms is a continuous t-norm, a t-norm is continuous iff it is isomorphic to an ordinal sum of $*_{\text{G}}$, $*_{\text{Ł}}$, and $*_{\Pi}$ (Mostert–Shields Theorem).*

The suitability of continuous t-norms for the truth functions of fuzzy conjunction is supported by the following property, which provides a natural candidate for the truth function of implication.

THEOREM 1.1.8. *For any continuous t-norm $*$ there is a unique binary operation $\Rightarrow_*$ on $[0,1]$ such that for all $x, y, z \in [0,1]$,*

$$z * x \leq y \quad \textit{iff} \quad z \leq x \Rightarrow_* y. \tag{1}$$

The operation $\Rightarrow_*$ is called the *residuum* (or *residual*) of $*$. The following theorem lists basic properties of the residua of continuous t-norms:

THEOREM 1.1.9. *For a continuous t-norm $*$, its residuum $\Rightarrow_*$, and any $x, y \in [0,1]$:*

1. $x \Rightarrow_* y = \max\{z \mid x * z \leq y\}$
2. *$x \Rightarrow_* -$ is a right adjoint to the functor $- * x$ in the lattice $[0,1]$ taken as a poset category*
3. *$x \Rightarrow_* y = 1$ iff $x \leq y$*
4. $0 \Rightarrow_* y = 1$
5. $1 \Rightarrow_* y = y$
6. $\min\{x, y\} = x * (x \Rightarrow_* y)$
7. $\max\{x, y\} = \min\{(x \Rightarrow_* y) \Rightarrow_* y, (y \Rightarrow_* x) \Rightarrow_* x\}$.

Theorem 1.1.9(1) provides an explicit definition of $\Rightarrow_*$, implicitly defined by the *residuation* condition (1) of Theorem 1.1.8. A residuated couple $*$ and $\Rightarrow_*$ equips the interval $[0, 1]$ with the structure of a (complete divisible linear bounded integral commutative) *residuated lattice* (see Definition 1.3.1). Because of the adjunction stated by Theorem 1.1.9(2), the residuated pair $*, \Rightarrow_*$ is often called an *adjoint pair.*

By Theorem 1.1.9(1), the residuum is the pointwise largest function such that $x * (x \Rightarrow_* y) \leq y$; or equivalently by Theorem 1.1.9(3), such that $x * (x \Rightarrow_* y) \Rightarrow_* y$ equals 1 for all $x, y \in [0, 1]$. The latter condition can be understood as an internalized fuzzy version of the modus ponens rule (if implication is interpreted as $\Rightarrow_*$ and, in accordance with previous motivation, $*$ interprets conjunction and 1 represents full truth). This justifies adopting $\Rightarrow_*$ for the truth function of implication ($\to$), as it is the logically weakest function that makes the fuzzy modus ponens generally valid (i.e., fully true for any truth values of fuzzy propositions).

By Theorem 1.1.9(3), the full truth of fuzzy implication internalizes the ordering of truth values by logical strength.[6] Theorem 1.1.9(4), which represents a fuzzy version of *ex falso quodlibet,* then corresponds to the designation of the full falsity as the logically strongest value, while Theorem 1.1.9(4) confirms full truth as a logically neutral assumption.

EXAMPLE 1.1.10. The residua of the three prominent continuous t-norms $*_{\mathrm{G}}$, $*_{\Pi}$, and $*_{\text{Ł}}$ are, respectively, the functions $\Rightarrow_{\mathrm{G}}$ *(Gödel implication),* $\Rightarrow_{\Pi}$ *(product implication,* also known as *Goguen implication),* and $\Rightarrow_{\text{Ł}}$ *(Łukasiewicz implication),* which have the following values for $x > y$:

$$x \Rightarrow_{\mathrm{G}} y = y$$
$$x \Rightarrow_{\Pi} y = y/x$$
$$x \Rightarrow_{\text{Ł}} y = 1 - x + y.$$

For $x \leq y$, all residua evaluate to 1 by Theorem 1.1.9(3). See Figure 2 for the graphs of $\Rightarrow_{\mathrm{G}}$, $\Rightarrow_{\Pi}$, and $\Rightarrow_{\text{Ł}}$.

By Theorems 1.1.9(6)–(7), the operations of minimum and maximum are definable in terms of $*$ and $\Rightarrow_*$. Since the operation of maximum extends the bivalent truth-table for classical disjunction on $\{0, 1\}$, it is a suitable truth-function for fuzzy disjunction ($\vee$). The operation of minimum, which like $*$ extends the classical truth-table for conjunction, becomes a second candidate for the truth-function of fuzzy conjunction. Since it is anyway definable from $*$ and $\Rightarrow_*$, it can be safely included in the propositional language of fuzzy logic, playing the rôle of another conjunctive connective ($\wedge$). Fuzzy logics will thus consider two conjunctions, one interpreted by a continuous t-norm $*$ (called *strong* or *residuated* conjunction) and the other interpreted by the minimum (called *weak, lattice,* or *minimum* conjunction). The two conjunctions coincide for the idempotent t-norm $*_{\mathrm{G}}$, but differ for all other t-norms; there are in fact deeper reasons for the presence of two conjunctions in fuzzy logic (see Section 3.1). In analogy to strong and weak

[6] Notice that fully true fuzzy implication preserves the quality of being true *at least* to the degree of its antecedent. This motivates Font's [65] definition of *truth degrees* (as opposed to truth values) as intervals $[\alpha, 1]$, for any truth value $\alpha \in [0, 1]$. Theorem 1.1.9(3) then embodies the fact that fully true fuzzy implication preserves truth *degrees* of propositions, and that a logically stronger proposition (i.e., one with the smaller truth value) always implies a logically weaker one.

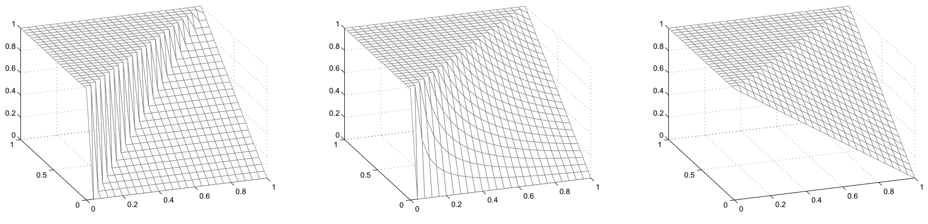

Figure 2. The graphs of the residua of the minimum, product, and Łukasiewicz t-norms. The function $\Rightarrow_{\mathrm{G}}$ is discontinuous at the points $\langle x, x\rangle$ for $x < 1$; the function $\Rightarrow_{\Pi}$ is discontinuous at the point $\langle 0, 0\rangle$; and the function $\Rightarrow_{\text{Ł}}$ is continuous.

conjunction, *strong disjunction* ($\oplus$) can be introduced in some systems of fuzzy logic (see 2.2.2 or Chapter VI). The disjunction interpreted by the maximum is then called *weak* (or *lattice*) disjunction. The terms *min-conjunction* and *max-disjunction* are also used for the two lattice connectives.

The truth-function of the equivalence connective ($\leftrightarrow$) can suitably be defined as the *bi-residuum* $\Leftrightarrow_*$ of $*$:

$$x \Leftrightarrow_* y = \min\{x \Rightarrow_* y, y \Rightarrow_* x\}, \tag{2}$$

for all $x, y \in [0, 1]$. It is irrelevant whether weak or strong conjunction is used in (2), as $(x \Rightarrow_* y) * (y \Rightarrow_* x) = \min\{x \Rightarrow_* y, y \Rightarrow_* x\}$ for all $x, y \in [0, 1]$, by Theorems 1.1.9(3) and 1.1.9(5).

Negation ($\neg$) can conveniently be interpreted by the function $\neg_*\colon [0, 1] \to [0, 1]$ defined as $\neg_* x = x \Rightarrow_* 0$, which represents a fuzzy version of the *reductio ad absurdum.* To distinguish it from an additional involutive negation introduced in some systems of fuzzy logic (see Section 2.2.2), this negation is sometimes called *residual* negation.

EXAMPLE 1.1.11. Residual negations for the three prominent continuous t-norms from Example 1.1.6 come out as follows, for any $x \in [0, 1]$:

$$\begin{aligned} \neg_{\text{Ł}} x &= 1 - x \\ \neg_{\mathrm{G}} x = \neg_{\Pi} x &= \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0. \end{cases} \end{aligned}$$

Observe that $\neg_{\text{Ł}}$ is continuous and involutive (i.e., $\neg_{\text{Ł}}\neg_{\text{Ł}} x = x$ for all x), while $\neg_{\mathrm{G}}$ is bivalent and coincides with $\neg_{\Pi}$. The bivalent negation $\neg_{\mathrm{G}} = \neg_{\Pi}$ is called *strict* (or Gödel) negation.

The function $\neg_*$ in fact coincides with $\neg_{\mathrm{G}}$ (i.e., is strict) unless $*$ has a first component isomorphic to $*_{\text{Ł}}$ on $[0, a]$ for some $a \in (0, 1]$, in which case $\neg_*$ is isomorphic to $\neg_{\text{Ł}}$ on $[0, a]$ and $\neg_* x = 0$ for $x \geq a$. All other continuous t-norms (i.e., those whose first component is isomorphic to either $*_{\mathrm{G}}$ or $*_{\Pi}$, and those that have no first component in the ordinal sum) have strict residual negation.

In this way, every continuous t-norm $*$ determines a family of truth functions $*$, $\Rightarrow_*$, $\min$, $\max$, $\Leftrightarrow_*$, $\neg_*$, which are natural candidates for the semantics of basic propositional connectives $\&, \to, \wedge, \vee, \leftrightarrow, \neg$ of fuzzy logic. These functions equip the interval $[0,1]$ with an algebraic structure that can be used for a standard definition of algebraic semantics for fuzzy logic. We shall call them *t-algebras* and denote them by $[0,1]_*$ (see Definition 1.1.12 below); later we show that t-algebras are examples of the so-called BL-algebras (see Definition 1.3.1). Notice that the constants 0 and 1 are added to the signature of $[0,1]_*$, since 0 is used in the definition of $\neg_*$, and 1 is anyway definable as $\neg_* 0$ for any $*$ by Theorem 1.1.9(4); they will respectively represent the propositional constants $\overline{0}$ for full falsity and $\overline{1}$ for full truth.

DEFINITION 1.1.12. *For a continuous t-norm $*$, we define the* t-algebra of $*$ *as the algebra*

$$[0,1]_* = \langle [0,1], *, \Rightarrow_*, \min, \max, 0, 1 \rangle,$$

where $\Rightarrow_*$ is the residuum of $*$ and $\min, \max$ *are the minimum and maximum in the usual order of reals. The operations $\Leftrightarrow_*$ and $\neg_*$ are defined in each t-algebra by setting $x \Leftrightarrow_* y = \min\{x \Rightarrow_* y, y \Rightarrow_* x\}$ and $\neg_* x = x \Rightarrow_* 0$ for all $x, y \in [0,1]$.*

By convention, for any set K of continuous t-norms we denote the corresponding set of t-algebras by $\mathbb{K}$, and vice versa.

Now we can define the syntax and standard semantics of logics based on continuous t-norms as follows:

DEFINITION 1.1.13 (Syntax and standard semantics of logics of continuous t-norms). *The* language $\mathcal{L}$ *of the* propositional fuzzy logic L_K *of the set K of continuous t-norms consists of the propositional variables $p, q, r, \ldots$, the binary propositional connectives $\&$ (strong conjunction), $\to$ (implication), $\wedge$ (weak conjunction), $\vee$ (weak disjunction), and $\leftrightarrow$ (equivalence), the unary propositional connective $\neg$ (negation), and the propositional constants $\overline{1}$ (truth) and $\overline{0}$ (falsity).*

The formulae *of $\mathcal{L}$ are formed as usual; by convention, $\to$ and $\leftrightarrow$ have the lowest and unary connective(s) the highest priority. The (denumerable) set of propositional variables of $\mathcal{L}$ will be denoted by Var and the set of all formulae of $\mathcal{L}$ by $Fm_{\mathcal{L}}$.*

We shall use Greek letters $\varphi, \psi, \chi, \ldots$ for formulae and $\Gamma, \Delta, \ldots$ for sets of formulae. We shall also use the following abbreviations: $\varphi^0 \equiv_{\mathrm{df}} \overline{1}$ and $\varphi^{n+1} \equiv_{\mathrm{df}} \varphi^n \,\&\, \varphi$, for all $n \in \mathsf{N}$.

A $[0,1]$-evaluation of propositional variables *is a mapping $e \colon Var \to [0,1]$. For any continuous t-norm $*$, the evaluation e of propositional variables extends uniquely to the $*$-evaluation $e_* \colon Fm_{\mathcal{L}} \to [0,1]$ of all formulae by the following recursive definition ('Tarski conditions'), for any proposition variable p and any formulae φ, ψ:*

$$\begin{aligned} e_*(p) &= e(p) & e_*(\varphi \,\&\, \psi) &= e_*(\varphi) * e_*(\psi) \\ e_*(\overline{0}) &= 0 & e_*(\varphi \to \psi) &= e_*(\varphi) \Rightarrow_* e_*(\psi) \\ e_*(\overline{1}) &= 1 & e_*(\varphi \wedge \psi) &= \min\{e_*(\varphi), e_*(\psi)\} \\ e_*(\neg\varphi) &= \neg_*(e_*(\varphi)) & e_*(\varphi \vee \psi) &= \max\{e_*(\varphi), e_*(\psi)\} \\ & & e_*(\varphi \leftrightarrow \psi) &= e_*(\varphi) \Leftrightarrow_* e_*(\psi). \end{aligned}$$

By Theorem 1.1.9 and the definitions of $\Leftrightarrow_*$ and $\neg_*$, we can note that obviously:

$$\begin{aligned}
e_*(\varphi \wedge \psi) &= e_*(\varphi \mathbin{\&} (\varphi \to \psi)) \\
e_*(\varphi \vee \psi) &= e_*(((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)) \\
e_*(\varphi \leftrightarrow \psi) &= e_*((\varphi \to \psi) \mathbin{\&} (\psi \to \varphi)) \\
e_*(\neg\varphi) &= e_*((\varphi \to \overline{0})) \\
e_*(\overline{1}) &= e_*(\neg\overline{0}).
\end{aligned}$$

Thus the only necessary connectives in any logic of continuous t-norms are $\&$, $\to$, and $\overline{0}$; the remaining ones are definable by the above definitions.

It can be observed that given a continuous t-norm $*$, some formulae obtain the truth value 1 for any $*$-evaluation of propositional variables; these can be regarded as the *tautologies* of the fuzzy logic based on the continuous t-norm $*$. We shall call such formulae *$*$-tautologies*. Some formulae turn out to be $*$-tautologies for any continuous t-norm $*$ in a given set K; these are tautologies of the logic L_K:

DEFINITION 1.1.14. *Let K be a set of continuous t-norms. A formula φ is a* tautology *of the logic L_K if $e_*(\varphi) = 1$ for any $* \in K$ and any $*$-evaluation e_*.*

*We shall use the expressions K-*tautology, $\mathbb{K}$-tautology, *and* L_K-tautology *as synonyms for* tautology of L_K. *We shall also write just $*$-tautology if $K = \{*\}$. If K is the set of all continuous t-norms, then K-tautologies are also referred to as* t-tautologies.

The *semantic consequence relation* of L_K is the relation between sets of formulae and formulae which preserves the truth value 1 (representing full truth) from premises to conclusions under each evaluation of propositions in each t-algebra from $\mathbb{K}$:

DEFINITION 1.1.15. *A $*$-evaluation e_* is called a* $*$-model *of a set Γ of formulae if $e_*(\psi) = 1$ for all $\psi \in \Gamma$. A formula φ is a* semantic consequence *of Γ in L_K (written $\Gamma \models_K \varphi$) if for each $* \in K$, all $*$-models e_* of Γ are $*$-models of $\{\varphi\}$. The relation $\models_K$ is called the* semantical consequence relation of L_K.

Sometimes we shall write $\models_{\mathbb{K}}$ instead of $\models_K$, and just $\models_*$ if $K = \{*\}$.

In classical logic, the set of tautologies straightforwardly encodes the semantic consequence relation, via the compactness and deduction theorems valid for this logic (i.e., $\Gamma \models \varphi$ iff there is a finite set $\{\psi_1, \ldots, \psi_n\} \subseteq \Gamma$ such that the formula $\psi_1 \wedge \cdots \wedge \psi_n \to \varphi$ is a tautology). We shall see, however, that most fuzzy logics lack this suitable form of the deduction theorem, and that their semantical consequence given by the standard (intended) semantics is seldom finitary (i.e., axiomatizable by deduction rules with finitely many premises). Thus in many fuzzy logics, the relationship between the set of tautologies and the consequence relation is not straightforward. In fact, in many fuzzy logics, the set of tautologies does not determine the consequence relation, as shown by the following example.

EXAMPLE 1.1.16. For a given set K of continuous t-norms, an alternative semantic consequence relation $\models'_K$ can be defined in the following way [20]:[7]

$$\Gamma \models'_K \varphi \text{ iff } \inf_{\psi\in\Gamma} e_*(\psi) \leq e_*(\varphi) \text{ for each } * \in K \text{ and each } *\text{-evaluation } e_*.$$

It can be easily shown that, e.g., for each φ we have $\models_{Ł} \varphi$ iff $\models'_{Ł} \varphi$, but $\varphi, \neg\varphi \models_{Ł} \overline{0}$ whereas $\varphi, \neg\varphi \not\models'_{Ł} \overline{0}$.

On the other hand, the consequence relation does determine the set of tautologies, as obviously φ is a $\mathbb{K}$-tautology if $\emptyset \models_{\mathbb{K}} \varphi$. It is therefore more appropriate to identify fuzzy logics (as mathematical objects) with their logical consequence relations rather than just their sets of tautologies.

Unfortunately, as hinted above, the consequence relation $\models_{\mathbb{K}}$ cannot be, in general, axiomatized in a finitary way (i.e., by deduction rules with *finitely* many premises), as follows from this counterexample:

EXAMPLE 1.1.17. Let $*$ be either Ł or Π. We consider a theory $\Gamma = \{q \to p^n \mid n \in \mathsf{N}\}$. We shall show that $\Gamma \models_* p \vee \neg q$, but for no finite $\Gamma' \subseteq \Gamma$ holds: $\Gamma' \models_* p \vee \neg q$.

We present the proof for $* = \Pi$; the proof for $* =$ Ł is analogous. The first claim is proved by contradiction: assume that there is a $*$-evaluation e such that $e(p \vee \neg q) < 1$ and $e(q \to p^n) = 1$ for each n. Then obviously $e(q) > 0$, $e(p) < 1$, and $e(q) \leq (e(p))^n$, which is clearly impossible (since $\inf_{n\in\mathsf{N}}(e(p))^n = 0$). The second claim: let m be the maximal n such that $q \to p^n \in \Gamma'$. Let $e(q) = a$ and $e(p) = \sqrt[m]{a}$, for an arbitrary $a \in (0,1)$. Then clearly $e(p \vee \neg q) = \sqrt[m]{a} < 1$ and $e(q \to p^n) = 1$ for each $n \leq m$.

Even though it is not in general possible to axiomatize the consequence relation $\models_{\mathbb{K}}$ by rules with finitely many premises, it turns out that its *finitary companion* $\models^{\mathrm{fin}}_{\mathbb{K}}$, i.e., the relation defined as

$$\Gamma \models^{\mathrm{fin}}_{\mathbb{K}} \varphi \quad \text{iff} \quad \text{there is a finite set } \Gamma' \subseteq \Gamma \text{ such that } \Gamma' \models_{\mathbb{K}} \varphi,$$

is finitely axiomatizable for any set $\mathbb{K}$ of t-algebras, with modus ponens (from φ and $\varphi \to \psi$ infer ψ) as the only deduction rule (see the next subsection). Note that for finite sets of premises the relations $\models_{\mathbb{K}}$ and $\models^{\mathrm{fin}}_{\mathbb{K}}$ coincide, thus in particular the *tautologies* of these two consequence relations are always the same. On the other hand, the previous example demonstrates that these two relations differ in general. In fact, even more can be proven:

THEOREM 1.1.18 ([51, 104]). *Let $\mathbb{K}$ be a set of t-algebras. Then $\models_{\mathbb{K}}$ and $\models^{\mathrm{fin}}_{\mathbb{K}}$ coincide iff $\mathbb{K} = \{[0,1]_{\mathrm{G}}\}$.*

In order to have the correspondence between the consequence relation and its axiomatic presentation, by the *logic of* $\mathbb{K}$ we shall, therefore, call the finitary consequence relation $\models^{\mathrm{fin}}_{\mathbb{K}}$:

[7] Note, however, that this consequence relation has rather bad properties: e.g., the modus ponens rule is not satisfied by $\models'_K$, unless $K = \{\mathrm{G}\}$.

DEFINITION 1.1.19. *The* logic of a set K of continuous t-norms *(or equivalently the logic of* $\mathbb{K}$*) will be identified with the finitary consequence relation* $\models_{\mathbb{K}}^{\mathrm{fin}}$ *and denoted by* L_K.

The logics of $*_{Ł}$*,* $*_{\Pi}$*, and* $*_{\mathrm{G}}$ *are, respectively, called* Łukasiewicz, product, *and* Gödel (fuzzy) logic *and denoted by* Ł*,* Π*, and* G.[8] *The logic of all continuous t-norms is called* basic logic *and denoted by* BL.[9] *The logic of all continuous t-norms whose residual negation is strict is called* SBL *(for 'strict basic logic').*

The infinitary consequence relation $\models_{\mathbb{K}}$ *can be called the* infinitary logic of $\mathbb{K}$*, with the particular cases of* infinitary Łukasiewicz, Gödel, product*, and* (strict) basic logic.

For the logic L_K*, the t-algebras of* $\mathbb{K}$ *are called the* standard L_K-algebras. *(For* general L_K*-algebras see Section 1.3.)*

Obviously, the (infinitary) logics (and so the sets of tautologies as well) of isomorphic continuous t-norms coincide, and similarly for sets of continuous t-norms differing only by isomorphism. The converse, however, is not true, as for instance the logic BL of all continuous t-norms coincides with the logic of any single t-norm which is an ordinal sum containing infinitely many isomorphic copies of $*_{Ł}$ having a first component isomorphic to $*_{Ł}$.

Thus there are sets K and K' of continuous t-norms such that $K \neq K'$, but $\mathrm{L}_K = \mathrm{L}_{K'}$; thus both $\mathbb{K}$ and $\mathbb{K}'$ are the sets of standard L_K-algebras. The notion of 'standard' algebra, i.e., that of *intended* semantics of a given logic, is clearly a matter of choice that is essentially ad hoc; therefore we will leave this discrepancy unresolved.

Note that for $\mathrm{L} \in \{Ł, \Pi, \mathrm{G}\}$ we have that $\mathrm{L} = \mathrm{L}_*$ if and only $*$ is isomorphic to $*_{\mathrm{L}}$. Therefore, we postulate that the t-algebras $[0,1]_{\Pi}$, $[0,1]_{Ł}$, and $[0,1]_{\mathrm{G}}$ (being the intended semantics of Łukasiewicz, product, and Gödel logic) are the only algebras to be called, respectively, the *standard product algebra,* the *standard* MV*-algebra,*[10] and the *standard Gödel algebra.*[11] Finally we postulate that all t-algebras $[0,1]_*$ are *standard* BL*-algebras* (or *standard* SBL*-algebras* if $*$ has strict residual negation).

Although the logics G, Ł, and Π are incomparable in strength (see Section 1.2 for examples of formulae in which they differ), it is not the case for all logics L_* of particular continuous t-norms. The following theorem displays some of the relationships between the logics L_K, for K a set of continuous t-norms (the first claim was proved in [104], the remaining ones are folklore).

---

[8] Two of these logics had been known before the advent of mathematical fuzzy logic. The logic Ł was introduced in 1930 by Łukasiewicz and Tarski [129] (its three-valued variant already in 1920 by Łukasiewicz [128]) and studied by Hay [106], Rose [163], and others. The logic G was implicitly defined in Gödel's 1932 paper [74] and extensively studied esp. by Dummett [47] and Horn [115]. Hence they have been known in the literature as the *infinite-valued (propositional) logic of Łukasiewicz* and *Dummett's* (or *Gödel–Dummett's*) *propositional logic.*

[9] At least three other logics are also called 'basic logic' in the literature; two related to intuitionistic logic [166, 175] and one to relevant logics [165]. When a misunderstanding could arise, the logic of continuous t-norms should better be called *basic fuzzy logic* or *Hájek's basic logic.*

[10] Algebras for Łukasiewicz logic are traditionally called MV-algebras, where MV stands for "many-valued", since Łukasiewicz logic was for long considered a paradigmatic example of many-valued logic.

[11] This exception actually does not make a difference in the case of the standard Gödel algebra, as $\mathrm{G} = \mathrm{L}_K$ if and only if $K = \{*_{\mathrm{G}}\}$, and so $[0,1]_{\mathrm{G}}$ is the only standard Gödel algebra by either definition.

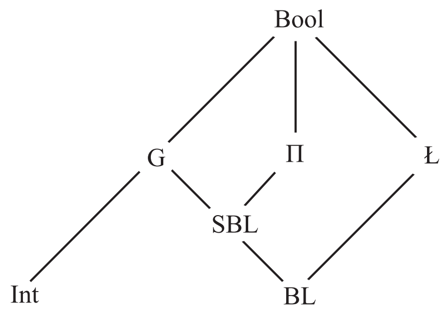

Figure 3. The prominent logics of continuous t-norms and their position with respect to classical logic (Bool) and intuitionistic logic (Int). Note that similar Hasse diagrams in this chapter only capture the relative strength of the logics (increasing upwards), rather than the lattice relationships between the logics: e.g., SBL *is not* the intersection of the logics G and Π.

THEOREM 1.1.20. *Let K, K' be sets of continuous t-norms and $*, *'$ continuous t-norms. Then the following properties hold:*

1. *There is a finite set of continuous t-norms $\hat{K}$ such that $\mathrm{L}_K = \mathrm{L}_{\hat{K}}$.*
2. *If $K \subseteq K'$, then $\mathrm{L}_{K'} \subseteq \mathrm{L}_K$.*
3. *$\mathrm{L}_{*\oplus*'} \subseteq \mathrm{L}_*$.*
4. *If $*$ has infinitely many idempotent elements, then $\mathrm{L}_* \subseteq \mathrm{G}$.*
5. *$\mathrm{BL} \subseteq \mathrm{L}_K$, and $\mathrm{L}_K \subseteq \text{Ł}$ or $\mathrm{L}_K \subseteq \Pi$ or $\mathrm{L}_K \subseteq \mathrm{G}$.*

1.2 Axiomatic systems for logics of continuous t-norms

Now we shall deal with the axiomatic facet of fuzzy logics, following the exposition of [83]. First let us recall some standard definitions pertaining to the notion of (finitary) Hilbert-style calculus:[12]

DEFINITION 1.2.1. *A* Hilbert-style calculus *(or* axiomatic system*) is given by a set of axioms and a set of derivation rules.* Axioms *are selected formulae in a given language.* Derivation rules *are pairs consisting of a finite set of formulae (called the* premises *of the rule) and a single formula (called the* conclusion *of the rule).*

The axioms and derivation rules of a Hilbert-style calculus are usually given in the form of *schemata*—i.e., with formulae containing placeholders to be replaced by arbitrary formulae of the language. Particular axioms and derivation rules of the calculus

[12] In this chapter we restrict ourselves to *finitary* axiomatic systems, i.e., such that all of their derivation rules have finite sets of premises. Some definitions (esp. that of the notion of proof) would need to be modified for infinitary axiomatic systems (see Chapter II).

are then *instances* of the schema resulting from such a replacement. We say that an axiomatic system is *finite* if it has finitely many schemata of axioms and derivation rules. We say that a calculus L' is an *axiomatic extension* of L if both have the same language and L' arises by adding some set of axiom schemata (but no deduction rules) to L.

DEFINITION 1.2.2. *A* proof *of a formula φ from a theory*[13] *T in a given Hilbert-style calculus L is a finite sequence of formulae whose last member is φ and whose every member is either (i) an axiom of the calculus, (ii) an element of T, or (iii) is derived from previous members of the sequence by a derivation rule of the calculus (i.e., is the conclusion of some derivation rule whose all premises are among the predecessors of the formula in the sequence).*

If there is a proof of φ from T in L, we say that φ is provable *from T in the calculus L (written $T \vdash_{\mathrm{L}} \varphi$). If $\emptyset \vdash_{\mathrm{L}} \varphi$, we say that φ is a* theorem *of the calculus L (written just $\vdash_{\mathrm{L}} \varphi$). The relation $\vdash_{\mathrm{L}}$, i.e., the set of all pairs $\langle T, \varphi\rangle$ such that φ is provable from T in L, is called the* provability relation *of the calculus L.*

It turns out (see [29, 58, 83, 104]) that for each set K of continuous t-norms, there is a finite axiomatic system such that the logic L_K (regarded here as a finitary consequence relation, cf. Def. 1.1.19) coincides with the provability relation given by this axiomatic system. An explicit formulation of this claim for the logics BL, SBL, Ł, Π, and G will be given in Theorem 1.2.4.

Recall that the language of BL consists of the primitive binary connectives $\to$ and $\&$, and the truth constant $\overline{0}$. Further it contains the following derived connectives defined as:

$$
\begin{aligned}
\varphi \wedge \psi &\equiv_{\mathrm{df}} \varphi \,\&\, (\varphi \to \psi)\\
\varphi \vee \psi &\equiv_{\mathrm{df}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)\\
\varphi \leftrightarrow \psi &\equiv_{\mathrm{df}} (\varphi \to \psi) \,\&\, (\psi \to \varphi)\\
\neg\varphi &\equiv_{\mathrm{df}} \varphi \to \overline{0}\\
\overline{1} &\equiv_{\mathrm{df}} \neg\overline{0}.
\end{aligned}
$$

In this section we shall show that the following set of axioms,[14]

$(\mathrm{BL1})$	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$(\mathrm{BL4})$	$\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\psi \to \varphi)$
$(\mathrm{BL5a})$	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
$(\mathrm{BL5b})$	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$
$(\mathrm{BL6})$	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
$(\mathrm{BL7})$	$\overline{0} \to \varphi,$

[13] A *theory* is just a set of formulae. In the literature the term theory is sometimes used for sets of formulae that are closed under the provability relation of a given logic. In this chapter we opted for the simpler meaning, but the latter meaning is used in Chapter II.

[14] Note that in fact we use axiomatic *schemata,* i.e., for an arbitrary formula φ, the formula $\overline{0} \to \varphi$ is an *instance* of the axiom schema $(\mathrm{BL7})$, and similarly for other axioms.

together with the deduction rule of modus ponens,

(MP) from φ and $\varphi \to \psi$ infer ψ,

is a sound and complete axiomatization of the logic BL. Before we formulate this claim formally, we shall discuss the rôles of these axioms and give axiomatic systems for some of the stronger logics introduced in the previous section.

The axiom $(\mathrm{BL1})$, also called *suffixing,* ensures the transitivity of implication (cf. the relationship between $\to$ and the ordering of truth values: see Theorem 1.1.9(3) and Section 3.3). The axiom $(\mathrm{BL4})$ ensures the commutativity of the defined min-conjunction $\wedge$; it is also called the axiom of *divisibility,* as together with the other axioms and definitions it ensures the divisibility condition in BL-algebras (see Definition 1.3.1). The mutually converse implications $(\mathrm{BL5a})$ and $(\mathrm{BL5b})$ express the residuation condition (cf. Theorem 1.1.8(1) and Definition 1.3.1). The axiom $(\mathrm{BL6})$, which is equivalent to $(\varphi \to \psi) \vee (\psi \to \varphi)$, expresses the property of *prelinearity* (cf. Definition 1.3.1). Finally, the axiom $(\mathrm{BL7})$ is the *ex falso quodlibet* law. This is a minimal independent set of axioms [26]. The numbering of axioms is due to the original numbering in [83], which included two more axioms (later proved redundant, see [26]):

$(\mathrm{BL2})$ $\varphi \mathbin{\&} \psi \to \varphi$

$(\mathrm{BL3})$ $\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$.

The logics G, Ł, and Π are axiomatized by adding some of the following axioms to the axiomatic system of BL:

(G) $\varphi \to \varphi \mathbin{\&} \varphi$

(Ł) $\neg\neg\varphi \to \varphi$

(Π) $\neg\varphi \vee ((\varphi \to \varphi \mathbin{\&} \psi) \to \psi)$.

Thus, $\mathrm{G} = \mathrm{BL} + (\mathrm{G})$; $\text{Ł} = \mathrm{BL} + (\text{Ł})$; and $\Pi = \mathrm{BL} + (\Pi)$. The axiom (G) expresses the idempotence of the residuated conjunction (note that the converse implication is a special case of $(\mathrm{BL2})$), and so its coincidence with the minimum conjunction. The axiom (Ł) of double negation expresses the involutiveness of negation (cf. Example 1.1.11); its converse implication is provable already in BL: see theorem $(\mathrm{T}_{\mathrm{BL}}40)$ on p. 18. Product logic can equivalently be axiomatized by adding the following two axioms to BL:

(S) $\neg(\varphi \wedge \neg\varphi)$

(Π_{S}) $\neg\neg\chi \to ((\varphi \mathbin{\&} \chi \to \psi \mathbin{\&} \chi) \to (\varphi \to \psi))$.

The axiom (S) expresses the strictness of residual negation (Example 1.1.11). Adding just the axiom (S) to BL yields the logic SBL of continuous t-norms with strict residual negation (see Definition 1.1.19). The logic SBL extends BL and is extended by both Π and G as well as by all other logics L_K where K is a set of t-norms with strict negations, i.e., t-norms which are not of the form Ł$\oplus*$ (cf. Example 1.1.11 and Theorem 1.1.20). In SBL, the axiom (Π_{S}) expresses the cancellativity of the t-norm: provided χ is non-zero (which in SBL is expressed by $\neg\neg\chi$), it can be canceled from $\varphi \mathbin{\&} \chi \to \psi \mathbin{\&} \chi$.

The logics of other continuous t-norms (or even sets of continuous t-norms) can also be axiomatized by adding certain finite sets of axioms to BL; however, since most of these axioms are not very intuitive, we do not present them here; they can be found in [58]. Extending BL (or any logic of continuous t-norms) by the law of excluded middle,

$$(\mathrm{LEM}) \qquad \varphi \vee \neg\varphi,$$

already yields classical logic.

The completeness theorem for logics of continuous t-norms can be formulated in a compact way by stating the equality of the logic (defined semantically as finitary semantic consequence relations, see Definition 1.1.19) and the corresponding syntactic provability relation (of Definition 1.2.2):

THEOREM 1.2.3. *Let* L *be any of* Ł, G, Π, BL, *and* SBL. *Then for every theory* Γ *and formula* φ *holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models^{\mathrm{fin}}_{\mathrm{L}} \varphi.$$

An expanded (traditional) formulation of this so-called *finite strong standard completeness theorem* for the prominent logics of continuous t-norms is as follows:

THEOREM 1.2.4 (Finite strong standard completeness). *Let* L *be one of the* Ł, G, Π, BL, SBL. *Then the following are equivalent for any* finite *theory* Γ *and a formula* φ*:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$.
2. $e(\varphi) = 1$ *for each standard* L*-algebra* $[0,1]_*$ *and any* $*$*-model* e *of* Γ.

Notice that Theorem 1.1.18 tells us that the restriction to finite theories Γ can only be dropped in the case of Gödel logic. The fact that $\Gamma \vdash_{\mathrm{L}} \varphi$ implies the second claim of Theorem 1.2.4 (usually called the *soundness* of the axiomatic system of L w.r.t. its standard semantics) is easily obtained by simple computation in [0,1]. We will not show the proof of the converse implication (usually called the *completeness* of the axiomatic system of L w.r.t. its standard semantics) here;[15] for particular logics it can be found in Chapters V, VI, and VII of this Handbook. Nevertheless, in the next subsection we will show the proof of a weaker claim, namely the completeness w.r.t. the (broader) class of *linearly ordered* L-algebras.

Let us now list some important formulae provable in the logics we have just defined, and comment on their rôle and importance. We start with theorems of BL (which by Theorem 1.2.4 coincide with t-tautologies) and list them with their customary names. Most of them are proved (and others are easily derivable from those proved) in [83]. First, the logic BL proves the following simple properties of implication:

[15] For BL this was proved in [29] by showing the redundancy of two additional axioms needed in Hájek's completeness proof [82]; for SBL in [29]; for Łukasiewicz logic in [106] (a weaker result, showing only the coincidence of tautologies and theorems of Łukasiewicz logic, was proved before independently in [24, 164]); for Gödel logic in [47]; and for product logic in [99].

$(\mathrm{T_{BL}}1)$	$\varphi \to \varphi$	reflexivity of $\to$
$(\mathrm{T_{BL}}2)$	$\varphi \to (\psi \to \varphi)$	weakening
$(\mathrm{T_{BL}}3)$	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
$(\mathrm{T_{BL}}4)$	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	prefixing
$(\mathrm{T_{BL}}5)$	$\varphi \to ((\varphi \to \psi) \to \psi)$.	assertion

We continue with the properties of the residuated conjunction $\&$. Note that the first two theorems were originally included among the axioms of BL (see page 15), and are therefore often referred to as $(\mathrm{BL2})$ and $(\mathrm{BL3})$:

$(\mathrm{T_{BL}}6)$	$\varphi \,\&\, \psi \to \varphi$	(&-form of) weakening
$(\mathrm{T_{BL}}7)$	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$	commutativity of $\&$
$(\mathrm{T_{BL}}8)$	$(\varphi \to \psi) \to (\varphi \,\&\, \chi \to \psi \,\&\, \chi)$	monotonicity of $\&$
$(\mathrm{T_{BL}}9)$	$(\varphi \,\&\, \psi) \,\&\, \chi \leftrightarrow \varphi \,\&\, (\psi \,\&\, \chi)$	associativity of $\&$
$(\mathrm{T_{BL}}10)$	$\varphi \to (\psi \to \varphi \,\&\, \psi)$.	

Next we deal with properties of min-conjunction $\wedge$ and max-disjunction $\vee$. Note that the minimum conjunction $\wedge$ differs from the residuated one $\&$, i.a., by its idempotence. Also note that the commutativity of $\wedge$ is stated in axiom $\mathrm{BL4}$.

$(\mathrm{T_{BL}}11)$	$\varphi \,\&\, \psi \to \varphi \wedge \psi$	
$(\mathrm{T_{BL}}12)$	$\varphi \wedge \psi \to \varphi$	
$(\mathrm{T_{BL}}13)$	$\varphi \leftrightarrow \varphi \wedge \varphi$	idempotency of $\wedge$
$(\mathrm{T_{BL}}14)$	$(\varphi \wedge \psi) \wedge \chi \leftrightarrow \varphi \wedge (\psi \wedge \chi)$	associativity of $\wedge$
$(\mathrm{T_{BL}}15)$	$(\varphi \to \psi) \to (\varphi \wedge \chi \to \psi \wedge \chi)$	monotonicity of $\wedge$
$(\mathrm{T_{BL}}16)$	$(\varphi \to \psi) \to (\varphi \leftrightarrow \varphi \wedge \psi)$.	

Theorems $(\mathrm{T_{BL}}11)$ and $(\mathrm{T_{BL}}10)$ allow us to obtain (by a double application of modus ponens) a *derived deduction rule* of BL called $\wedge$*-adjunction*: $\varphi, \psi \vdash_{\mathrm{BL}} \varphi \wedge \psi$ (thus in particular: $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{BL}} \varphi \leftrightarrow \psi$).

$(\mathrm{T_{BL}}17)$	$\varphi \to \varphi \vee \psi$	
$(\mathrm{T_{BL}}18)$	$\varphi \leftrightarrow \varphi \vee \varphi$	idempotency of $\vee$
$(\mathrm{T_{BL}}19)$	$\varphi \vee \psi \to \psi \vee \varphi$	commutativity of $\vee$
$(\mathrm{T_{BL}}20)$	$(\varphi \vee \psi) \vee \chi \leftrightarrow \varphi \vee (\psi \vee \chi)$	associativity of $\vee$
$(\mathrm{T_{BL}}21)$	$(\varphi \to \psi) \to (\varphi \vee \chi \to \psi \vee \chi)$	monotonicity of $\vee$
$(\mathrm{T_{BL}}22)$	$(\varphi \to \psi) \to (\psi \leftrightarrow \varphi \vee \psi)$	
$(\mathrm{T_{BL}}23)$	$(\varphi \to \psi) \vee (\psi \to \varphi)$.	prelinearity

Furthermore we list several forms of distributivity laws between the connectives:

($T_{BL}24$) $(\chi \to \varphi \wedge \psi) \leftrightarrow (\chi \to \varphi) \wedge (\chi \to \psi)$ distributivity of $\wedge$ over $\to$

($T_{BL}25$) $(\chi \to \varphi \vee \psi) \leftrightarrow (\chi \to \varphi) \vee (\chi \to \psi)$ distributivity of $\vee$ over $\to$

($T_{BL}26$) $(\varphi \wedge \psi \to \chi) \leftrightarrow (\varphi \to \chi) \vee (\psi \to \chi)$ 'distributivity' of $\wedge$ over $\to$

($T_{BL}27$) $(\varphi \vee \psi \to \chi) \leftrightarrow (\varphi \to \chi) \wedge (\psi \to \chi)$ 'distributivity' of $\vee$ over $\to$

($T_{BL}28$) $\varphi \mathbin{\&} (\psi \wedge \chi) \leftrightarrow (\varphi \mathbin{\&} \psi) \wedge (\varphi \mathbin{\&} \chi)$ distributivity of $\&$ over $\wedge$

($T_{BL}29$) $\varphi \mathbin{\&} (\psi \vee \chi) \leftrightarrow (\varphi \mathbin{\&} \psi) \vee (\varphi \mathbin{\&} \chi)$ distributivity of $\&$ over $\vee$

($T_{BL}30$) $\varphi \vee (\psi \wedge \chi) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$ distributivity of $\vee$ over $\wedge$

($T_{BL}31$) $\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$. distributivity of $\wedge$ over $\vee$

The following properties of truth constants $\overline{0}$ and $\overline{1}$ are provable in BL:

($T_{BL}32$) $\varphi \mathbin{\&} \overline{1} \leftrightarrow \varphi$ $\overline{1}$ is the unit of $\&$

($T_{BL}33$) $\varphi \wedge \overline{1} \leftrightarrow \varphi$ $\overline{1}$ is the unit of $\wedge$

($T_{BL}34$) $\varphi \mathbin{\&} \overline{0} \leftrightarrow \overline{0}$ $\overline{0}$ is the annihilator of $\&$

($T_{BL}35$) $\varphi \wedge \overline{0} \leftrightarrow \overline{0}$ $\overline{0}$ is the annihilator of $\wedge$

($T_{BL}36$) $(\overline{1} \to \varphi) \leftrightarrow \varphi$. push and pop

Finally we give some BL-provable properties of residual negation. Note that the properties of negation are weaker than those known from the classical logic: in fact, they more resemble those of intuitionistic logic; later we show that BL indeed does not prove their classical variants.

($T_{BL}37$) $\neg\varphi \to (\varphi \to \psi)$ ex falso quodlibet

($T_{BL}38$) $\neg(\varphi \mathbin{\&} \neg\varphi)$ law of non-contradiction

($T_{BL}39$) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ antitonicity of $\neg$

($T_{BL}40$) $\varphi \to \neg\neg\varphi$ 'weak' double negation law

($T_{BL}41$) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$ 'weak' De Morgan law

($T_{BL}42$) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$. 'weak' De Morgan law

Observe that the monotonicity properties proved for all the connectives can be used to prove the corresponding properties of congruence w.r.t. equivalence—e.g.:

($T_{BL}43$) $(\varphi \leftrightarrow \psi) \to ((\chi \to \varphi) \leftrightarrow (\chi \to \psi))$ right-congruence of $\to$

($T_{BL}44$) $(\varphi \leftrightarrow \psi) \to ((\varphi \to \chi) \leftrightarrow (\psi \to \chi))$ left-congruence of $\to$

($T_{BL}45$) $(\varphi \leftrightarrow \psi) \to ((\varphi \mathbin{\&} \chi) \leftrightarrow (\psi \mathbin{\&} \chi))$. congruence of $\&$

Thus proceeding inductively (and using modus ponens) we obtain the following theorem, which later plays a crucial rôle in the proof of the completeness theorem (see Definition 1.3.8).

THEOREM 1.2.5 (Intersubstitutivity). *Let χ be a formula and let χ' be a formula resulting from χ by replacing some occurrences of its subformula φ by a formula ψ. Then:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{BL}} \chi \leftrightarrow \chi'. \tag{Cong}$$

The theorem holds analogously in all extensions of BL. By Theorem 1.2.5, we also obtain: $\varphi \leftrightarrow \psi, \chi \vdash_{\mathrm{BL}} \chi'$. Provably equivalent formulae can thus be freely intersubstituted in the context of any formula without changing its provability.

Now let us show some theorems of the prominent axiomatic extensions of BL:

THEOREM 1.2.6. *The logic* SBL *can be equivalently axiomatized by adding any of the following (*SBL*-provable) formulae to the axioms of* BL*:*

$(\mathrm{T_{SBL}}1)$	$\neg\varphi \vee \neg\neg\varphi$	weak law of excluded middle
$(\mathrm{T_{SBL}}2)$	$(\varphi \to \neg\varphi) \to \neg\varphi$	
$(\mathrm{T_{SBL}}3)$	$\neg(\varphi \mathbin{\&} \psi) \to \neg(\varphi \wedge \psi)$.	

EXAMPLE 1.2.7. The axiom (S), and thus (by the previous theorem) also the formulae $(\mathrm{T_{SBL}}1)$–$(\mathrm{T_{SBL}}3)$, are not provable in Łukasiewicz logic. Indeed, consider any evaluation e such that $e(\varphi) = \frac{1}{2}$: then $e_{\text{Ł}}(\neg\varphi) = \frac{1}{2}$, thus $e_{\text{Ł}}(\neg(\varphi \wedge \neg\varphi)) = \frac{1}{2} \neq 1$.

THEOREM 1.2.8. *Łukasiewicz logic can be equivalently axiomatized by adding any of the following* Ł*-provable formulae to the axioms of* BL*:*

$(\mathrm{T_{Ł}}1)$	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	converse contraposition law
$(\mathrm{T_{Ł}}2)$	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom
$(\mathrm{T_{Ł}}3)$	$\varphi \mathbin{\&} \psi \leftrightarrow \neg(\varphi \to \neg\psi)$	definability of $\&$
$(\mathrm{T_{Ł}}4)$	$(\varphi \to \psi) \leftrightarrow \neg(\varphi \mathbin{\&} \neg\psi)$	definability of $\to$
$(\mathrm{T_{Ł}}5)$	$\varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$	De Morgan law
$(\mathrm{T_{Ł}}6)$	$\varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$.	De Morgan law

EXAMPLE 1.2.9. The axiom (Ł), and thus (by the previous theorem) also the formulae $(\mathrm{T_{Ł}}1)$–$(\mathrm{T_{Ł}}6)$, are provable neither in Gödel logic nor product logic (and also not in SBL and BL). Indeed, consider any evaluation $e(\varphi) = \frac{1}{2}$. In both G and Π we obtain $e(\neg\neg\varphi) = 1$, thus $e(\neg\neg\varphi \to \varphi) = \frac{1}{2} \neq 1$.

Notice that the theorems $(\mathrm{T_{Ł}}3)$–$(\mathrm{T_{Ł}}4)$ allow us to further reduce the set of basic connectives in Łukasiewicz logic, to either $\{\to, \overline{0}\}$, or $\{\&, \neg\}$, or $\{\to, \neg\}$. The last one is the set of connectives used by Łukasiewicz and Tarski in their original axiomatization of Łukasiewicz logic (see, e.g., [83]); they used an equivalent system of axioms, composed of $(\mathrm{BL}1)$, $(\mathrm{T_{BL}}2)$, $(\mathrm{T_{Ł}}1)$, $(\mathrm{T_{Ł}}2)$, and modus ponens.

So far we have seen four notable axiomatic extensions of BL, namely, Ł, G, Π, and SBL, shown by Theorem 1.2.4 to be the logics of certain (sets of) continuous t-norms. Clearly not all axiomatic extensions of BL are 't-norm based' in the latter sense: for example, even though classical (Boolean) logic is the axiomatic extension of BL by the law $\varphi \vee \neg\varphi$ of excluded middle, it is not complete w.r.t. any set of t-algebras, since for any $*$-evaluation e in any t-algebra $[0,1]_*$, $e(\varphi \vee \neg\varphi) = 1$ iff $e(\varphi) = 0$ or $e(\varphi) = 1$.

All axiomatic extensions of BL enjoy a so-called *local deduction theorem* (for a proof see Chapter II):

THEOREM 1.2.10 (Local deduction theorem). *Let* L *be an axiomatic extension of* BL. *Then the following holds for each theory* Γ *and formulae* φ, ψ*:*

$$T, \varphi \vdash_{\mathrm{L}} \psi \quad \textit{iff} \quad \textit{there is a natural } n \textit{ such that } T \vdash \varphi^n \to \psi.$$

The word *local* in the name of the theorem refers to the dependence of the multiplicity n in the right-hand side of the equivalence on the formula φ. An axiomatic extension L of BL enjoys the classical deduction theorem ($T, \varphi \vdash_{\mathrm{L}} \psi$ iff $T \vdash_{\mathrm{L}} \varphi \to \psi$) if and only if L extends Gödel logic. Thus (see Theorem 1.1.20) Gödel logic is the *only* logic of continuous t-norms which enjoys the classical deduction theorem.

The following two properties of axiomatic extensions of BL, the proof by cases property and the semilinearity property, were implicitly proved (though not mentioned by these names) in [83]. While the former is widely used and studied in the general logical literature, the latter was first explicitly formulated in [34] and called there 'the prelinearity property'; later in [41] it was renamed to its current name (for reasons for this change see the comments preceding Definition 3.3.9). The semilinearity property plays a crucial rôle in the proof of the so-called *linear completeness* of fuzzy logics (see Theorem 1.3.12), and arguably in whole Mathematical Fuzzy Logic (cf. [13]); therefore we sketch its proof here.

THEOREM 1.2.11. *Let* L *be an axiomatic extension of* BL. *Then* L *satisfies the following two properties, for each theory* Γ *and formulae* φ, ψ, χ*:*

1. The proof by cases property (PCP):

$$\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi \quad \textit{whenever} \quad \Gamma, \varphi \vdash_{\mathrm{L}} \chi \textit{ and } \Gamma, \psi \vdash_{\mathrm{L}} \chi.$$

2. The semilinearity property (SLP):

$$\Gamma \vdash_{\mathrm{L}} \chi \quad \textit{whenever} \quad \Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \textit{ and } \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi.$$

Proof. The first claim is a straightforward application of the local deduction theorem and theorems of BL: From the assumptions we obtain some m and n such that $\Gamma \vdash_{\mathrm{L}} \varphi^n \to \chi$ and $\Gamma \vdash_{\mathrm{L}} \psi^m \to \chi$, and so by ($\mathrm{T_{BL}}27$) we have $\Gamma \vdash_{\mathrm{L}} \varphi^n \vee \psi^m \to \chi$. If we show that BL proves: $(\varphi \vee \psi)^{n+m} \to \varphi^n \vee \psi^m$, the proof is done. Using ($\mathrm{T_{BL}}29$) and the associativity and commutativity of $\vee$ we obtain $(\varphi \vee \psi)^{n+m} \leftrightarrow \bigvee_{i=0}^{m+n} \varphi^i \mathbin{\&} \psi^{m+n-i}$. Observe that for each $i \leq m+n$ we have $\varphi^i \mathbin{\&} \psi^{m+n-i} \to \varphi^n$ or $\varphi^i \mathbin{\&} \psi^{m+n-i} \to \psi^m$ (using (BL2) and ($\mathrm{T_{BL}}8$)), Theorems ($\mathrm{T_{BL}}17$) and ($\mathrm{T_{BL}}29$) complete the proof.

The second claim is then a simple corollary of theorem ($\mathrm{T_{BL}}23$), $(\varphi \to \psi) \vee (\psi \to \varphi)$. □

It can be proved that if $\Gamma \nvdash_{\mathrm{L}} \chi$, then there is a theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_{\mathrm{L}} \chi$ and $\Gamma', \varphi \vdash_{\mathrm{L}} \chi$ for each $\varphi \notin \Gamma'$.[16] Note than in classical logic such a theory Γ' would be 'complete' (i.e., for each formula φ, either φ of $\neg\varphi$ would be provable from Γ). In fuzzy logic, however, only a weaker statement can be proved: $\Gamma \vdash_{\mathrm{L}} \varphi \to \psi$ and $\Gamma \vdash_{\mathrm{L}} \psi \to \varphi$ for each φ, ψ.[17] The theories satisfying this property have in [83] and subsequent papers been called 'complete theories', since the rôle they play in the proof of a completeness theorem for fuzzy logics is similar to that of complete theories in the completeness proof for classical logic. However, this name can be misleading, as even though this property entails completeness (in the above sense of $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$ for each φ) in classical logic, in fuzzy logics it differs from this notion of completeness. Thus in recent papers, the term 'linear theory' has been coined for this notion (for the rationale of the term 'linear' see Lemma 1.3.9).

Besides linear theories, we also define the notion of prime theory, which is well known, e.g., from the study of super-intuitionistic logics.

DEFINITION 1.2.12 ([34]). *We say that a theory* Γ *is*

- Linear *in* L *if for each* φ, ψ*:* $\Gamma \vdash_{\mathrm{L}} \varphi \to \psi$ *or* $\Gamma \vdash_{\mathrm{L}} \psi \to \varphi$.
- Prime *in* L *if for each* φ, ψ*:* $\Gamma \vdash_{\mathrm{L}} \varphi \vee \psi$ *implies* $\Gamma \vdash_{\mathrm{L}} \varphi$ *or* $\Gamma \vdash_{\mathrm{L}} \psi$.

It can be shown [176] that the notions of prime and linear theory coincide in any axiomatic extension of BL (but they differ, e.g., in intuitionistic logic where, due to the disjunction property, the set of theorems is a prime, but clearly not linear theory). Later we will see that this equality is one of the characteristic properties of fuzzy logics (in a certain setting, see Chapter II for details). As hinted above, the following theorem of [83] is crucial for the proof of the so-called *linear completeness* of fuzzy logics, described in the next Section 1.3.

THEOREM 1.2.13 (Linear / prime extension principle). *Let* L *be an axiomatic extension of* BL*. Let* Γ *be a theory and* φ *a formula such that* $\Gamma \nvdash_{\mathrm{L}} \chi$*. Then there is a linear (prime) theory* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{L}} \chi$.

1.3 Algebraic semantics

In this subsection we introduce a more general algebraic semantics of the logics we have considered so far. It will be, in a certain specific sense, the most general algebraic semantics, and so we will be able to easily prove that it is also a complete semantics. The proof of the so-called *general completeness,* together with its strengthening, the *linear completeness,* is the crucial first step in the proof of the *standard completeness* mentioned in the previous section. Unlike in the standard case, we are able to show that the semantical consequence relations given by general and linear semantics are finitary, and the axiomatic systems presented in the previous subsection are complete w.r.t. these semantics for *all* (possibly infinite) sets of formulae.

[16] The proof consists in a simple application of Zorn's Lemma, see Chapter II.

[17] The proof is easy: Assume otherwise, then from the 'maximality' of Γ' we obtain $\Gamma', \varphi \to \psi \vdash_{\mathrm{L}} \chi$ and $\Gamma', \psi \to \varphi \vdash_{\mathrm{L}} \chi$ thus the semilinearity property will give us $\Gamma' \vdash_{\mathrm{L}} \chi$—a contradiction.

General algebras for the logic BL are called BL-algebras [83]. The notion of BL-algebra can be defined in several equivalent ways; the definition given below puts BL-algebras in the context of the well known and deeply studied class of residuated lattices (see, e.g., [67]). For simplicity we use the same symbols for the connectives of L and the operations in the algebras, although we keep using the alternative symbols, introduced in Section 1.1, for realizations of connectives in the standard semantics. If necessary, they can be disambiguated by superscripting the name of the algebra to the operation symbol.

DEFINITION 1.3.1 ([67]). *A* bounded integral commutative residuated lattice*, or an* $\mathrm{FL_{ew}}$-algebra,[18] *is an algebra* $\boldsymbol{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1}\rangle$ *such that:*

1. $\langle A, \wedge, \vee, \overline{0}, \overline{1}\rangle$ *is a bounded lattice*
2. $\langle A, \&, \overline{1}\rangle$ *is a commutative monoid*
3. $\rightarrow$ *is the residuum of* $\&$*, i.e., for each* $x, y, z \in A$ *holds:* $x \,\&\, y \leq z$ *iff* $x \leq y \rightarrow z$.

The class of all $\mathrm{FL_{ew}}$*-algebras will be denoted by* $\mathbb{FL}_{\mathrm{ew}}$*. The operations* $\neg$ *and* $\leftrightarrow$ *are defined in each* $\mathrm{FL_{ew}}$*-algebra by setting for all* $x, y \in [0,1]$*:*

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$$
$$\neg x = x \rightarrow \overline{0}.$$

We say that a $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$ *is*

- Linearly ordered *if the order induced by its lattice reduct is total.*
- Prelinear *if it satisfies the identity* $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ *for all* $x, y \in A$.
- Divisible *if it satisfies the identity* $x \wedge y = x \,\&\, (x \rightarrow y)$ *for all* $x, y \in A$.

Divisible prelinear $\mathrm{FL_{ew}}$*-algebras are simply called* BL-algebras *and their class is denoted by* $\mathbb{BL}$*; linearly ordered* BL*-algebras are called* BL-chains.

In earlier literature on mathematical fuzzy logic (prominently in [83], where the reader can also find the proofs of the following theorems), $\mathrm{FL_{ew}}$-algebras were called just *residuated lattices.*

If we assume that an algebra $\boldsymbol{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1}\rangle$ satisfies conditions 1 and 2 of the definition of a $\mathrm{FL_{ew}}$-algebra, then the residuation condition 3 is equivalent to the following pair of identities:

$$(x \,\&\, (x \rightarrow z \wedge y)) \vee z = z$$
$$(x \rightarrow (x \,\&\, y \vee z)) \wedge y = y,$$

for all $x, y, z \in A$. Thus both $\mathbb{FL}_{\mathrm{ew}}$ and $\mathbb{BL}$ are *varieties* of algebras. Before we show the relation of the just defined class of BL-algebras and the logic BL, let us give some of its basic properties and define its important subvarieties related to other fuzzy logics introduced in the previous section.

[18] See Section 3.1 for the motivation of this terminology. Note that bounded integral residuated lattices can be regarded as $\mathrm{FL_{ew}}$-algebras (see Definition 2.1.6), since $\mathrm{FL_{ew}}$-algebras the constant $\overline{0}$ coincides with the least element.

THEOREM 1.3.2. *Let* $\boldsymbol{A}$ *be a* $\mathrm{FL_{ew}}$*-algebra. Then:*

1. $x \leq y$ *iff* $x \to y = \overline{1}$
2. $x = y$ *iff* $x \leftrightarrow y = \overline{1}$.

It is very easy to check that every t-algebra is a BL-algebra; in fact, even the converse claim can be proved:

THEOREM 1.3.3. *A* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A} = \langle [0,1], \&, \to, \min, \max, 0, 1\rangle$ *is a* BL*-algebra if and only if* $\&$ *is a continuous t-norm and* $\to$ *is its residuum.*

For any $\mathrm{FL_{ew}}$-algebra $\boldsymbol{A}$ we define the notion of $\boldsymbol{A}$-evaluation and $\boldsymbol{A}$-model in the same way as in the case of t-algebras (see Definitions 1.1.13–1.1.15). The next theorem shows that BL-algebras form a *sound* semantics of the logic BL.

THEOREM 1.3.4 (Soundness). *For any theory* Γ *and a formula* φ *such that* $\Gamma \vdash_{\mathrm{BL}} \varphi$ *we have* $e(\varphi) = \overline{1}$ *for each* BL*-algebra* $\boldsymbol{A}$ *and any* $\boldsymbol{A}$*-model* e *of* Γ.

In fact even more can be proved, namely that the BL-algebras are the *maximal sound semantics* of BL among all algebras with the same signature (see Section 3.3 for details). In particular, if $\boldsymbol{A}$ is a $\mathrm{FL_{ew}}$-algebra, then $\boldsymbol{A}$ is a BL-algebra if and only if for any theory Γ and a formula φ such that $\Gamma \vdash_{\mathrm{BL}} \varphi$ holds that $e(\varphi) = \overline{1}$ for each $\boldsymbol{A}$-model e of Γ.

DEFINITION 1.3.5. *A* BL*-algebra* $\boldsymbol{A}$ *is called:*

- *An* SBL-algebra *if it satisfies the identity* $x \wedge \neg x = \overline{0}$ *for all* $x \in A$.
- *An* MV-algebra[19] *if it satisfies the identity* $\neg\neg x = x$ *for all* $x \in A$.
- *A* product algebra *if it satisfies the identity* $\neg x \vee ((x \to x \,\&\, y) \to y) = \overline{1}$ *for all* $x, y \in A$.
- *A* Gödel algebra *if it satisfies the identity* $x \,\&\, x = x$ *for all* $x \in A$.

The corresponding varieties of algebras will be denoted by $\mathbb{SBL}$*,* $\mathbb{MV}$*,* $\mathbb{P}$*, and* $\mathbb{G}$*.*

Notice that in the definition of product algebras we simply use the defining axiom (Π), and put it always equal to $\overline{1}$. Analogously the defining identities of SBL-, MV-, and Gödel algebras could be equivalently replaced by $\neg(x \wedge \neg x) = \overline{1}$, $\neg\neg x \to x = \overline{1}$, and $x \to x \,\&\, x = \overline{1}$, respectively, i.e., by the validity of the axioms (S), (Ł), and (G). Conversely, each subvariety of BL-algebras determines an axiomatic extension of BL. We give a general definition:

[19] As mentioned in footnote 10 on p. 12, we speak about MV-algebras rather than Łukasiewicz algebras for historical reasons. Strictly speaking the MV-algebras are usually presented with a different signature, but can be shown termwise equivalent to our definition of MV-algebras, cast as a subvariety of $\mathbb{FL}_{\mathrm{ew}}$. MV-algebras can also be, termwise equivalently, presented as *Wajsberg* algebras, with only one binary connective $\to$ and the truth constant $\overline{0}$. (Recall that we have seen in Theorem 1.2.8 that $\&$ is definable in Łukasiewicz logic, and the remaining connectives of $\mathrm{FL_{ew}}$-algebras can already be defined in BL.)

DEFINITION 1.3.6. *Let* L *be an axiomatic extension of* BL. *We define a subvariety* $\mathbb{V}_{\mathrm{L}}$ *of* BL*-algebras as those satisfying, for each* L*-theorem* φ*, the identity* $\varphi = \overline{1}$ *for all values of propositional variables. The algebras from* $\mathbb{V}_{\mathrm{L}}$ *are called* L*-algebras.*

Let L be an axiomatic extension of BL by a set of axioms $\mathcal{A}$; then we can simply prove (by induction on the complexity of the proof) that $\boldsymbol{A}$ is an L-algebra iff it satisfies, for each $\varphi \in \mathcal{A}$, the identity $\varphi = \overline{1}$ for all values of propositional letters. In particular, for the logics from the previous subsection we have: $\mathbb{SBL} = \mathbb{V}_{\mathrm{SBL}}, \mathbb{MV} = \mathbb{V}_{\text{Ł}}, \mathbb{P} = \mathbb{V}_{\Pi}$, and $\mathbb{G} = \mathbb{V}_{\mathrm{G}}$. Consequently, we will sometimes write $\mathbb{L}$ instead of $\mathbb{V}_{\mathrm{L}}$.

THEOREM 1.3.7 (General completeness). *Let* L *be an axiomatic extension of* BL*. Then the following are equivalent for each theory* Γ *and formula* φ*:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$.
2. $e(\varphi) = \overline{1}$ *for each* L*-algebra* $\boldsymbol{A}$ *and any* $\boldsymbol{A}$*-model* e *of* Γ.

Below we give a hint of the proof of this fundamental theorem. First we need one important definition and a lemma.

DEFINITION 1.3.8 (Lindenbaum–Tarski algebra). *Let* L *be an axiomatic extension of* BL *and* Γ *a theory. For every formula* φ*, we define the set*

$$[\varphi]_\Gamma = \{\psi \mid \Gamma \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi\}.$$

The Lindenbaum–Tarski algebra of the theory Γ*, denoted by* $\boldsymbol{LT}_\Gamma$*, is the algebra with the domain* $\{[\varphi]_\Gamma \mid \varphi$ *a formula of* $\mathrm{BL}\}$ *and the operations defined as follows:*

$$\begin{aligned}
[\varphi]_\Gamma \,\&^{\boldsymbol{LT}_\Gamma}\, [\psi]_\Gamma &= [\varphi \,\&\, \psi]_\Gamma \\
[\varphi]_\Gamma \rightarrow^{\boldsymbol{LT}_\Gamma} [\psi]_\Gamma &= [\varphi \rightarrow \psi]_\Gamma \\
[\varphi]_\Gamma \wedge^{\boldsymbol{LT}_\Gamma} [\psi]_\Gamma &= [\varphi \wedge \psi]_\Gamma \\
[\varphi]_\Gamma \vee^{\boldsymbol{LT}_\Gamma} [\psi]_\Gamma &= [\varphi \vee \psi]_\Gamma \\
\overline{0}^{\boldsymbol{LT}_\Gamma} &= [\overline{0}]_\Gamma \\
\overline{1}^{\boldsymbol{LT}_\Gamma} &= [\overline{1}]_\Gamma.
\end{aligned}$$

The soundness of the latter definition of operations follows from Theorem 1.2.5. The proof of the following lemma is straightforward.

LEMMA 1.3.9. *Let* L *be an axiomatic extension of* BL *and* Γ *a theory. Then:*

1. $\boldsymbol{LT}_\Gamma$ *is an* L*-algebra.*
2. $\boldsymbol{LT}_\Gamma$ *is an* L*-chain iff* Γ *is a linear theory.*
3. *The* $\boldsymbol{LT}_\Gamma$*-evaluation* $e(\varphi) = [\varphi]_\Gamma$ *is an* $\boldsymbol{LT}_\Gamma$*-model of* Γ.

Proof of Theorem 1.3.7. One implication is Theorem 1.3.4. We prove the converse implication counterpositively: assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ and consider the $\boldsymbol{LT}_\Gamma$-evaluation $e(\varphi) = [\varphi]_\Gamma$. We know that e is an $\boldsymbol{LT}_\Gamma$-model of Γ and obviously $e(\varphi) \neq \overline{1}^{\boldsymbol{LT}_\Gamma}$ (as otherwise $\Gamma \vdash \varphi \leftrightarrow \overline{1}$, and so $\Gamma \vdash \varphi$—a contradiction). □

Next we move to a characteristic topic of mathematical fuzzy logic: completeness w.r.t. linearly ordered algebras of truth values. Since all t-algebras are indeed linear, we have already shown a form of such completeness; however, recall that we have done so for finite theories only. Historically speaking, there are two possible approaches: one using the notion of subdirect product and another using the notion of linear theory. The next chapter will study the mutual relationship of these two approaches and show their equivalence; here we give the main theorems of both approaches.

DEFINITION 1.3.10. *We say that a* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$ *is* representable as a subdirect product *of the family of* $\mathrm{FL_{ew}}$*-algebras* $\{\boldsymbol{A}_i \mid i \in I\}$ *if there is an injective homomorphism* α *from* $\boldsymbol{A}$ *into the direct product* $\prod_{i\in I} \boldsymbol{A}_i$ *such that for every* $i \in I$*, the composition of* α *with the* i*-th projection,* $\pi_i \circ \alpha$*, is surjective. In this case,* α *is called a* subdirect representation *of* $\boldsymbol{A}$*; it is called* finite *if* I *is finite.*

A $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$ *is* (finitely) subdirectly irreducible *if for every (finite) subdirect representation* α *with a family* $\{\boldsymbol{A}_i \mid i \in I\}$ *there is* $i \in I$ *such that* $\pi_i \circ \alpha$ *is an isomorphism.*

THEOREM 1.3.11 (Linear subdirect representability). *Let* L *be a axiomatic extension of* BL*. Then any* L*-algebra is a subdirect product of a set of* L*-chains.*

THEOREM 1.3.12 (Linear completeness). *Let* L *be an axiomatic extension of* BL*. Then the following are equivalent for each theory* Γ *and formula* φ*:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$.
2. $e(\varphi) = \overline{1}$ *for each* L*-algebra* $\boldsymbol{A}$ *and* $\boldsymbol{A}$*-model* e *of* Γ.
3. $e(\varphi) = \overline{1}$ *for each* L-chain $\boldsymbol{A}$ *and* $\boldsymbol{A}$*-model* e *of* Γ.

Proof. We can either (straightforwardly) use the previous theorem to show that 3. implies 2., or we can counterpositively prove that 3. implies 1.: We start as in the proof of Theorem 1.3.7, and using Theorem 1.2.13 we obtain a linear theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_{\mathrm{L}} \varphi$. From Lemma 1.3.9 we know that $\boldsymbol{LT}_{\Gamma'}$ is an L-chain, and again the $\boldsymbol{LT}_{\Gamma'}$-evaluation $e(\varphi) = [\varphi]_{\Gamma}$ is an $\boldsymbol{LT}_{\Gamma'}$-model of Γ such that obviously $e(\varphi) \neq \overline{1}^{\boldsymbol{LT}_{\Gamma'}}$. □

Theorem 1.3.11 can be strengthened in the following way:

THEOREM 1.3.13 ([145]). *Let* L *be an axiomatic extension of* BL*. Then any* L*-algebra is finitely subdirectly irreducible iff it is an* L*-chain.*

By a well known algebraic fact, the class of subvarieties of $\mathbb{BL}$ as well as the set of axiomatic extensions of BL form lattices. In Definition 1.3.6 we have introduced the mapping $\mathbb{V}_{\mathrm{L}}$ that assigns to each axiomatic extension of BL the corresponding subvariety of $\mathbb{BL}$. This mapping is in fact a dual isomorphism of these two lattices, whose inverse $\Lambda_{\mathbb{V}}$ assigns to $\mathbb{V}$ the extension of BL by the set of axioms $\mathcal{A}_{\mathbb{V}} = \{\varphi \leftrightarrow \psi \mid$ the identity $\varphi = \psi$ is valid in $\mathbb{V}\}$. This fact is formalized in the next theorem, which for BL-algebras is folklore; in the general framework of *algebraizable* logic it was proved by Blok and Pigozzi in [18] (see Section 3.3, and Theorem 3.3.8 in particular, for more details on this topic).

THEOREM 1.3.14. *Let* L, L' *be axiomatic extensions of* BL *and let* $\mathbb{U}, \mathbb{U}'$ *be subvarieties of* $\mathbb{BL}$. *Then:*

1. $\mathbb{U} = \mathbb{V}_{\Lambda_{\mathbb{U}}}$ *and* $\mathrm{L} = \Lambda_{\mathbb{V}_{\mathrm{L}}}$
2. $\mathbb{U} \cap \mathbb{U}' = \Lambda_{\mathbb{U}} \cup \Lambda_{\mathbb{U}'}$ *and* $\mathrm{L} \cup \mathrm{L}' = \mathbb{V}_{\mathrm{L}} \cap \mathbb{V}_{\mathrm{L}'}$.

2 Variations of basic propositional fuzzy logics

In Section 1 we introduced basic systems of fuzzy logic based on continuous t-norms. In this section we shall indicate the directions in which the apparatus of fuzzy logic can be extended. The principal ways of altering the logics of continuous t-norms are the following:

- *Discarding axioms or rules* of a fuzzy logic, enlarging thus the class of its models
- *Adding new connectives,* thus increasing the expressive power of the logic
- *Discarding connectives* present in a fuzzy logic and dealing with its fragments
- *Adding axioms or rules* to obtain stronger logics with more specific models.

These types of alterations can further be combined—e.g., some axioms can first be discarded and some others (possibly weaker or incomparably strong) added back, or connectives be added to a previously weakened logic, etc. Variations of the above moves generate a rich landscape of mathematical fuzzy logic, inhabited by dozens of different logical systems and their classes, with complex interrelations and interesting metamathematical properties.

In the subsections of the present section, we shall briefly introduce the main fuzzy logics resulting from these modifications of the logics of continuous t-norms. The way of extending propositional fuzzy logics to predicate fuzzy logics of the first or higher order is postponed to Section 5.

The new fuzzy logics will mainly be introduced by their standard semantics, or by syntactic manipulation with their language or axiomatic system. In order to avoid many repetitive definitions, we shall employ the following terminological convention (for a general formal definition of L-algebras and other notions introduced by this convention see Section 3.3):

CONVENTION 2.0.1. *For all newly introduced logics* L, *the corresponding algebras for which* L *is sound will be called* L-algebras. *A terminological exception are modifications of Łukasiewicz logic, where (subscripted)* MV- *is used instead of* $\mathrm{Ł}$- *(cf. footnote 10 on p. 12).*

More precisely, we say that $\boldsymbol{A}$ *is an* L*-algebra if* $\Gamma \vdash_{\mathrm{L}} \varphi$ *implies that any* $\boldsymbol{A}$*-model of all formulae from* Γ *is an* $\boldsymbol{A}$*-model of* φ, *where an evaluation* e *is an* $\boldsymbol{A}$-model *of* φ *if* $e(\varphi) = \overline{1}$, *unless said otherwise.*[20]

[20] A different specification of $\boldsymbol{A}$-models is employed, e.g., in Section 2.1.2. As in the previous section, the class of L-algebras will usually admit some explicit (often equational) description.

We say that an L*-algebra* $\boldsymbol{A}$ *is:*

- *An* L-chain *if the order determined by the lattice connectives of* L *is linear.*[21]
- *A* real-valued L*-algebra if its lattice reduct is the real interval, either closed* $[0, 1]$ *or half-open* $(0, 1]$*, with the usual ordering of reals.*
- *A* (finitely) subdirectly irreducible L*-algebra if it cannot be obtained as a non-trivial (finite) subdirect product of a family of* L*-algebras (for a formal definition in the case of* BL*-algebras see Definition 1.3.10).*

In the previous section we have seen that particular real-valued L-algebras may be specified as the *standard* L-algebras. These algebras are the *intended semantics* of the logic. By default, all real-valued algebras will be considered standard. In some cases, however, we may want to select a narrower class of real-valued algebras as standard: for instance, if all real-valued L-algebras are mutually isomorphic (as is the case, e.g., in G, Ł, and Π, see p. 12), we pick one particular representative.

CONVENTION 2.0.2 (Standard algebras). *Unless specified otherwise, all (and only) real-valued* L*-algebras are the* standard L-algebras.

The logic need not be complete with respect to its standard algebras;[22] if it is, we speak about the *standard completeness* of the fuzzy logic. As we have seen in Sections 1.1–1.2, the logics Ł and Π enjoy standard completeness for *finite* theories only; Gödel logic, on the other hand, enjoys standard completeness even for infinite theories. This motivates the distinction made in the following convention.

CONVENTION 2.0.3 (Standard completeness). *We say that* L *enjoys* finite strong standard completeness *if the following conditions are equivalent for each formula* φ *and each* finite *theory* T*:*

- $T \vdash_{\mathrm{L}} \varphi$.
- *For each standard* L*-algebra* $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-model* e *of* T*,* e *is an* $\boldsymbol{A}$*-model of* φ*.*

If the equivalence holds for all theories, we say that L *enjoys* strong standard completeness, *and we speak just about* (weak) standard completeness *if the equivalence holds for* $T = \emptyset$.

All logics L that will be introduced in this section enjoy several important properties that have been discussed in the previous section for logic BL and its axiomatic extensions:

- The *intersubstitutivity of equivalent subformulae* (cf. Theorem 1.2.5)
- The *proof by cases property* PCP and the *semilinearity property* SLP (cf. Theorem 1.2.11)

[21] If the logic L does not possess lattice connectives, the order can be defined by means of implication, see Section 3.3.

[22] Incompleteness with respect to the intended semantics is not unusual in logic: cf., e.g., the essential incompleteness of Peano arithmetic with respect to the standard model of natural numbers.

- The *linear (or prime) extension property* (cf. Theorem 1.2.13)
- The *general completeness theorem* (i.e., completeness w.r.t. all L-algebras, cf. Theorem 1.3.7)
- The *linear completeness theorem* (i.e., completeness w.r.t. all L-chains, cf. Theorem 1.3.12)
- The *linear subdirect decomposition property* (cf. Theorems 1.3.11 and 1.3.13).

Therefore we are not going to mention these properties repeatedly, and will omit stating that each of the upcoming logics possesses them. On the other hand, the logics will differ in what form of the (local) deduction theorem (cf. Theorem 1.2.10) holds for them; also the form of *standard* completeness (see Convention 2.0.3) will change from logic to logic.

Finally, let us remark that the intersubstitutivity of equivalent subformulae and general completeness are properties possessed by a broad range of logics (namely the so-called *weakly implicative logics,* which include, i.a., intuitionistic logic or normal modal logics). The remaining properties from the above list, on the other hand, are 'characteristic' of fuzzy logics: namely, they are satisfied by nearly all logics studied under the name 'fuzzy logic' in the literature, and rarely satisfied by a logic that is not commonly classified as 'fuzzy'. This remark will be made more precise in Section 3 and Chapter II.

2.1 Discarding axioms or rules

Even though the conditions adopted in Section 1.1 for propositional connectives, with the ensuing axioms of Section 1.2, are reasonable assumptions on generalized versions of classical propositional connectives, not all of them are necessary for generating a meaningful system of truth-functional fuzzy logic. In this subsection we shall describe several systems arising from dropping some of the properties of logics of continuous t-norms. The algebraic semantics of logics introduced in this subsection is thoroughly studied in Chapter IV.

2.1.1 Logics of left-continuous t-norms

In Section 1.1 we assumed that the t-norm representing conjunction is continuous. This assumption ensured the existence of a unique residuum (see Theorem 1.1.8), resulting in a good interplay between conjunction and implication. It turns out, however, that continuity is unnecessarily strong a condition for the existence of a unique residuum, the minimal condition for residuation being just the *left*-continuity of the t-norm.

Recall that a unary function is *left-continuous* (or *lower-semicontinuous*) if and only if it commutes with suprema: $\sup_{x \leq a} f(x) = f(a)$. By a *left-continuous t-norm* we mean a t-norm that is left-continuous in either argument (see [123]). Unlike for continuous t-norms, no characterization similar to Theorem 1.1.7(5) (the Mostert–Shields Theorem) is known for left-continuous t-norms. Prominent examples of left-continuous t-norms that are not continuous are the weak nilpotent minimum t-norms $*_{\mathrm{WNM}(n)}$:

EXAMPLE 2.1.1 ([53]). An order-reversing function $n\colon [0,1] \to [0,1]$ with $n(n(x)) \geq x$ for all $x \in [0,1]$ and $n(1) = 0$ will be called a *weak negation.* Given a weak negation n, the *weak nilpotent minimum* t-norm $*_{\mathrm{WNM}(n)}$ is defined as follows:

$$x *_{\mathrm{WNM}(n)} y = \begin{cases} 0 & \text{if } x \leq n(y) \\ \min\{x, y\} & \text{otherwise.} \end{cases} \tag{3}$$

Each $*_{\mathrm{WNM}(n)}$ is nilpotent and left-continuous, but not (right-)continuous.

A weak negation n that is involutive, i.e., $n(n(x)) = x$ for all $x \in [0,1]$, will be called a *strong negation.* If n is a strong negation, then $*_{\mathrm{WNM}(n)}$ is called the *nilpotent minimum* t-norm pertaining to n and can be denoted by $*_{\mathrm{NM}(n)}$. For the standard involution $n(x) = 1 - x$, the nilpotent minimum $*_{\mathrm{NM}(n)}$ is called the *standard nilpotent minimum* t-norm and denoted by $*_{\mathrm{NM}}$ (see Figure 4 for the graph). Historically, this t-norm was the first known example of left-continuous, but not continuous t-norm [64].

Theorem 1.1.8 holds for left-continuous t-norms as well, and ensures the unique existence of its residuum satisfying the condition (1) of the Theorem. Also Theorem 1.1.9 holds equally well for left-continuous t-norms, except the claim 6 (the definability of $\min$ in terms of $*$ and $\Rightarrow$): due to its failure, the minimum conjunction has to be included among primitive connectives of the logic of left-continuous t-norms.

EXAMPLE 2.1.2 ([53]). The residuum $\Rightarrow_{\mathrm{WNM}(n)}$ of any weak nilpotent minimum t-norm $*_{\mathrm{WNM}(n)}$ comes out as

$$x \Rightarrow_{\mathrm{WNM}(n)} y = \begin{cases} 1 & \text{if } x \leq y \\ \max\{n(x), y\} & \text{otherwise.} \end{cases}$$

The residual negation $\neg_{\mathrm{WNM}(n)}$ pertaining to the weak nilpotent minimum t-norm $*_{\mathrm{WNM}(n)}$ coincides with n.

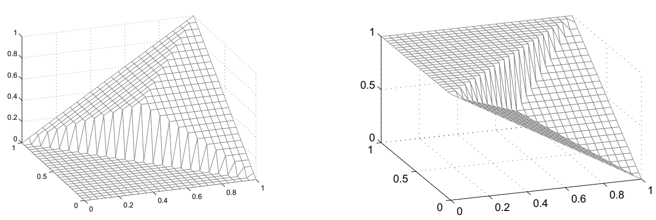

Figure 4. The graphs of the standard nilpotent minimum t-norm and its residuum. The function $*_{\mathrm{NM}}$ is discontinuous at the points $\langle x, 1-x \rangle$ for $0 < x < 1$, and the function $\Rightarrow_{\mathrm{NM}}$ is discontinuous at the points $\langle x, x \rangle$ for $0 < x < 1$.

The definition of standard semantics for the logics of left-continuous t-norms can thus run along the same lines as for the logics of continuous t-norms described in Section 1.1. The notions of tautologicity and (finitary) consequence relation with respect to any set of left-continuous t-norms (or the *logic* of the set of left-continuous t-norms) can be defined in the same way as for continuous t-norms (see Definitions 1.1.15 and 1.1.19; the notational conventions introduced in the definitions will be extended to left-continuous t-norms as well).

The logic of *all* left-continuous t-norms is called *monoidal t-norm logic,* or MTL, and was introduced in [53]. The primitive connectives of the logic MTL are $\&$, $\to$, $\wedge$, and $\overline{0}$. Its derived connectives $\vee$, $\leftrightarrow$, $\neg$, and $\overline{1}$ are defined in the same way as in BL (see Section 1.2). Also the axioms and rules of MTL are the same as those of BL, only the divisibility axiom $(\mathrm{BL4})$ is replaced by the following three axioms describing the properties of $\wedge$:

$(\mathrm{MTL4a})$	$\varphi \wedge \psi \to \varphi$
$(\mathrm{MTL4b})$	$\varphi \wedge \psi \to \psi \wedge \varphi$
$(\mathrm{MTL4c})$	$\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi.$

Clearly, the logic BL extends MTL by the converse of $(\mathrm{MTL4c})$. The logic MTL shares many metamathematical properties with the logic BL: besides those mentioned in the introduction to this section, also all formulae $(\mathrm{T_{BL}}1)$–$(\mathrm{T_{BL}}45)$ are theorems of MTL as well, and MTL enjoys the same variant of local deduction theorem (cf. Theorem 1.2.10).

MTL-algebras (i.e., the algebras for which the logic MTL is sound) can be characterized as prelinear bounded integral commutative residuated lattices, or prelinear $\mathrm{FL_{ew}}$-algebras (see Definition 1.3.1);[23] the class of MTL-algebras is thus a variety. Unlike in BL, where the standard completeness could be proved only for *finite* theories, in MTL it can be proved for *all* theories [117];[24] i.e., the logic MTL enjoys the full strong standard completeness.

2.1.2 Uninorm fuzzy logics

Another way of relaxing the conditions on t-norm fuzzy logics is dropping the requirement that the unit element of the operation representing conjunction coincides with the largest element of the lattice of truth values. This leads to a generalization of t-norms, called the *uninorms:*

DEFINITION 2.1.3 ([180]). *A binary function* $*\colon [0,1]^2 \to [0,1]$ *is a* uninorm *if it is commutative, associative, monotone, and has a unit element* $e \in [0,1]$*. The uninorm* $*$ *is* conjunctive *if* $0 * 1 = 0$*, and* disjunctive *if* $0 * 1 = 1$.

All uninorms are either conjunctive or disjunctive. T-norms are uninorms with the unit element $e = 1$. The unique residuum $\Rightarrow_*$ that satisfies the residuation condition (1) of Theorem 1.1.8 and is explicitly given by Theorem 1.1.9(1) exists if and only if $*$ is left-continuous and conjunctive.

[23] This class of algebras was independently introduced in [63] under the name *weak*-BL-*algebras.*

[24] This difference is caused by the fact that for the class $\mathbb{T}$ of all left-continuous t-norms, $\models_{\mathbb{T}} \;=\; \models^{\mathrm{fin}}_{\mathbb{T}}$ obtains, like in the case of the minimum t-norm, but unlike all other *continuous* t-norms (see Theorem 1.1.18).

All continuous conjunctive uninorms are already continuous t-norms. Prominent examples of left-continuous conjunctive uninorms that are not t-norms are the idempotent left-continuous conjunctive uninorms characterized by the following theorem:

THEOREM 2.1.4 ([43]). *The binary operation* $*\colon [0,1]^2 \to [0,1]$ *is an idempotent left-continuous conjunctive uninorm with the unit element* $e \in (0,1]$ *if and only if there is a weak negation (see Example 2.1.1)* $n\colon [0,1] \to [0,1]$ *such that* $n(e) = e$ *and*

$$x * y = \begin{cases} \min\{x,y\} & \textit{if } y \le n(x) \\ \max\{x,y\} & \textit{otherwise.} \end{cases} \tag{4}$$

The uninorm given by (4) will be denoted by $*_{\mathrm{IU}(n)}$*, or just* $*_{\mathrm{IU}}$ *if* $n(x) = 1 - x$.

The existence of unique residua makes it possible to construct the semantics of propositional calculus based on left-continuous conjunctive uninorms along similar lines as done for (left-)continuous t-norms in Section 1.1; the first paper studying logics of uninorms was [133]. The primitive binary connectives of uninorm logics are $\&$, $\to$, $\wedge$, and $\vee$, interpreted respectively by a left-continuous conjunctive uninorm $*$, its residuum $\Rightarrow_*$, and the lattice operations of minimum and maximum. Since the unit element e of $*$ need not equal 1, two different primitive truth constants $\overline{1}$ and $\top$ are distinguished, the former represented by e and the latter by 1. Similarly there are two different falsity-related primitive truth constants $\overline{0}$ and $\bot$,[25] the latter interpreted by 0 and the former by any fixed element $f \in [0,1]$. Negation $\neg\varphi$ is defined as $\varphi \to \overline{0}$ and equivalence $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

In logics based on t-norms, only the largest truth value 1 (which coincides with the unit element e of conjunction) is considered as representing the full truth of a proposition (i.e., is the designated truth value in the definitions of consequence and tautologicity). In logics based on left-continuous conjunctive uninorms (where $e \le 1$), *all* truth values $x \ge e$ are considered as representing the full truth of propositions.[26] Consequently, the formula φ is defined to be a tautology w.r.t. a set K of uninorms if $v_*(\varphi) \ge e$ (i.e., if $v_*(\varphi)$ is a *designated* truth value) for each valuation v in each uninorm algebra $[0,1]_* = \langle [0,1], *, \Rightarrow_*, \min, \max, 0, 1, e, f\rangle$ for $* \in K$. Similarly, the consequence relation $\models_K$ is defined by the transmission of the 'designatedness' property $v_*(\varphi) \ge e$ from the set of premises to the conclusion in each valuation v_* for each $* \in K$. Formally, we say that v is an $*$-model of Γ iff $v_*(\varphi) \ge e$ for each $\varphi \in \Gamma$, and then proceed as in Definition 1.1.15. The notion of finitary consequence relation w.r.t. a set K of uninorms, or the *logic* of K, is defined analogously to Definition 1.1.19.

Thus, while t-norm fuzzy logics recognize degrees of truth *between* the full truth $e = 1$ and the full falsity $f = 0$, uninorm logics furthermore recognize *degrees of full truth*[27] $x \ge e$ and degrees of full falsity $x \le f$. The truth constant $\top$ thus denotes the

[25] The symbols t and f are also often used in the literature for $\overline{1}$ and $\overline{0}$, respectively.

[26] Note, however, that the understanding of the values e and f in mathematical fuzzy logic differs from that used in engineering fuzzy methods and the theory of aggregation operators. In the latter areas, only the values 0 and 1 are understood, respectively, as full membership and full non-membership, while the value e just separates values with different aggregating behavior.

[27] Earlier than in the context of mathematical fuzzy logic, degrees of full truth were discussed by Casari in [23].

largest degree of full truth, while $\overline{1}$ denotes the *smallest* degree of full truth. Similarly $\overline{0}$ denotes the largest and $\bot$ the smallest full falsity. Observe that since no assumption is made on the value of f, uninorm logics leave room for *dialetheism* (see [157]): if $f \geq e$, then the truth values $x \in [e, f]$ represent propositions that are *both* fully true and fully false.

Unlike in t-norm logics, in uninorm logics $\wedge$ is not generally weaker than $\&$: if $x, y \geq e$, then on the contrary holds $x * y \geq (x \vee y) \geq x \wedge y$. In uninorm logics it is therefore preferable to use the names *lattice* and *residuated* conjunction instead of weak and strong conjunction. Another consequence of this fact is that the $\wedge$-adjunction rule (which is derivable in BL, see the comments after $(\mathrm{T_{BL}}16)$ in Section 1.2) has to be added as a primitive rule of the uninorm logic UL introduced below.

The logic UL of *all* left-continuous conjunctive uninorms can be axiomatized by the following axioms:

$(\mathrm{UL}1)$	$\varphi \to \varphi$
$(\mathrm{UL}2)$	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$(\mathrm{UL}3)$	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
$(\mathrm{UL}4)$	$(\varphi \mathbin{\&} \psi \to \chi) \leftrightarrow (\varphi \to (\psi \to \chi))$
$(\mathrm{UL}5a, b)$	$\varphi \wedge \psi \to \varphi, \quad \varphi \wedge \psi \to \psi$
$(\mathrm{UL}6)$	$(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$
$(\mathrm{UL}7a, b)$	$\varphi \to (\varphi \vee \psi), \quad \psi \to (\varphi \vee \psi)$
$(\mathrm{UL}8)$	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
$(\mathrm{UL}9)$	$\varphi \leftrightarrow (\overline{1} \to \varphi)$
$(\mathrm{UL}10a, b)$	$\bot \to \varphi, \quad \varphi \to \top$
$(\mathrm{UL}11)$	$((\varphi \to \psi) \wedge \overline{1}) \vee ((\psi \to \varphi) \wedge \overline{1})$

with the definitions:

$$\varphi \leftrightarrow \psi \equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$$
$$\neg\varphi \equiv \varphi \to \overline{0}$$

and the derivation rules of modus ponens and $\wedge$-adjunction:

(MP)	From φ and $\varphi \to \psi$ infer ψ
$(\wedge\text{-Adj})$	From φ and ψ infer $\varphi \wedge \psi$.

Further logics of various sets of left-continuous conjunctive uninorms will be introduced in Section 2.4. The logic MTL of left-continuous t-norms extends the logic UL of left-continuous conjunctive uninorms by the axiom stating that $\overline{0}$ is the smallest and $\overline{1}$ the largest truth value:

(W) $\quad (\overline{0} \to \varphi) \wedge (\varphi \to \overline{1})$.

Unlike MTL, the logic UL differs more radically from BL: e.g., the formulae $(\mathrm{T_{BL}}6)$, $(\mathrm{T_{BL}}2)$, and $(\mathrm{T_{BL}}11)$ are not theorems of UL, and the local deduction theorem of BL (cf. Theorem 1.2.10) is not valid for UL. However we can prove a modified variant of the deduction theorem:

$$T, \varphi \vdash_{\mathrm{L}} \psi \quad \text{iff} \quad \text{there is natural } n \text{ such that } T \vdash (\varphi \wedge \overline{1})^n \to \psi.$$

The logic UL enjoys strong standard completeness.

The algebras for which the logic UL is sound (i.e., UL-algebras) can be characterized as pointed semilinear[28] commutative bounded residuated lattices:

DEFINITION 2.1.5 ([67, 133]). *A* bounded pointed commutative residuated lattice, *or an* $\mathrm{FL_e}$-algebra,[29] *is an algebra* $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$ *such that:*

1. $\langle A, \wedge, \vee, \bot, \top\rangle$ *is a bounded lattice*
2. $\langle A, \&, \overline{1}\rangle$ *is a commutative monoid*
3. $\to$ *is the residuum of* $\&$*, i.e., for each* $x, y, z \in A$ *holds:* $x \,\&\, y \le z$ *iff* $x \le y \to z$.

A bounded pointed commutative residuated lattice $\boldsymbol{A}$ *is a* UL-algebra *if it satisfies the identity* $((x \to y) \wedge \overline{1}) \vee ((y \to x) \wedge \overline{1}) = \overline{1}$ *for all* $x, y \in A$.

2.1.3 Fuzzy logics with non-commutative conjunction

Now we shall survey fuzzy logics that omit the assumption that strong conjunction is commutative, i.e., that $x * y = y * x$ for all truth values x and y. The residuation condition (1) of Theorem 1.1.8 then has two non-equivalent variants:

$$z * x \le y \quad \text{iff} \quad z \le x \Rightarrow y$$
$$x * z \le y \quad \text{iff} \quad z \le x \Rightarrow y.$$

Each of them corresponds to a distinct residuum, respectively denoted by / and \, and called the *left* and *right* residuum. The residuation condition then reads:

$$x * y \le z \quad \text{iff} \quad x \le z \,/\, y \quad \text{iff} \quad y \le x \setminus z. \tag{5}$$

The general non-commutative case of residuated lattices (cf. Definitions 1.3.1 and 2.1.5) is thus defined as follows:

DEFINITION 2.1.6 ([67]). *A* residuated lattice *is a lattice equipped with a monotone monoidal operation* $*$ *(with a unit* e*) and a pair of operations* $/, \setminus$ *satisfying the residuation condition (5).*

A pointed residuated lattice, *or an* FL-algebra *(cf. Section 3.1) is a residuated lattice expanded by a constant* $\overline{0}$ *(with no conditions imposed on* $\overline{0}$*).*

---

[28] A class of algebras is called *semilinear* (a term *representable* is sometimes used instead) if all its subdirectly irreducible members (cf. Definition 1.3.10) are linearly ordered. Note that in BL- and MTL-algebras, the axiom $(x \to y) \vee (y \to x) = \overline{1}$ called *prelinearity* was sufficient to enforce semilinearity (which can equivalently be formulated as the subdirect representation theorem, cf. Theorem 1.3.11). In UL and $\mathrm{psMTL^r}$ (introduced in Section 2.1.3 below) this is no longer the case, and more complex axioms are needed to ensure semilinearity (cf. Definition 2.1.5 and footnote 31 on p. 34). See Section 3.3 for more details and Chapter II for a thorough exposition of this phenomenon.

[29] See Section 3.1 for an explanation of this terminology.

A residuated lattice or an FL*-algebra is called:*

- Commutative *if its monoidal operation* $*$ *is commutative.*
- Bounded *if its lattice reduct is a bounded lattice.*
- Integral *if the unit of* $*$ *is its greatest element.*

Clearly $*$ is commutative iff $/$ and $\backslash$ coincide, in which case they can be denoted by $\to$ as usual. In $[0,1]$, or any residuated lattice, the two residua can be explicitly defined as follows (cf. Theorem 1.1.9(1)):

$$z \,/\, y = \max\{x \mid x * y \leq z\}$$
$$x \setminus z = \max\{y \mid x * y \leq z\}$$

The residua $/$ and $\backslash$ are the pointwise largest (i.e., logically weakest) functions validating the two versions of 'internalized modus ponens':

$$(z \,/\, y) * y \leq z \tag{6}$$
$$x * (x \setminus z) \leq z \tag{7}$$

Integral bounded residuated lattices on $[0,1]$ are given by *left-continuous pseudo-t-norms* ([63]), i.e., binary operations $*\colon [0,1]^2 \to [0,1]$ which are associative, non-decreasing in both arguments, have 1 as both the left and right unit element, and are left-continuous in both arguments.

EXAMPLE 2.1.7. Let $0 < a < b < 1$ and let

$$x * y = \begin{cases} 0 & \text{if } x \leq a \text{ and } y \leq b \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

This is a non-commutative pseudo-t-norm which is left-continuous in both arguments.

The existence of the left and right residua of left-continuous pseudo-t-norms makes it possible to define the propositional logics of sets of left-continuous pseudo-t-norms along the lines of Section 1.1, with the main difference of having *two* implications: $\swarrow$ interpreted by $/$, and $\searrow$ interpreted by $\backslash$.[30] The *logic of all left-continuous pseudo-t-norms* has been denoted by $\mathrm{psMTL^r}$ and called *representable pseudo-*MTL.[31] It can be

[30] In the literature on substructural logics, the implications are usually denoted by the same signs $/, \backslash$ as the residua. In the literature on non-commutative fuzzy logic, $/, \swarrow$ are often denoted by $\to$ (with swapped arguments) and $\backslash, \searrow$ by $\rightsquigarrow$. Here we use the signs $\swarrow, \searrow$ suggested by L.N. Stout, since besides indicating the side of conjoining the antecedent in (6)–(7) they also mark the direction of the implication from the antecedent to the succedent.

[31] The superscript $^{\mathrm{r}}$ and the adjective *representable* come from the historical fact that an ill-motivated weaker logic lacking the axioms ($\mathrm{psMTL^r}$8a,b) was originally named psMTL. The latter logic was introduced in [88], following the terminology for previously introduced logic psBL [87], whose name was in turn based on previously defined corresponding algebraic structures [46], originally motivated by non-commutative version of MV-algebras [69]. The logic psMTL does not enjoy standard completeness nor linear subdirect representation (i.e., semilinearity, cf. footnote 28 on p. 33). The 'representability' means recovering the latter property in $\mathrm{psMTL^r}$ by adding the axioms ($\mathrm{psMTL^r}$8a,b) to the original psMTL. The fact that these axioms ensure this property was explicitly proved in [126]; it is also a corollary of the characterization of representable residuated lattices given in [119]. A systematic name $\mathrm{FL}^{\ell}_{\mathrm{w}}$ for $\mathrm{psMTL^r}$ will be introduced in Section 3.

axiomatized as follows:

$(\mathrm{psMTL^r1a,b})$	$(\varphi \searrow \psi) \searrow ((\chi \searrow \varphi) \searrow (\chi \searrow \psi)),\ \ ((\psi \swarrow \chi) \swarrow (\varphi \swarrow \chi)) \swarrow (\psi \swarrow \varphi)$
$(\mathrm{psMTL^r2a,b})$	$\varphi \,\&\, \psi \searrow \varphi, \quad \varphi \,\&\, \psi \searrow \psi$
$(\mathrm{psMTL^r3})$	$\varphi \vee \psi \searrow \psi \vee \varphi$
$(\mathrm{psMTL^r4a})$	$\varphi \wedge \psi \searrow \varphi$
$(\mathrm{psMTL^r4b})$	$\varphi \wedge \psi \searrow \psi \wedge \varphi$
$(\mathrm{psMTL^r4c,d})$	$\varphi \,\&\, (\varphi \searrow \psi) \searrow \varphi \wedge \psi,\ \ \varphi \wedge \psi \swarrow (\psi \swarrow \varphi) \,\&\, \varphi$
$(\mathrm{psMTL^r5a,b})$	$(\varphi \searrow (\psi \searrow \chi) \mathrel{\searrow\!\!\nwarrow} (\psi \,\&\, \varphi \searrow \chi)),\ \ ((\chi \swarrow \psi) \swarrow \varphi) \mathrel{\nearrow\!\!\swarrow} (\chi \swarrow \varphi \,\&\, \psi)$
$(\mathrm{psMTL^r6a})$	$((\varphi \searrow \psi) \searrow \chi) \searrow (((\psi \searrow \varphi) \searrow \chi) \searrow \chi)$
$(\mathrm{psMTL^r6b})$	$(\chi \swarrow (\chi \swarrow (\varphi \swarrow \psi))) \swarrow (\chi \swarrow (\psi \swarrow \varphi))$
$(\mathrm{psMTL^r7})$	$\overline{0} \searrow \varphi$
$(\mathrm{psMTL^r8a,b})$	$(\varphi \searrow \psi) \vee (\chi \,\&\, (\psi \searrow \varphi) \swarrow \chi),\ \ (\psi \swarrow \varphi) \vee (\chi \searrow (\varphi \swarrow \psi) \,\&\, \chi)$

with the definitions:

$$
\begin{aligned}
\varphi \vee \psi &\equiv ((\psi \swarrow \varphi) \searrow \psi) \wedge (\varphi \swarrow (\psi \searrow \varphi)) \\
\varphi \mathrel{\searrow\!\!\nwarrow} \psi &\equiv (\varphi \searrow \psi) \wedge (\psi \searrow \varphi) \\
\varphi \mathrel{\nearrow\!\!\swarrow} \psi &\equiv (\varphi \swarrow \psi) \wedge (\psi \swarrow \varphi)
\end{aligned}
$$

and the derivation rules:

(MP)	from φ and $\varphi \searrow \psi$ infer ψ
(Imp a, b)	from $\varphi \searrow \psi$ infer $\psi \swarrow \varphi$, from $\psi \swarrow \varphi$ infer $\varphi \searrow \psi$.

The logic $\mathrm{psMTL^r}$ also enjoys a variant of the local deduction theorem, but we do not give it here due to its complexity (see Chapter II).

The algebras for which the logic $\mathrm{psMTL^r}$ is sound are called *representable* psMTL-*algebras,* or $\mathrm{psMTL^r}$-algebras; they can be characterized as semilinear bounded integral residuated lattices (see Definition 2.1.6). Their lattice reduct is $[0,1]$ if and only if $*$ is a left-continuous pseudo-t-norm (then they are called *standard* $\mathrm{psMTL^r}$-algebras). The strong standard completeness theorem holds for $\mathrm{psMTL^r}$ [118].

A similar non-commutative generalization can be done with other logics. In this way we obtain, e.g., the logic $\mathrm{psBL^r}$ that extends $\mathrm{psMTL^r}$ by the converse of the axioms $(\mathrm{psMTL^r4c,d})$,[32] or the logic psUL of semilinear bounded residuated lattices. In both cases, however, the standard completeness is lost: in $\mathrm{psBL^r}$ it is simply because any *continuous* pseudo-t-norm is already a continuous *t-norm* [63] and it can be shown [177] that psUL is not the logic of residuated pseudo-uninorms either (for details see Chapter IV).

[32] Consequently, $\mathrm{psBL^r}$-algebras are $\mathrm{psMTL^r}$-algebras satisfying the identity $x * (x \setminus y) = (y / x) * x = x \wedge y$.

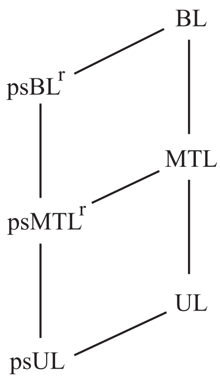

Figure 5. Relative position of fuzzy logics introduced in Section 2.1

2.2 Adding new connectives

We shall now survey another direction in varying the apparatus of propositional fuzzy logics, namely expanding its expressive power by adding new primitive connectives. It can be observed that truth functions definable by means of the basic propositional connectives (&, $\to$, $\wedge$, $\vee$, $\leftrightarrow$, $\neg$, and $\overline{0}$) of the logic MTL in a given standard MTL-algebra form just a limited subset of all truth functions $[0,1]^n \to [0,1]$ (or $A^n \to A$ in a general MTL-algebra $\boldsymbol{A}$). The same is true about other fuzzy logics treated in previous sections (for characterizations of truth functions representable by formulae in various fuzzy logics see Section 4.2). Various important propositional concepts are actually unexpressible by these basic connectives, and if they are to be available in a system of fuzzy logic, new primitive connectives with suitable truth functions need be added. The logics introduced in this subsection are studied in detail in Chapter VIII.

2.2.1 The Delta connective

One of the concepts expressible in neither MTL nor other logics of left-continuous t-norms (except some of their finitely valued variants, see Section 2.4.3) is the notion of full truth of a proposition, formally represented by a truth function $\triangle\colon \boldsymbol{A} \to \boldsymbol{A}$ such that the conditions $\triangle(x) = \overline{0}^{\boldsymbol{A}}$ for all $x < \overline{1}^{\boldsymbol{A}}$ and $\triangle(\overline{1}^{\boldsymbol{A}}) = \overline{1}^{\boldsymbol{A}}$ would hold at least in the intended semantics (i.e., standard or linear): for instance, such a function is clearly not definable in the standard MV-algebra, due to the continuity of its primitive truth functions. This deficiency can be remedied by adding a new primitive connective $\triangle$,[33] with the standard and linear semantics given as:

$$\triangle(x) = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } x = \overline{1}^{\boldsymbol{A}} \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases} \tag{8}$$

[33] The connective first appeared in Monteiro's paper [142] in the context of intuitionistic and Gödel logics, and was later extensively studied by M. Baaz; therefore it is often called *Baaz Delta.* Its axiomatization was generalized for other fuzzy logics in [83].

Tarski conditions (see Definition 1.1.13) are then extended by the clause:

$$e_*(\triangle\varphi) = \triangle(e_*(\varphi)).$$

Let us fix (for the rest of this subsection) an axiomatic extension L of MTL. A fuzzy logic L expanded by this connective is called “L with $\triangle$” and denoted by $L_\triangle$. It can be axiomatized by adding the following axioms and rule to the axiomatic system for the logic L:

($\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$
($\triangle$2) $\triangle(\varphi \vee \psi) \to \triangle\varphi \vee \triangle\psi$
($\triangle$3) $\triangle\varphi \to \varphi$
($\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$
($\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

and the rule of $\triangle$-necessitation,

($\triangle$-Nec) from φ infer $\triangle\varphi$.

Linear $L_\triangle$-algebras are linear L-algebras expanded by the operation $\triangle$ defined by (8) above.[34] The usual linear and general completeness theorems hold for $L_\triangle$, as well as the linear subdirect representation theorem. In consequence of these properties, $L_\triangle$ extends L conservatively. The form of the deduction theorem for $L_\triangle$ is different from that for L (see Theorem 1.2.10):

THEOREM 2.2.1 ($\triangle$-deduction theorem). *Let* L *be an axiomatic extension of* MTL. *Then for any set of formulae T and formulae φ, ψ of $L_\triangle$, the following equivalence holds:*

$$T, \varphi \vdash_{L_\triangle} \psi \quad \textit{iff} \quad T \vdash_{L_\triangle} \triangle\varphi \to \psi.$$

The *standard* $L_\triangle$-algebras are defined (cf. Convention 2.0.2) as those $L_\triangle$-algebras whose lattice reduct is $[0,1]$, with the exception of $Ł_\triangle$ and $\Pi_\triangle$, where only the single t-algebras $[0,1]_Ł$ resp. $[0,1]_\Pi$ expanded by $\triangle$ are considered standard. The standard $L_\triangle$-algebras are thus the standard L-algebras expanded by the operation $\triangle$ defined by (8). The logic $L_\triangle$ is strong (resp., finite strong) standard complete if and only if L is strong (resp., finite strong) standard complete [35].

Besides the notion of full truth, also the ordering of truth values is internalized in $L_\triangle$, as $\triangle(x \to y) = \overline{1}$ iff $x \leq y$. The connective $\triangle$ makes it possible to interpret classical logic in $L_\triangle$ by prefixing each propositional variable by $\triangle$. Moreover, the logic $G_\triangle$ can be interpreted in any $L_\triangle$ (so already in $MTL_\triangle$), as Gödel conjunction is the lattice conjunction of $MTL_\triangle$ and Gödel implication is definable in $MTL_\triangle$ by setting:

$$\varphi \to_G \psi \equiv_{df} \triangle(\varphi \to \psi) \vee \psi.$$

[34] The semantics of $\triangle$ in non-linear $L_\triangle$ algebras, however, need not be that given by (8): rather, general $L_\triangle$-algebras are L-algebras $\boldsymbol{B}$ expanded by the operation defined as (8) *in every component* $\boldsymbol{A}$ of the linear subdirect representation of $\boldsymbol{B}$.

2.2.2 Fuzzy logics with additional involutive negation

Another frequent expansion of basic t-norm logics is by adding an extra unary connective $\sim$ which is order-reversing (i.e., if $x \leq y$ then $\sim x \geq \sim y$ for all truth values x, y) and involutive (i.e., $\sim\sim x = x$ for all truth values x). The connective is usually called *involutive negation,* or simply *involution.* The expansion is done in a similar way as in the case of $\triangle$.

Let us again fix (for the rest of this subsection) an axiomatic extension L of MTL or $\mathrm{MTL}_\triangle$. The logic L expanded by involutive negation is called "L with involution" and denoted by $\mathrm{L}_\sim$.[35] In logics with involution we depart from Convention 2.0.2 and call *standard* $\mathrm{L}_\sim$-algebras only the expansions of standard L-algebras by the *standard involutive negation* $\sim x = 1 - x$. Thus there are just single standard $\Pi_\sim$- and $\mathrm{G}_\sim$-algebras, and the standard $\mathrm{MV}_\sim$-algebra is (modulo the signature) identical with the standard MV-algebra $[0,1]_Ł$, as in the former both negations ($\neg$ and $\sim$) coincide.

The logic $\mathrm{L}_\sim$ is axiomatized by adding the following axiom and rule to the logic L:

$(\neg\neg)$	$\sim\sim\varphi \leftrightarrow \varphi$
$(\mathrm{Rev}\text{-}\vee)$	From $(\varphi \to \psi) \vee \chi$ infer $(\sim\psi \to \sim\varphi) \vee \chi$.

In logics with $\triangle$, the rule $(\mathrm{Rev}\text{-}\vee)$ can equivalently be replaced by the axiom

$(\mathrm{Rev}\text{-}\triangle)$	$\triangle(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$.

If $\neg(\varphi \wedge \neg\varphi)$ is a theorem of L (i.e., the residual negation $\neg$ of L is strict, as, e.g., in G, Π, or SBL; cf. Example 1.1.11), then $(\mathrm{Rev}\text{-}\vee)$ can be replaced by the following two axioms and rule [30]:

$(\mathrm{Rev\text{-}S1})$	$\neg\sim(\varphi \to \psi) \to \neg\sim(\sim\psi \to \sim\varphi)$
$(\mathrm{Rev\text{-}S2})$	$\neg\varphi \to \sim\psi$
$(\mathrm{Rev\text{-}S3})$	From φ infer $\neg\sim\varphi$.

In such logics, the connective $\triangle$ is definable by setting $\triangle\varphi \equiv \neg\sim\varphi$. Thus if L has strict residual negation, then $\mathrm{L}_\sim$ includes $\mathrm{L}_\triangle$; in particular, $\mathrm{G}_\triangle \subseteq \mathrm{G}_\sim$ and $\Pi_\triangle \subseteq \Pi_\sim$. Furthermore, in this case the logic $\mathrm{L}_\sim$ enjoys the same deduction theorem as $\mathrm{L}_\triangle$ (cf. Theorem 2.2.1)

A (form of) standard completeness of $\mathrm{L}_\sim$ can be proved for a broad class of t-norm logics satisfying certain criteria (for details see [62] or Chapter VIII), among them the logics $\mathrm{MTL}_\sim$, $\mathrm{BL}_\sim$, $\mathrm{SBL}_\sim$, and $\mathrm{G}_\sim$. As in the case of $\mathrm{L}_\triangle$, the logic $\mathrm{L}_\sim$ is strong (resp., finite strong) standard complete if and only if L is strong (resp., finite strong) standard complete.

[35] As in the case of Baaz Delta, the additional involutive negation first appeared in [142] in the context of intuitionistic and Gödel logics. Later is was studied in [56] in logics L extending SBL (in which the connective $\triangle$, needed for the rule (Rev-$\triangle$) below, becomes definable in $\mathrm{L}_\sim$). The next generalization [62] required that the logic L extend $\mathrm{MTL}_\triangle$; the logic called $\mathrm{MTL}_\sim$ in [62] thus in fact denoted the logic $\mathrm{MTL}_{\triangle\sim}$. We are changing the terminology here to a more systematic one, since the $\triangle$ can be dispensed with by using the rule (Rev-$\vee$); cf. Chapter II. Note also that this generalization also allows us to add an additional involution $\sim$ to a logic in which already the residual negation $\neg$ is involutive, e.g., IMTL or Ł.

The logics $Ł_\sim$ and $\Pi_\sim$ fail to be standard complete, i.e., complete w.r.t. the (single) standard MV-algebra $[0,1]_Ł$ resp. Π-algebra $[0,1]_\Pi$ expanded by the *single* standard involution $1-x$.[36] Nevertheless, they do enjoy what is sometimes called the *semi-standard* completeness [56], i.e., completeness w.r.t. the (single) standard MV or Π-algebra expanded by *arbitrary* involutive negation $\sim$ on $[0,1]$. Interestingly, both these logics are also complete w.r.t. the class of all *real-valued* MV or Π-algebras expanded by the *standard* involutive negation $1-x$ (this is a simple corollary of the fact that all involutive negations on $[0,1]$ are isomorphic). Thus in $Ł_\sim$ and $\Pi_\sim$ we can, roughly speaking, fix either the 'standard' t-norm or the 'standard' negation, but not both at once. Clearly the logic of the standard $\mathrm{MV}_\sim$-algebra is the axiomatic extension of $Ł_\sim$ by the axiom $\sim\varphi \leftrightarrow \neg\varphi$. The case of $\Pi_\sim$ is more involved and interesting; we will comment on it further in Section 2.2.3. Axiomatic extensions of $\Pi_\sim$ were studied in detail in [40] and [105].

An important feature of the logics $\mathrm{L}_\sim$ is the definability of the so-called *strong disjunction,* i.e., a connective $\oplus$ dual to strong conjunction:

$$\varphi \oplus \psi \equiv {\sim}({\sim}\varphi \mathbin{\&} {\sim}\psi).$$

The standard semantics of $\oplus$ on $[0,1]$ is that of a *t-conorm:*

DEFINITION 2.2.2 ([123]). *A binary function $S\colon [0,1]^2 \to [0,1]$ is called a* triangular conorm *(a* t-conorm *for short) if it is commutative, associative, monotone, and* 0 *is its neutral element.*

T-norms and t-conorms are mutually dual via the standard involution $1-x$: every t-norm T has the dual t-conorm $S_T(x,y) = 1 - T(1-x, 1-y)$, and every t-conorm S has the dual t-norm $T_S(x,y) = 1 - S(1-x, 1-y)$, with $S_{T_S} = S$ and $T_{S_T} = T$.

EXAMPLE 2.2.3. The t-conorms dual to the three prominent continuous t-norms come out as follows:

- The t-conorm dual to $*_Ł$ is the *bounded sum* $x \oplus_Ł y = \max\{x+y, 1\}$.
- The t-conorm dual to $*_\Pi$ is the *probabilistic sum* $x \oplus_\Pi y = x + y - xy$.
- The t-conorm dual to $*_\mathrm{G}$ is the *maximum.* Consequently, due to the standard completeness of $\mathrm{G}_\sim$, strong disjunction $\oplus$ coincides with lattice disjunction $\vee$ in $\mathrm{G}_\sim$.

The presence of two disjunctions alongside two conjunctions in fuzzy logics is natural in view of their relation to substructural logics (see Section 3.1). Various concepts can be expressed by means of strong disjunction and involution, including the so-called *S-implication,* $I_S(x,y) = {\sim}x \oplus y$, often encountered in applied fuzzy logic, or the *Q-implication,* $I_Q(x,y) = {\sim}x \oplus (x \mathbin{\&} y)$, related to the implication used in quantum logic. De Morgan laws with $\sim$ hold not only for $\&$ and $\oplus$ (by the definition of $\oplus$), but also for $\wedge$ and $\vee$ in $\mathrm{MTL}_\sim$.

[36] This failure is obvious in the case of $Ł_\sim$, where standard completeness would entail that $\sim$ coincides with residual negation, which clearly is not the case; the failure in $\Pi_\sim$ is shown in [56].

2.2.3 Fuzzy logics with intermediate truth constants

Another way of enriching the expressive power of a fuzzy logic is expansion by truth constants (i.e., nullary connectives) other than $\overline{0}$ and $\overline{1}$. If a new truth constant is intended to have a fixed standard semantics $r \in [0,1]$, it is denoted by $\overline{r}$; in this case, Convention 2.0.2 is strengthened so as to admit as standard only those algebras that indeed interpret $\overline{r}$ as r.

EXAMPLE 2.2.4. Łukasiewicz logic can easily be expanded by a truth constant with the standard semantics of one half: since $\neg_{Ł}0.5 = 0.5$, the axiom $\neg\overline{0.5} \leftrightarrow \overline{0.5}$ enforces that the truth constant $\overline{0.5}$ be interpreted by 0.5 in the standard MV-algebra $[0,1]_{Ł}$. In the general semantics, the constant $\overline{0.5}$ has to be interpreted by the fixed point of $\neg$. Only those MV-algebras that do have this fixed point can be expanded to models of Ł with $\overline{0.5}$. The expansion therefore excludes, i.a., the two-element (Boolean) MV-algebra $\{0,1\}$, and so ensures the many-valuedness of the logic.

An important logic is the expansion of Łukasiewicz logic by truth constants for all rational numbers in $[0,1]$. The idea of using truth constants denoting truth degrees in the language of fuzzy logic goes back to Pavelka [154], who used truth constants for all reals from $[0,1]$. Later it turned out, though, that for the main results to work well it is sufficient to introduce truth constants just for the rational numbers of $[0,1]$, which does not force the language to be uncountable.

DEFINITION 2.2.5 ([83]). *The* Rational Pavelka logic, *denoted by* RPL*, is the expansion of Łukasiewicz logic by the truth constants* $\overline{r}$ *for each rational* $r \in [0,1]$ *and the* 'bookkeeping axioms' *for each rational* $r, s \in [0,1]$:

$$\overline{r} \,\&\, \overline{s} \leftrightarrow \overline{r * s}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}.$$

The standard RPL-algebra *is the expansion of the standard* MV*-algebra* $[0,1]_{Ł}$ *with each* $\overline{r}$ *interpreted as* r.

The logic RPL is a conservative extension of Łukasiewicz logic and possesses the same deduction theorem. It has the finite strong standard completeness. Moreover, RPL enjoys the following property called *Pavelka-style completeness* for *arbitrary* sets of formulae:

THEOREM 2.2.6 (Pavelka-style completeness of RPL, [83]). *Let* T *be a set of formulae (possibly infinite) and* φ *a formula of* RPL*. Let the* provability degree *of* φ *in the theory* T *be the real number*

$$|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}} \overline{r} \to \varphi\}$$

and the truth degree $\|\varphi\|_T$ *of* φ *in* T *be the infimum of truth degrees of* φ *in all standard models of* T*. Then*

$$|\varphi|_T = \|\varphi\|_T\,.$$

The continuity of $\to$ is essential for Pavelka-style completeness; in expansions by rational truth constants of t-norm logics other than Łukasiewicz, additional infinitary rules have to be added in order for these logics to enjoy this style of completeness (see [33, 56, 57] for details). It is, nevertheless, possible to consider just the usual style of completeness for these logics; see e.g. [51, 59, 167] or Chapter VIII for a detailed exposition of known results.

Finally, Pavelka-style logics are closely related to the so-called *fuzzy logics with evaluated syntax,* which also incorporate the truth values of formulae directly into the syntax of the logic. An evaluated formula is a pair $\langle r, \varphi \rangle$, which expresses the fact that the truth value of φ is at least r. The rules of fuzzy logics with evaluated syntax operate on evaluated formulae: e.g., the evaluated rule of modus ponens has the form:

$$\text{From } \langle r, \varphi \rangle \text{ and } \langle s, \varphi \to \psi \rangle \text{ infer } \langle r *_{\text{Ł}} s, \psi \rangle.$$

The logic with evaluated syntax based on Łukasiewicz logic, denoted by Ł_{Ev}, can be interpreted in RPL, by translating an evaluated formula $\langle r, \varphi \rangle$ as the RPL-formula $\overline{r} \to \varphi$. The variant $\text{Ł}\Pi_{\text{Ev}}$ based on the logic $\text{Ł}\Pi$ introduced below has also been considered. See [150] for a comprehensive treatment of these logics and further references.

2.2.4 Fuzzy logics with multiple sets of t-norm connectives

As stated above, one of the main reasons for expanding t-norm logics by additional connectives is the fact that only a limited set of truth functions is in general available in the logic of any particular left-continuous t-norm (cf. Section 4.2). Thus, for example, Łukasiewicz logic only possesses connectives interpreted by additive arithmetical operations, while in product logic we are only in possession of 'multiplicative' connectives. A solution to the need of possessing a fuller arithmetic power over truth degrees is to combine connectives pertaining to several left-continuous t-norms in one logic.

Several logics of this kind, differing in expressive power, have been described in the literature. Most of them add connectives pertaining to the product t-norm to Łukasiewicz logic. Adding just the product conjunction to Ł or $\text{Ł}_\triangle$, with various strength of axioms, leads to logics PŁ and $\text{PŁ}'$ (possibly with $\triangle$, see [112])[37] and the propositional logic of Takeuti–Titani [171]. These logics have remarkable logical properties: the logic PŁ is not even weakly standard complete (see Convention 2.0.3) w.r.t. the standard MV-algebra expanded by the connective $\&_\Pi$ interpreted as $*_\Pi$. The logic $\text{PŁ}'$, which extends PŁ by the deduction rule $\neg(\varphi \&_\Pi \varphi) \vdash \neg\varphi$ does have the finite strong standard completeness, but the additional rule cannot be replaced by axioms. Consequently, $\text{PŁ}'$ lacks the deduction theorem (cf. Theorem 1.2.10) and the class of $\text{PŁ}'$-algebras is not a variety of algebras.

Despite these interesting properties we will not describe these logics in detail here and will rather concentrate on the (much better behaving) logic $\text{Ł}\Pi$, which arises from adding both product conjunction and implication to Łukasiewicz logic, and its expansion $\text{Ł}\Pi\frac{1}{2}$ by the truth constant $\frac{1}{2}$ (cf. Example 2.2.4).[38] The logic $\text{Ł}\Pi\frac{1}{2}$ is one of the expres-

[37] The algebras for these logics are studied under the name PMV-algebras, see, e.g., [44, 137, 141].

[38] A more systematic denotation for the connective represented by the value $\frac{1}{2}$ in the standard semantics would be $\overline{\frac{1}{2}}$, with the expansion of $\text{Ł}\Pi$ by this connective denoted by $\text{Ł}\Pi\overline{\frac{1}{2}}$. For typographical reasons, however, we shall follow the tradition of denoting the constant simply by $\frac{1}{2}$ and the logic by $\text{Ł}\Pi\frac{1}{2}$.

sively strongest fuzzy logics studied in the literature; it includes many other t-norm fuzzy logics and contains a broad class of definable connectives, while still possessing good metamathematical properties. In this sense, the logic $\text{Ł}\Pi\frac{1}{2}$ can be viewed as an over-arching system for a large class of t-norm fuzzy logics.

DEFINITION 2.2.7 ([31, 52, 57]). *The primitive connectives of the* logic ŁΠ *are the truth constant* $\overline{0}$*, Łukasiewicz implication* $\to_{\text{Ł}}$*, product implication* $\to_{\Pi}$*, and product conjunction* $\&_{\Pi}$*. Furthermore we define the following derived connectives of* ŁΠ:

$$
\begin{aligned}
\neg_{\text{Ł}}\varphi &\equiv \varphi \to_{\text{Ł}} \overline{0} \\
\neg_{\Pi}\varphi &\equiv \varphi \to_{\Pi} \overline{0} \\
\overline{1} &\equiv \neg_{\text{Ł}}\overline{0} \\
\triangle\varphi &\equiv \neg_{\Pi}\neg_{\text{Ł}}\varphi \\
\varphi \,\&_{\text{Ł}}\, \psi &\equiv \neg_{\text{Ł}}(\varphi \to_{\text{Ł}} \neg_{\text{Ł}}\psi) \\
\varphi \oplus \psi &\equiv \neg_{\text{Ł}}\varphi \to_{\text{Ł}} \psi \\
\varphi \ominus \psi &\equiv \varphi \,\&_{\text{Ł}}\, \neg_{\text{Ł}}\psi \\
\varphi \wedge \psi &\equiv \varphi \,\&_{\text{Ł}}\, (\varphi \to_{\text{Ł}} \psi) \\
\varphi \vee \psi &\equiv (\varphi \to_{\text{Ł}} \psi) \to_{\text{Ł}} \psi \\
\varphi \to_{\mathrm{G}} \psi &\equiv \triangle(\varphi \to_{\text{Ł}} \psi) \vee \psi
\end{aligned}
$$

and $\varphi \leftrightarrow_{*} \psi \equiv (\varphi \to_{*} \psi) \wedge (\psi \to_{*} \varphi)$ *for* $* \in \{\text{Ł}, \Pi, \mathrm{G}\}$. *The axioms and rules of the logic* ŁΠ *are the following:*

(Ł)	the axioms and rules of Ł (with Ł-subscripted connectives)
(Π)	the axioms and rules of Π (with Π-subscripted connectives)
($\triangle$-Nec)	from φ infer $\triangle\varphi$
($\to_{\text{ŁΠ}}$ a, b)	$\triangle(\varphi \to_{\text{Ł}} \psi) \to_{\text{Ł}} (\varphi \to_{\Pi} \psi)$, $\quad \triangle(\varphi \to_{\Pi} \psi) \to_{\text{Ł}} (\varphi \to_{\text{Ł}} \psi)$
(Distr)	$\varphi \,\&_{\Pi}\, (\psi \ominus \chi) \leftrightarrow_{\text{Ł}} (\varphi \,\&_{\Pi}\, \psi) \ominus (\varphi \,\&_{\Pi}\, \chi)$.

The standard ŁΠ-algebra $[0,1]_{\text{ŁΠ}}$ *is the standard* MV*-algebra* $[0,1]_{\text{Ł}}$ *expanded by the product t-norm and its residuum.*

In the standard ŁΠ-algebra $[0,1]_{\text{ŁΠ}}$, the connectives subscripted by Ł, Π, or G have the standard semantics of the corresponding t-norm logics. The connectives $\oplus$ and $\ominus$, definable already in Ł, have the standard semantics of bounded addition and subtraction.

The tautologicity and the (finitary) consequence relation are defined along the usual lines (cf. Section 1.1), with 1 as the only designated truth value. The ŁΠ*-algebras,* i.e., general algebras for which the logic ŁΠ is sound (see Convention 2.0.1), can be characterized as MV-algebras expanded by the product adjoint pair (so that the appropriate reduct is a Π-algebra) and satisfying the distributivity axiom (Distr). The logic ŁΠ enjoys only the finite strong standard completeness theorem [57].

Due to the presence of the basic connectives of all the three logics Ł, Π, and G as well as $\triangle$ and an involutive negation $\neg_{\text{Ł}}$, the logic ŁΠ contains the logics $\text{Ł}_{\triangle}$, $\Pi_{\sim}$, and $\mathrm{G}_{\sim}$ (thus also $\Pi_{\triangle}$, $\mathrm{G}_{\triangle}$, and all weaker logics). It can moreover be shown to extend $\mathrm{G}_{\sim}$, Ł, and Π (both with or without △) conservatively. Several alternative axiomatizations of ŁΠ are known; for instance, it can be viewed as an *axiomatic* extension of $\Pi_\sim$ by the axiom (BL1) for Łukasiewicz implication defined as $\varphi \to_{Ł} \psi \equiv \sim(\varphi \mathbin{\&} \sim(\varphi \to \psi))$ [30], or by the axiom (BL3) for Łukasiewicz conjunction defined as $\varphi \mathbin{\&}_{Ł} \psi \equiv \varphi \mathbin{\&} \sim(\varphi \to \sim\psi)$ [174], where the connectives of $\Pi_\sim$ are taken for the product connectives and $\neg_{Ł}$ of ŁΠ.

The logic ŁΠ can further be expanded by the truth constant $\frac{1}{2}$ (see footnote 38 on p. 41) satisfying the axiom $\frac{1}{2} \leftrightarrow_{Ł} \neg_{Ł}\frac{1}{2}$, with the standard semantics 0.5 (cf. Example 2.2.4). This not only excludes the two-valued ŁΠ-algebra, in which the connectives coincide with the classical bivalent ones, but also increases significantly the expressive power of the logic, as all rational truth constants $\overline{r}$ are definable and their bookkeeping axioms are provable in ŁΠ$\frac{1}{2}$ (cf. Definition 2.2.5); ŁΠ$\frac{1}{2}$ thus contains the logic RPL. The metamathematical properties of ŁΠ mentioned in the previous paragraphs hold for ŁΠ$\frac{1}{2}$ as well.

The logic ŁΠ$\frac{1}{2}$ thus contains connectives corresponding in standard semantics to all basic arithmetical operations (namely, $\wedge$, $\vee$, $\oplus$, $\ominus$, $\&_{\Pi}$, and $\to_{\Pi}$) and comparison relations (namely, $\triangle(\varphi \to_{Ł} \psi)$ and $\triangle(\varphi \leftrightarrow_{Ł} \psi)$) as well as all rational numbers in $[0,1]$. Consequently, a broad class of truth functions on $[0,1]$ is representable in the standard semantics of ŁΠ$\frac{1}{2}$ (see Section 4.2). The ŁΠ$\frac{1}{2}$-representable functions include not only the truth functions of basic connectives of the three salient continuous t-norms Ł, Π, and G, but also of all their finite ordinal sums:

THEOREM 2.2.8 ([32]). *Let $*$ be a finite ordinal sum of $*_{Ł}$, $*_{\Pi}$, and $*_{G}$ on rational subintervals of $[0,1]$ and $\Rightarrow_*$ be its residuum. Then there are connectives $\&_*$ and $\to_*$ definable in ŁΠ$\frac{1}{2}$ with the standard semantics of $*$ and $\Rightarrow_*$. Moreover, if φ is provable in the logic L_* of $*$, then the formula φ_* obtained by replacing the connectives $\&$ and $\to$ of L_* by the defined connectives $\&_*$ and $\to_*$ of ŁΠ$\frac{1}{2}$ is provable in ŁΠ$\frac{1}{2}$.*

The logic ŁΠ$\frac{1}{2}$ thus contains the logics of all finite ordinal sums of the basic continuous t-norms, and also many particular left-continuous t-norms (e.g., the nilpotent minimum of Example 2.1.1) and uninorms (see [130] for details).

The logics ŁΠ and ŁΠ$\frac{1}{2}$ (as well as the logics PŁ and PŁ$'$ mentioned in the beginning of this subsection, either with or without △) can be extended by rational truth constants (which in ŁΠ$\frac{1}{2}$ are already definable) and appropriate infinitary rules to obtain their Pavelka-style extensions (containing RPL and contained in the Pavelka-style extension RŁΠ of ŁΠ) that enjoy Pavelka-style completeness (cf. Theorem 2.2.6).

2.3 Discarding connectives

Various fragments of fuzzy logics in restricted languages have been studied. The most important ones are certain natural expansions of the logic BCK (for which see, e.g., [116, 152] or [67, §2.3.2]). An extensive study of these fragments is given in [38]. In this subsection we restrict ourselves to the axiomatic extensions of MTL introduced in the previous sections and to languages containing implication (as implication-free fragments of our fuzzy logics are essentially classical, see [1]) and a subset of the connectives $\{\&, \wedge, \vee, \overline{0}\}$.

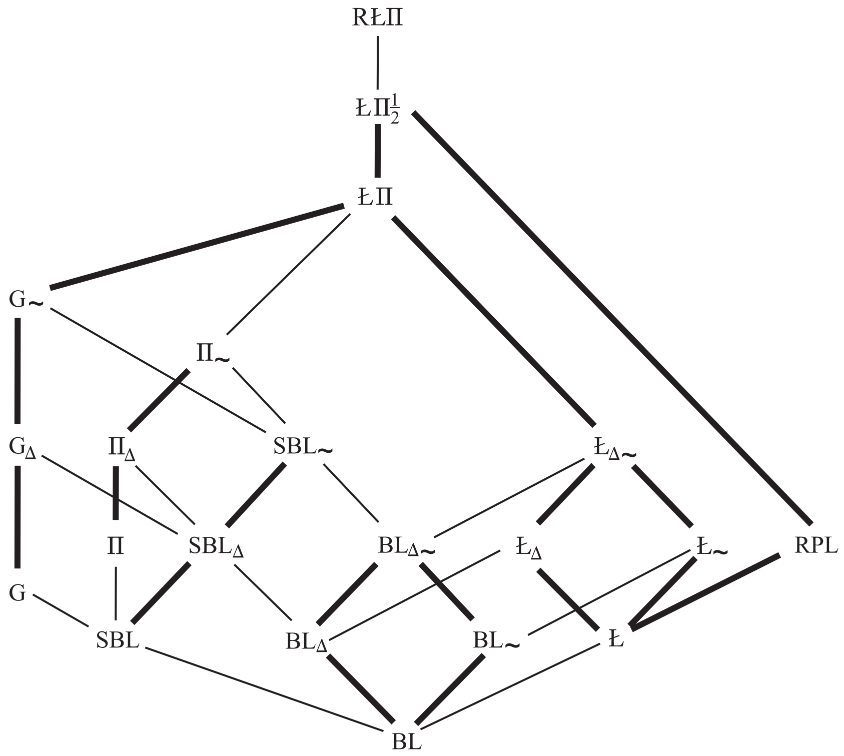

Figure 6. Relative positions of prominent logics introduced in Section 2.2. Thin lines indicate extensions in the same language (or just with addition of definable connectives), while thick lines indicate expansions by connectives not definable in the weaker logic.

Let $\mathcal{L}$ be a sublanguage of $\{\to, \&, \wedge, \vee, \overline{0}\}$ and L an axiomatic extension of MTL. The $\mathcal{L}$-fragment of L, denoted by $\mathrm{L}\restriction\mathcal{L}$, is the logic in the language $\mathcal{L}$ such that $\Gamma \vdash_{\mathrm{L}\restriction\mathcal{L}} \varphi$ iff $\Gamma \vdash_{\mathrm{L}} \varphi$, for every $\Gamma \cup \{\varphi\} \in Fm_{\mathcal{L}}$. Algebraically speaking, the $\mathcal{L}$-reduct of an algebra $\boldsymbol{A}$ is the algebra $\boldsymbol{A}\restriction\mathcal{L}$ with the same domain and operations from $\mathcal{L}$; i.e., we just 'forget' the operations not in $\mathcal{L}$. An $\mathcal{L}$-subreduct of $\boldsymbol{A}$ is just a subalgebra of the $\mathcal{L}$-reduct of $\boldsymbol{A}$. For a logic L in language $\mathcal{L}$ and a connective $c \in \mathcal{L}$, the logic $\mathrm{L}\restriction(\mathcal{L}\setminus\{c\})$ is sometimes called the c-free fragment of L, and an analogous convention is used for (sub)reducts of algebras.

2.3.1 Falsity-free fuzzy logics

Here we shall briefly discuss $\overline{0}$-free fragments of fuzzy logics. L-algebras for $\overline{0}$-free t-norm fuzzy logics L fall within the class of algebras known as *hoops*. Hoops were introduced in [21] and studied, e.g., in [17, 61]. The seminal paper on hoops in the context of fuzzy logic is [55], followed by [2]; see these two papers (where all results mentioned in this section could be found) or Chapter V for detailed references.

DEFINITION 2.3.1. *A structure* $\boldsymbol{H} = \langle H, \&, \to, \overline{1}\rangle$ *is a* hoop *if* $\&$ *is a commutative operation on* H *with the unit* $\overline{1}$ *and* $\to$ *is a binary operation satisfying*

$$\begin{aligned} x \to x &= \overline{1} \\ x \,\&\, (x \to y) &= y \,\&\, (y \to x) \\ x \to (y \to z) &= (x \,\&\, y) \to z \end{aligned}$$

for all $x, y, z \in H$. *Define* $x \leq y$ *iff* $x \to y = \overline{1}$. *A hoop is*

- Prelinear[39] *if* $(x \to y) \to z \leq ((y \to x) \to z) \to z)$ *for all* $x, y, z \in H$.
- Wajsberg *if* $(x \to y) \to y = (y \to x) \to x$ *for all* $x, y \in H$.
- Cancellative *if* $x \,\&\, y \leq x' \,\&\, y$ *implies* $x \leq x'$ *for all* $x, x', y \in H$.

One can prove that, in each hoop, $\leq$ is an ordering, $\&$ is associative and non-decreasing, $\to$ is the residuum of $\&$, and $\overline{1}$ is the largest element. Hoops can be characterized as divisible integral commutative residuated lattices (see Definition 1.3.1).

Prelinear hoops are precisely the $\overline{0}$-free (or falsity-free) subreducts (i.e., subalgebras of $\overline{0}$-free reducts) of BL-algebras, and Wajsberg hoops are falsity-free subreducts of Wajsberg algebras.[40] Note that each cancellative hoop is Wajsberg and each Wajsberg hoop is prelinear. On the other hand, each unbounded Wajsberg hoop is cancellative and bounded Wajsberg hoops are $\overline{0}$-free *reducts* of Wajsberg algebras. Linearly ordered Wajsberg hoops play an important rôle in the description of the structure of BL-chains, which can be decomposed into an ordinal sum of linearly ordered Wajsberg hoops in a similar (though slightly different) manner as can continuous t-norms be decomposed into an ordinal sum of the three basic continuous t-norms by Theorem 1.1.7(5); see Chapter V for details.

The logic BLH of prelinear hoops, or *basic hoop logic,* has the axioms $(\mathrm{BL1})$–$(\mathrm{BL6})$ of the logic BL and the rule of modus ponens; thus only the last axiom $(\mathrm{BL7})$ which speaks of $\overline{0}$ is deleted (see Section 1.2). Similarly the logic MTLH is the logic with the axioms of MTL except the last one (speaking of $\overline{0}$) and modus ponens. The logic GH extends BLH by the axiom (G) of Gödel logic. The logic ŁH of Wajsberg hoops extends BLH by the axiom

(WH) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

The logic $\Pi\mathrm{H}$ extends BLH by the following three axioms:

(ΠH1) $(\varphi \to \varphi \,\&\, \varphi) \to ((\psi \wedge (\psi \to \varphi)) \to \varphi)$
(ΠH2) $((\chi \to \upsilon) \to \upsilon) \,\&\, (\chi \,\&\, \varphi \to \chi \,\&\, \psi) \,\&\, (\upsilon \,\&\, \varphi \to \upsilon \,\&\, \psi) \to (\varphi \to \psi)$
(ΠH3) $((\psi \to \chi) \to ((\psi \to \varphi) \to \varphi)) \to (((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi))$.

[39]The term 'basic hoop' has been used in the literature instead [55]. However, that terminology is rather confusing, as the condition is equivalent to the *prelinearity* condition from the definition of BL- and MTL-algebras, while the divisibility condition (which needs to be added to MTL-algebras to obtain BL-algebras—the algebras for *basic* logic) is in fact satisfied in all hoops.

[40]Recall that Wajsberg algebras are termwise equivalent to MV-algebras, see footnote 19 on p. 23.

The logic of cancellative hoops, or *cancellative hoop logic* CHL, extends the logic BLH by the following axiom:

$$\text{(CH)} \qquad (\varphi \to \varphi \mathbin{\&} \psi) \to \psi.$$

The logic CHL is not a $\overline{0}$-free fragment of any logic that proves $\overline{0} \to \varphi$, since expanding CHL by $\overline{0}$ and the axiom $\overline{0} \to \varphi$ makes the resulting logic, $\mathrm{BL} + (\mathrm{CH})$, inconsistent (as easily shown by instantiating (CH) by $\overline{0}$ for φ and ψ). Product logic is not an extension of CHL, as (CH) is not valid in Π (one cannot cancel by $\overline{0}$); however, there is the following connection between CHL and Π:

THEOREM 2.3.2. *Let all propositional variables occurring in φ be among $p_1, \ldots, p_n$. Then* CHL *proves φ iff* Π *proves* $\neg\neg p_1 \wedge \cdots \wedge \neg\neg p_n \to \varphi$.

Moreover it can be shown that:

- The logic BLH is the $\overline{0}$-free fragment of both BL and SBL.
- The logic MTLH is the $\overline{0}$-free fragment of MTL, IMTL, and SMTL (introduced below in Section 2.4.1).
- The logic ŁH (or GH or ΠH, resp.) is the $\overline{0}$-free fragment of Ł (or G or Π, resp.).

The first two claims are perhaps surprising, as we have *different* logics which share the same $\overline{0}$-free fragment; this shows that these logics really differ only in the properties of negation.

The algebras for the logics MTLH, GH, and ΠH are respectively called *prelinear semihoops, Gödel hoops,* and *product hoops,* and they are $\overline{0}$-free subreducts of the corresponding MTL-, G-, and Π-algebras. Standard (semi-)hoops are just all real-valued (semi-)hoops.[41] The definition of standard hoops for stronger hoop logics is more complex:

- The *standard cancellative hoop* is the positive part of the standard product algebra (i.e., the half-open interval $(0, 1]$ with the standard product operations).
- The *standard Gödel, Wajsberg,* and *product hoop* is just the $\overline{0}$-free reduct of $[0, 1]_{\mathrm{G}}$, $[0, 1]_{\text{Ł}}$, and $[0, 1]_{\Pi}$, respectively.

The logics MTLH and GH enjoy strong standard completeness, whereas the logic BLH, ŁH, ΠH, and CHL only enjoy finite strong standard completeness.

What about deleting both the truth constant for falsity and the commutativity of conjunction from a t-norm fuzzy logic? Algebras of this kind have been studied under the name *pseudohoops.* A logic generalizing both the logic psMTL and the hoop logic MTLH was introduced under the name *flea logic.* See [70] for pseudohoops and [91] for fleas.

[41] Recall that in Convention 2.0.1 we allowed the real-unit interval to be half-open.

2.3.2 Discarding other connectives

In this subsection we deal with the remaining fragments of prominent axiomatic extensions of MTL. All results of this subsection appear in [38], where further references to the original sources of some particular results can be found. Our first goal is to present explicit axiomatic systems for these fragments. In order to be able to formulate many such results at once we introduce the following notion (particularized to our setting).

DEFINITION 2.3.3. *Let* L *be an axiomatic extension of* MTL *and let* $\mathcal{A}$ *be an axiomatic system for* L*. We say that* $\mathcal{A}$ *is* strongly separable *if for each propositional language* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \&, \wedge, \vee, \overline{0}\}$*, the* $\mathcal{L}$*-fragment of* L *is axiomatized by the axioms and rules from* $\mathcal{A}$ *that contain the connectives from* $\mathcal{L}$ *only.*

The axiomatic system $\mathrm{MTL_s}$, with modus ponens as the only deduction rule and axioms listed below, is a strongly separable axiomatic system of MTL:[42]

$(\mathrm{MTL_s}1)$	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$(\mathrm{MTL_s}2)$	$\varphi \to (\psi \to \varphi)$
$(\mathrm{MTL_s}3)$	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
$(\mathrm{MTL_s}6)$	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
$(\mathrm{MTL_s}5\mathrm{a})$	$(\varphi \to (\psi \to \chi)) \to (\varphi\&\psi \to \chi)$
$(\mathrm{MTL_s}5\mathrm{b})$	$(\varphi\&\psi \to \chi) \to (\varphi \to (\psi \to \chi))$
$(\mathrm{MTL_s}7)$	$\overline{0} \to \varphi$
$(\mathrm{MTL_s}4\mathrm{a})$	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
$(\mathrm{MTL_s}4\mathrm{b})$	$\varphi \wedge \psi \to \varphi$
$(\mathrm{MTL_s}4\mathrm{c})$	$\varphi \wedge \psi \to \psi \wedge \varphi$
$(\mathrm{MTL_s}4{\vee}\mathrm{a})$	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
$(\mathrm{MTL_s}4{\vee}\mathrm{b})$	$\varphi \to \varphi \vee \psi$
$(\mathrm{MTL_s}4{\vee}\mathrm{c})$	$\varphi \vee \psi \to \psi \vee \varphi.$

To obtain strongly separable axiomatic systems for prominent axiomatic extensions of MTL, we need ‘implicational’ forms of their characteristic axioms:

(Div)	$((\varphi \to \psi) \to (\varphi \to \chi)) \to ((\psi \to \varphi) \to (\psi \to \chi))$	divisibility
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom
(Contr)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(SBL)	$(\varphi \to \neg\varphi) \to \neg\varphi.$	strictness

[42]The numbering of axioms follows the numbers of corresponding axioms in the axiomatic systems of BL and MTL introduced in Section 1.2 and 2.1; the axioms are grouped according to the connectives involved. Note that the first three axioms with modus ponens constitute the axiomatic system of the well-known purely implicational logic BCK (see, e.g., [67, §2.3.2]). Its extension by the axiom $(\mathrm{MTL_s}6)$ is the implicational fragment of MTL, called ‘fuzzy BCK’, or FBCK, in [38].

Logic	Axioms added to $\mathrm{MTL_s}$
BL	(Div)
SBL	(Div), (SBL)
Ł	(Div), (Waj)
G	(Div), (Contr)
Π	(Div), (SBL), (ΠH1)–(ΠH3)

Table 1. Strongly separable axiomatic systems for some prominent extensions of MTL

Fragment:	$\to$	$\to,\wedge$	$\to,\vee$	$\to,\wedge,\vee$	$\to,\&$	$\to,\&,\vee$	$\to,\&,\wedge$
MTL		$\vee$				$\wedge$	$\vee$
BL, SBL, Π		$\vee$			$\wedge,\vee$	$\wedge$	$\vee$
G		$\&,\vee$		$\&$	$\wedge,\vee$	$\wedge$	$\vee$
Ł	$\vee$	$\vee$			$\wedge,\vee$	$\wedge$	$\vee$

Table 2. Definability of connectives in fragments without $\overline{0}$

Also recall the axioms (ΠH1)–(ΠH3) of product hoop logic from Section 2.3.1. We list strongly separable axiomatic systems for prominent extensions of MTL in Table 1.

We define standard $\mathrm{L}{\restriction}\mathcal{L}$-algebras as just the $\mathcal{L}$-reducts of standard L-algebras. Standard completeness theorems are then obviously inherited by the fragments: if L enjoys strong (resp., finite strong) standard completeness, then also $\mathrm{L}{\restriction}\mathcal{L}$ enjoys strong (resp., finite strong) standard completeness. Thus, for instance, $\mathrm{FBCK} = \mathrm{MTL} \restriction \{\to\}$ is the *logic of the residua of left-continuous t-norms.*

Next let us present some results on the definability of connectives in prominent fragments. We say that a connective c is definable in $\mathrm{L}{\restriction}\mathcal{L}$ if there is an $\mathcal{L}$-formula φ such that $c(p_1,\dots,p_n) \leftrightarrow \varphi(p_1,\dots,p_m)$ is a theorem of L. Note that the defining formula can have a different number of variables, as witnessed, e.g., by the definability of $\overline{1}$ in the $\{\to\}$-fragment of MTL: indeed $\overline{1} \leftrightarrow (p \to p)$ is a theorem of MTL. Let us list some known positive results:

- $\wedge$ is definable in $\mathrm{MTL} \restriction \{\to,\&,\vee\}$ by $(\varphi \mathbin{\&} (\varphi \to \psi)) \vee (\psi \mathbin{\&} (\psi \to \varphi))$
- $\vee$ is definable in $\mathrm{MTL} \restriction \{\to,\wedge\}$ by $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$
- $\wedge$ is definable in $\mathrm{BL} \restriction \{\to,\&\}$ by $\varphi \mathbin{\&} (\varphi \to \psi)$
- $\vee$ is definable in Ł $\restriction \{\to\}$ by $(\varphi \to \psi) \to \psi$
- $\&$ is definable in Ł $\restriction \{\to,\overline{0}\}$ by $\neg(\varphi \to \neg\psi)$
- $\&$ is definable in $\mathrm{G} \restriction \{\to,\wedge\}$ by $\varphi \wedge \psi$.

Of course a connective definable in a logic in a certain language is definable in all stronger logics and/or bigger languages. The definability of connectives in fragments

Fragment: $\overline{0}$,	$\to$	$\to, \wedge$	$\to, \vee$	$\to, \wedge, \vee$	$\to, \&$	$\to, \&, \vee$	$\to, \&, \wedge$
MTL		$\vee$				$\wedge$	$\vee$
BL, SBL, Π		$\vee$			$\wedge, \vee$	$\wedge$	$\vee$
G		$\&, \vee$		$\&$	$\wedge, \vee$	$\wedge$	$\vee$
Ł	$\&, \wedge, \vee$	$\&, \vee$	$\&, \wedge$	$\&$	$\wedge, \vee$	$\wedge$	$\vee$

Table 3. Definability of connectives in fragments with $\overline{0}$

of prominent extensions of MTL is summarized in Tables 2 and 3, which list all of the connectives $\&, \wedge, \vee, \overline{0}$ that are definable in each fragment (omitting those which are already present in the fragment's language). The constant $\overline{0}$ is not definable in any of the fragments listed (the empty column for the language $\to, \&, \wedge, \vee$ is therefore omitted in Table 2).

2.4 Adding axioms or rules

Another way of varying a fuzzy logic is by strengthening it by additional axioms or rules, thereby narrowing down the class of its algebraic models. In some cases, adding an axiom or rule leads to a previously introduced logic (e.g., MTL plus the axiom of idempotence of conjunction yields the same logic as BL plus the same axiom, namely Gödel logic). In other cases, however, a new fuzzy logic is obtained (e.g., MTL plus the axiom of double negation yields the logic IMTL, which is weaker than an analogous extension of BL, i.e., Łukasiewicz logic).

We shall discuss two main kinds of such axiomatic strengthening, namely imposing some restrictions on the behavior of logical connectives (esp. negation or conjunction) and limiting the set of truth values to finite cardinalities. We shall introduce main fuzzy logics arising by these kinds of strengthening, and briefly discuss their properties and mutual relationships.

2.4.1 Special properties of negation

Axioms or rules added to a given fuzzy logic can enforce special properties of residual negation. Prominent extensions of this kind are the logics IMTL, IUL, SMTL, and SBL, which enforce either involutiveness or strictness (see Example 1.1.11) of residual negation.

The *involutiveness* of residual negation can be ensured in any extension of the uninorm logic by adding the axiom of double negation

$$(\neg\neg) \qquad \neg\neg\varphi \to \varphi.$$

The converse implication $\varphi \to \neg\neg\varphi$ is provable in all extensions of UL (in fact, in all logics of pointed commutative residuated lattices, as $\varphi \to ((\varphi \to \overline{0}) \to \overline{0})$ follows by modus ponens from residuation). Adding this axiom to BL yields Łukasiewicz logic (see Section 1.2). The extensions by $(\neg\neg)$ of MTL and UL are called, respectively, IMTL [53] and IUL [133] (where the 'I' stands for 'involutive').[43] The logic IMTL en-

[43] The logic IMTL and its algebras were implicitly studied already in [23] (called there m-z-pregroups).

joys strong standard completeness, i.e., completeness w.r.t. all standard MTL-algebras $[0,1]_*$ with involutive residual negation [50]; the standard completeness of IUL is an open problem.

Like in Ł and the logics $L_\sim$, a strong disjunction $\oplus$ is definable in IMTL, by $\varphi \oplus \psi \equiv \neg(\neg\varphi \mathbin{\&} \neg\psi)$. Notice, however, that only a limited set of t-conorms is obtained by this definition as the standard semantics of $\oplus$ in IMTL, since the t-norm in standard IMTL-algebras has to be such as to yield involutive residual negation.

The logic IMTL shares many features with Łukasiewicz logic, which is its extension by the divisibility axiom (BL4). Like in Łukasiewicz logic (see Theorem 1.2.8), the implication and strong conjunction are mutually interdefinable and the contraposition law holds in IMTL. It can be shown that IMTL is equivalently axiomatized by adding any of the following IMTL-provable formulae to MTL: ($T_Ł1$), ($T_Ł3$), ($T_Ł4$), ($T_Ł5$), or ($T_Ł6$) of Theorem 1.2.8. Note, however, that the Wajsberg axiom ($T_Ł2$) is not provable in IMTL, and adding it to MTL in fact yields Łukasiewicz logic.

The *strictness* of residual negation (i.e., the fact that $\neg$ is a pseudocomplement) can be enforced in any extension of MTL by adding the axiom

(S) $\quad \neg(\varphi \wedge \neg\varphi).$

Analogously to the logic $\mathrm{SBL} = \mathrm{BL} + (\mathrm{S})$ introduced in Section 1, we can define the logic SMTL as the extension of MTL by (S). Like SBL (which extends SMTL by the divisibility axiom (BL4)), also SMTL can equivalently be axiomatized by any of the formulae ($T_{SBL}1$), ($T_{SBL}2$), or ($T_{SBL}3$) of Theorem 1.2.6. These theorems restrict the behavior of negation and conjunction in SMTL-chains. In particular, the theorem ($T_{SBL}1$), $\neg\varphi \vee \neg\neg\varphi$ or the weak law of excluded middle, expresses the *bivalence* of negation—i.e., the fact that $\neg\varphi$ is in each SMTL-chain evaluated either to $\overline{0}$ or $\overline{1}$. The theorem ($T_{SBL}3$), $\neg(\varphi \mathbin{\&} \psi) \to \neg(\varphi \wedge \psi)$, entails that SMTL-chains are exactly those MTL-chains that have no zero divisors (i.e., non-zero elements x, y such that $x * y = 0$).

Adding the axiom (S) to IMTL or the axiom ($\neg\neg$) to SMTL, already makes the logic classical. Recall, however, from Section 2.2.2 that an additional involutive negation $\sim$ can be naturally added to any logic containing SMTL, by the axioms ($\neg\neg$), (Rev-S1), (Rev-S2), and (Rev-S3), and that $\triangle$ is definable in $\mathrm{SMTL}_\sim$ by $\triangle\varphi \equiv \neg{\sim}\varphi$. Strict negation $\neg_S$ is, on the other hand, definable already in $\mathrm{MTL}_\triangle$ by $\neg_S\varphi \equiv \triangle\neg\varphi$. Both strict and involutive negations are also present in logics containing $\mathrm{SMTL}_\sim$ or $\mathrm{IMTL}_\triangle$ (incl., e.g., $Ł_\triangle$ or ŁΠ). Thus, although it is not possible to have a fuzzy negation that is both involutive and strict, one can have both negations in richer fuzzy logics.

2.4.2 Special properties of conjunction

Various properties of conjunction $\&$ correspond to special axioms added to usual systems of fuzzy logics. For instance, the *idempotence* of $\&$ corresponds to the validity of the axiom

(C) $\quad \varphi \leftrightarrow \varphi \mathbin{\&} \varphi$

in any extension of the uninorm logic UL. Adding the axiom (C) to MTL (or any logic between MTL and G, e.g., BL, SBL, or SMTL) yields Gödel logic; adding (C) to IMTL makes the logic classical. Adding the axiom to the uninorm logic UL yields the logic called UML, or *uninorm mingle logic.* UML is the *logic of idempotent residuated uninorms,* characterized by Theorem 2.1.4. Extending the logic UML further by the axioms $\neg\neg\overline{0} \to \overline{0}$ and $\overline{0} \leftrightarrow \overline{1}$ yields the logic IUML of the single idempotent uninorm $*_{\mathrm{IU}}$ introduced in Theorem 2.1.4; in both cases, strong standard completeness of the logic can be proved [133]. Notice that while (C) entails the coincidence of $\&$ and $\wedge$ in all t-norm logics, this is not in general so in uninorm logics, as idempotent residuated uninorms need not coincide with the minimum.

The axiom (C) can be called the axiom of *contraction,* as it expresses the fact that any conjunction $\varphi \mathbin{\&} \cdots \mathbin{\&} \varphi$ of the same conjuncts φ can be contracted to a single φ.[44] Writing the axiom (C) as $\varphi^2 \leftrightarrow \varphi$ suggests a natural generalization

$$(\mathrm{C}_n) \qquad \varphi^n \leftrightarrow \varphi^{n-1}$$

for any $n \geq 2$, called n*-contraction.*[45] Semantically, the validity of (C_n) is equivalent to the condition that x^{n-1} is an idempotent element of $\&$ for each x. For any axiomatic extension L of MTL we define the logic $\mathrm{C}_n\mathrm{L}$ as $\mathrm{L} + (\mathrm{C}_n)$. Clearly, (C_n) implies (C_m) in MTL for all $m \geq n$; thus $\mathrm{C}_m\mathrm{L} \subseteq \mathrm{C}_n\mathrm{L}$ if $m \geq n$. In the logic $\mathrm{C}_n\mathrm{L}$, a global bound can obviously be given in the deduction theorem:

$$\Gamma, \varphi \vdash_{\mathrm{C}_n\mathrm{L}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{C}_n\mathrm{L}} \varphi^{n-1} \to \psi.$$

Since $\mathrm{C}_2\mathrm{MTL}$ is Gödel logic, the logics $\mathrm{C}_n\mathrm{MTL}$ are intermediary between MTL and G. It can be proved that $\mathrm{C}_{n+1}\mathrm{MTL} \subsetneq \mathrm{C}_n\mathrm{MTL}$ and that $\mathrm{MTL} = \bigcap_n \mathrm{C}_n\mathrm{MTL}$, and similarly for $\mathrm{C}_n\mathrm{IMTL}$, $\mathrm{C}_n\mathrm{BL}$, and C_nŁ. For each n, the logics $\mathrm{C}_n\mathrm{MTL}$ and $\mathrm{C}_n\mathrm{IMTL}$ are strong standard complete [27]. This is not the case with $\mathrm{C}_n\mathrm{BL}$ and C_nŁ, though, as the only standard $\mathrm{C}_n\mathrm{BL}$-chain is $[0,1]_{\mathrm{G}}$ and the only C_nŁ-chains are MV-chains with less than n elements.

Let us say that an MTL-algebra $\boldsymbol{A}$ is n-contractive if (C_n) is valid in $\boldsymbol{A}$. It can be observed that, for instance, every finite MTL-algebra is n-contractive for some n. All weak nilpotent minimum t-algebras (see Example 2.1.1) are 3-contractive and standard Gödel algebra is clearly a weak nilpotent minimum t-algebra. Therefore the logic WNM of *weak nilpotent minima,* introduced in [53], is intermediary between $\mathrm{C}_3\mathrm{MTL}$ and $\mathrm{C}_2\mathrm{MTL} = \mathrm{G}$; it can be easily shown that both inclusions are proper. The logic WNM can be axiomatized by adding the axiom[46]

$$(\mathrm{WNM}) \qquad \neg(\varphi \mathbin{\&} \psi) \vee (\varphi \wedge \psi \to \varphi \mathbin{\&} \psi)$$

to MTL. The strong standard completeness of WNM can be proved [53].

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[44] In substructural logics, usually (e.g., in [67]) just the implication $\varphi \to \varphi \mathbin{\&} \varphi$ is called the axiom of contraction (as it corresponds to the structural rule of contraction, cf. Section 3.1), while the converse implication $\varphi \mathbin{\&} \varphi \to \varphi$ is called the axiom of expansion. Nevertheless, since the latter is a theorem of MTL, the distinction does not matter in MTL or stronger logics.

[45] Again, in substructural logics the term n-contraction is used just for the implication $\varphi^{n+1} \to \varphi^n$ and the property (C_n) is (e.g., in [67]) called n*-potence*. In the context of Łukasiewicz logic, n-contraction has been studied in [158]; the systematic study of n-contraction in mathematical fuzzy logic has originated with [27].

[46] Observe that its first disjunct represents the nilpotent part and the second disjunct the idempotent part of a weak nilpotent minimum t-norm.

The logic NM of *nilpotent minima* (i.e., weak nilpotent minima given by an involutive negation, see Example 2.1.1), introduced in [53], can be shown to extend WNM by the axiom $(\neg\neg)$, $\neg\neg\varphi \to \varphi$; thus NM is an extension (also proper) of $\mathrm{C_3IMTL}$. All NM-algebras on the real unit interval $[0,1]$ are mutually isomorphic. Because of this fact, only the t-algebra $[0,1]_{\mathrm{NM}}$ of the standard nilpotent minimum t-norm $*_{\mathrm{NM}}$ (given by the standard involution $1-x$) is called the *standard* NM-algebra.[47] The strong standard completeness theorem can be proved for the logic NM [53].

The logic NM expanded by $\triangle$ is in a close relationship to Gödel logic with involution, as the standard connectives of $\mathrm{NM}_\triangle$ and $\mathrm{G}_\sim$ turn out to be definable in each other by mutually inverse translations. Thus, due to the standard completeness of both logics, $\mathrm{NM}_\triangle$ and $\mathrm{G}_\sim$ have the same logical strength and can be considered notational variants of each other.

Another property of $\&$ that can be axiomatically enforced in extensions of MTL is *n-nilpotence,* i.e., the identity $x^n = \overline{0}$ for each $x \neq 1$. For any given $n \geq 2$, the $(n-1)$-nilpotence of $\&$ is ensured by the axiom

$$(\mathrm{S}_n) \qquad \varphi^{n-1} \vee \neg\varphi.$$

For any extension L of MTL, the logic $\mathrm{L} + (\mathrm{S}_n)$ is denoted by $\mathrm{S}_n\mathrm{L}$. These logics were studied in [113]; the axioms (S_n) were studied under the name *n-excluded middle* in [124] in the context of substructural logics, and first brought to mathematical fuzzy logic in [73]). The name (unrelated to SBL and SMTL) and the numbering of the axioms (S_n) is motivated by the relation (described below) between the logics $\mathrm{S}_n\mathrm{MTL}$ and $\mathrm{C}_n\mathrm{MTL}$. Clearly $\mathrm{S}_m\mathrm{L} \subseteq \mathrm{S}_n\mathrm{L}$ for $m \geq n$, as MTL proves $(\mathrm{S}_n) \to (\mathrm{S}_m)$ if $n \leq m$. The axiom (S_2) is the law of excluded middle, thus all logics $\mathrm{S}_2\mathrm{L}$ coincide with classical logic.

Recall that an algebra $\boldsymbol{A}$ is *simple* if it has only trivial congruences (i.e., its only congruences are A^2 and the identity on A), and *semisimple* if it is a subdirect product of simple algebras. $\mathrm{S}_n\mathrm{MTL}$-algebras can be characterized as n-contractive semisimple MTL-algebras. Consequently, $\mathrm{S}_n\mathrm{L}$ extends $\mathrm{C}_n\mathrm{L}$ for each n; it can be shown that for $\mathrm{L} = \mathrm{MTL}$ the inclusions are strict for any $n \geq 3$, but on the other hand, $\mathrm{S}_n\text{Ł} = \mathrm{C}_n\text{Ł}$ for all n.

It can be observed that $\mathrm{S_3MTL}$-chains are just those in which $\&$ is the so-called *drastic product,* i.e., $x \,\&\, y = 0$ for all $x, y < 1$. Since the drastic product on $[0,1]$ is not left-continuous, there are no standard $\mathrm{S_3MTL}$-algebras. Generally it can be shown that every simple n-contractive MTL-chain must have a co-atom (i.e., the largest element smaller than $\overline{1}$); thus there are no standard $\mathrm{S}_n\mathrm{MTL}$-algebras for any n. Consequently, none of the logics $\mathrm{S}_n\mathrm{MTL}$ (nor $\mathrm{S}_n\mathrm{IMTL}$) can enjoy any kind of standard completeness.

Earlier we have seen that the logic IMTL arises by adding to MTL the characteristic axiom $(\neg\neg)$ of Łukasiewicz logic over BL (cf. Section 1.2), while extending MTL by the characteristic axiom $(\mathrm{G}) = (\mathrm{C}_2)$ of Gödel logic over BL already yields Gödel logic itself. We can now ask what logic arises by adding the characteristic axiom of product logic over BL,

$$(\Pi) \qquad \neg\psi \vee ((\psi \to \varphi \,\&\, \psi) \to \varphi),$$

[47] Cf. the similarly motivated departure from Convention 2.0.2 in the cases of Łukasiewicz and product logic.

to MTL. The resulting logic $\Pi\mathrm{MTL} = \mathrm{MTL} + (\Pi)$ is the logic of *cancellative* MTL-algebras (i.e., those validating cancellation by non-zero elements: if $z \neq \overline{0}$ and $x * z = y * z$, then $x = y$, for all x, y, and z), just like $\Pi = \mathrm{BL} + (\Pi)$ is the logic of cancellative BL-algebras. This logic was introduced in [86]. Also just like over BL (see Section 1.2), the axiom (Π) can equivalently be replaced by the two axioms (S) and (Π_S); thus $\Pi\mathrm{MTL} = \mathrm{MTL} + (\mathrm{S}) + (\Pi_\mathrm{S})$, too, and the logic is intermediary between SMTL and Π (both inclusions are strict). The logic $\Pi\mathrm{MTL}$ is finitely strong standard complete [108] (though not strong standard complete [110]).

The logic SBL contains both G and Π, since it is based on a common property of $*_\mathrm{G}$ and $*_\Pi$, namely the strictness of their residual negation (see Example 1.1.11), but it turns out to be strictly weaker that the intersection of these two logics. Similarly there is a common property of $*_\text{Ł}$ and $*_\Pi$, related to the fact that $*_\text{Ł}$ is isomorphic to truncated $*_\Pi$ (see Theorem 1.1.7(4)), namely their cancellativity restricted to elements with non-zero conjunction. This property is called *weak cancellativity:*

DEFINITION 2.4.1. *An* MTL*-chain* $\boldsymbol{A} = \langle L, *, \Rightarrow, \wedge, \vee, 0, 1\rangle$ *is* weakly cancellative *if* $x * z = y * z \neq 0$ *implies* $x = y$ *for all* $x, y, z \in A$.

Weak cancellativity is in MTL characterized by the following axiom:

$$(\mathrm{WC}) \qquad \neg(\varphi \mathbin{\&} \psi) \vee ((\psi \to \varphi \mathbin{\&} \psi) \to \varphi).$$

For any extension of MTL, the logic $\mathrm{L} + (\mathrm{WC})$ is denoted by WCL; the logics WCBL and WCMTL were introduced in [140]. The following facts about the logics WCL can be shown [140, 145]:

- The logic WCBL is exactly the intersection of Ł and Π.
- $\mathrm{S}_n\mathrm{L} = \mathrm{C}_n\mathrm{L}$ for any axiomatic extension L of WCMTL.
- Adding (WC) to IMTL already yields Łukasiewicz logic.
- The logics WCBL and WCMTL enjoy finite strong standard completeness.

Note that the first two claims entail the already mentioned fact that $\mathrm{S}_n\text{Ł} = \mathrm{C}_n\text{Ł}$.

2.4.3 Finitely-valued fuzzy logics

Let the logic L be an axiomatic extension of MTL and $n \geq 1$ a natural number. We define the logic $\mathrm{L}_{\leq n}$ as the extension of L by the axiom

$$(\leq n) \qquad \bigvee_{i=1}^{n} (\varphi_{i-1} \to \varphi_i).$$

It can be easily demonstrated that an L-chain $\boldsymbol{A}$ is an $\mathrm{L}_{\leq n}$-chain if and only if A has at most n elements: just observe that the axiom $(\leq n)$ is *not* satisfied by an $\boldsymbol{A}$-evaluation e iff $e(\varphi_0) > e(\varphi_1) > \cdots > e(\varphi_n)$, thus it holds for all evaluations and all formulae φ_i iff there are at most n values in $\boldsymbol{A}$.

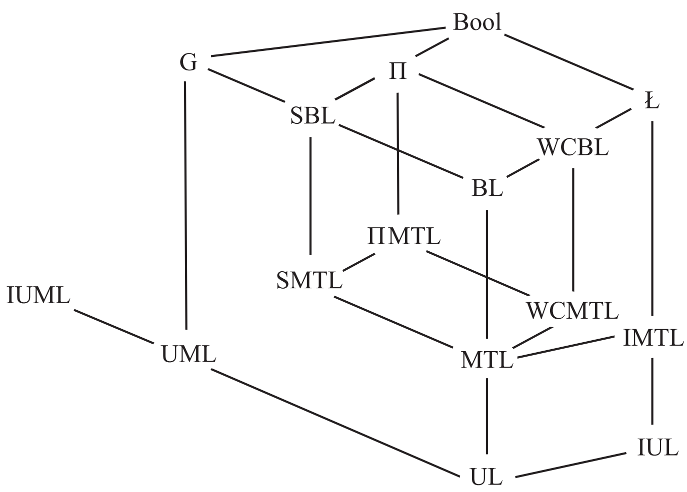

Figure 7. Relative positions of prominent logics introduced in Section 2.4

Clearly $\mathrm{L}_{\leq n}$ cannot enjoy standard completeness, but it satisfies a bunch of other interesting properties [35]:

- *Strong finiteness:* There is a finite set $\mathbb{K}$ of finite algebras (namely, the set of all $\mathrm{L}_{\leq n}$-chains on subsets of $\{1, \ldots, n\}$) such that $\{\vdash_{\mathrm{L}_{\leq n}}\} = \models_{\mathbb{K}}$.
- *Tabularity:* There is a finite $\mathrm{L}_{\leq n}$-algebra $\boldsymbol{F}$ such that $\{\vdash_{\mathrm{L}_{\leq n}}\} = \models_{\boldsymbol{F}}$.
- *Finite embeddability property:* Every finite subset of any $\mathrm{L}_{\leq n}$-algebra can be partially embedded into a finite $\mathrm{L}_{\leq n}$-algebra.

Note that $\mathrm{L}_{\leq n}$ extends $\mathrm{C}_n\mathrm{L}$ (see Section 2.4.2). Let us now briefly discuss the situation in particular logics. First let us note that the only finite $\Pi\mathrm{MTL}$-chains (so a fortiori the only finite product chains) are two-element Boolean algebras. Consequently, the only finite $\Pi\mathrm{MTL}$-algebras are finite Boolean algebras, and so there are no non-trivial (i.e., non-classical) finitely-valued $\Pi\mathrm{MTL}$ (nor product) logics. Actually even more can be shown: the only proper extensions (not necessarily axiomatic!) of product logic are either classical or inconsistent.

Further let us observe that for each n there is only one (up to isomorphism) n-valued MV and G-chain. Let $Ł_n$ and G_n denote the corresponding logics semantically induced by these algebras.

First let us describe the situation in finitely-valued Gödel logics [76]:

- $\mathrm{G}_n = \mathrm{G}_{\leq n}$ and $\mathrm{G} = \bigcap_{n=1}^{\infty} \mathrm{G}_n$
- $\mathrm{G}_n \subseteq \mathrm{G}_m$ iff $m \leq n$
- For each proper axiomatic extension L of G there is n such that $\mathrm{L} = \mathrm{G}_n$.

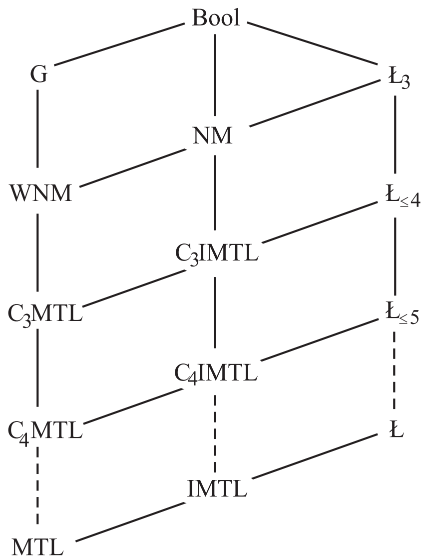

Figure 8. Relative positions of further prominent logics of Section 2.4

The situation in finitely-valued Łukasiewicz logics is more complex. While it is still true that $\text{Ł} = \bigcap_{n=1}^{\infty} \text{Ł}_n$, it is not true that $\text{Ł}_n \subseteq \text{Ł}_m$ iff $m \leq n$. For instance, $(2p)^4 \leftrightarrow 4(p^2)$ and $\neg((p \leftrightarrow \neg p)^3)$ are tautologies of Ł_4, though not of Ł_3 (where $p^n \equiv \&_{i=1}^{n} p$ as usual and $np \equiv \bigoplus_{i=1}^{n} p$). Therefore clearly $\text{Ł}_n \neq \text{Ł}_{\leq n}$ for $n \geq 4$, although for $n < 4$ they coincide. The (non-linear) ordering by strength of the logics Ł_n is described by the following theorem which in [129] is attributed to Lindenbaum:

THEOREM 2.4.2. $\text{Ł}_n \subseteq \text{Ł}_m$ *iff* $n - 1$ *divides* $m - 1$, *for any* $m, n \geq 2$.

Unlike in Gödel logic, there are proper axiomatic extensions of Łukasiewicz logic that differ from $\models_{\mathbb{K}}$ for any set $\mathbb{K}$ of finite MV-chains (e.g., the logic of the so-called *Chang* MV*-algebra,* see Chapter VI).

3 Families of fuzzy logics in the logical landscape

In previous sections we introduced numerous prominent members of the broad family of logical systems studied in mathematical fuzzy logic, their axiomatic systems, and their general, linear, and standard algebraic semantics. The multitude of fuzzy logics calls for a general unifying (meta)theory. Indeed, many metamathematical properties, such as general and linear completeness theorems or the linear subdirect representation theorem, can be proved generally for large classes of logics delimited by simple (e.g., syntactic) criteria.

In this section we shall study the position of fuzzy logics in the logical landscape—i.e., their relationship to well-known broader families of propositional logics (such as substructural or algebraizable) as well as to particular prominent non-classical logics.

We shall briefly survey characteristic properties of fuzzy logics, both those shared with the mentioned broad classes of logics and those particular to fuzzy logic.

In the first subsection we show the position of fuzzy logics among substructural logics. Then, in Section 3.2, we shall describe two important classes of fuzzy logics, namely the core and $\triangle$-core fuzzy logics. These classes will play an important rôle in the remaining sections of this chapter as well as in some subsequent chapters. Finally, in Section 3.3, we shall recall some basic notions of Abstract Algebraic Logic, particularized for our needs (this theory is covered in much more detail in Chapter II).

3.1 Fuzzy logics among substructural logics

In this subsection we observe that fuzzy logics introduced in Section 2.1 can be seen as the logics of (suitable classes of) FL-algebras (or pointed residuated lattices, see Definitions 1.3.1, 2.1.5, and 2.1.6).

(Pointed) residuated lattices form the algebraic semantics for so-called (intuitionistic) *substructural logics.* Substructural logics provide a unifying framework for several kinds of logics, such as relevance logics, variants of Girard's linear logic, the Lambek calculus, the logic BCK, etc. In a specific sense (suggested by Ono in [151]), substructural logics can be delimited as the logics of varieties of (pointed) residuated lattices. As we have seen, many fuzzy logics fall within this delimitation, and can thus be seen as a special kind of substructural logics. Substructural logics thus form a neighborhood of fuzzy logics in the landscape of non-classical logics. In this section we shall briefly introduce the class of (intuitionistic) substructural logics, indicate the position of fuzzy logics within this family, and describe the relationship of fuzzy logics to other substructural logics. For details on substructural logics, including the relationship of some prominent fuzzy logics to this class, see esp. [54, 67, 151, 161].

The logic of all (pointed) residuated lattices (with elements $x \geq \overline{1}$ taken as designated in the definition of logical consequence) is called the *full Lambek calculus* FL. The name comes from the fact that its conjunction–implication fragment is the well-known Lambek calculus (an important tool in the study of categorial grammars). The full Lambek calculus is an expansion of the Lambek calculus to the 'full' language, containing also the lattice connectives, the defined connectives of negation and equivalence, and the propositional constants for truth and falsity.

Proof-theoretically, the logic FL arises by removing the structural rules of exchange, weakening, and contraction from the Gentzen-style calculus LJ of intuitionistic logic.[48] In the absence of structural rules, certain equally well motivated variants of the operational rules for propositional connectives become non-equivalent, and thus define different connectives. This explains why in substructural logics (and consequently in fuzzy logics as their special kind) propositional connectives known from intuitionistic or classical logic naturally split into pairs of different connectives (that have to be included in

[48] Recall that Gentzen-style sequent calculi have two kinds of rules: *operational rules* for introduction of propositional connectives, and *structural rules* for manipulation with whole formulae in sequents. The Gentzen-style calculi LJ and LK for intuitionistic resp. classical logic have (besides the indispensable rule introducing the axiom sequents and the eliminable rule of cut) exactly the structural rules of exchange, weakening, and contraction; the logic FL is obtained by removing all these three structural rules from LJ. For more details on Gentzen-style calculi LJ and LK, their structural rules, and substructural logics in general, see, e.g., [67, 151, 153, 161].

the general signature of a pointed residuated lattice). In particular, implication and negation split into two variants (left and right) in the absence of exchange (cf. the connectives $\swarrow$ and $\searrow$ in the non-commutative fuzzy logics of Section 2.1.3, and the corresponding negations $\overline{0} \swarrow \varphi$ and $\varphi \searrow \overline{0}$); the constants for truth and falsity split each into two in the absence of weakening (cf. the distinction between $\overline{1}, \overline{0}$ and $\top, \bot$ in uninorm fuzzy logics, Section 2.1.2); and conjunction splits into two in the absence of contraction or weakening (cf. the presence of lattice conjunction $\wedge$ and residuated conjunction $\&$ in t-norm fuzzy logics), as does disjunction ($\vee$ vs. $\oplus$) in contraction-free logics with involutive negation.[49] The full propositional language of the logics of residuated lattices (including fuzzy logics) is therefore assumed to contain all of these connectives.[50] Under the presence of the appropriate structural rules (or the equivalent axioms), these connectives collapse into the single variants known from classical (or intuitionistic) logic.

The *basic substructural logics* are obtained by extending FL by a subset of the following Hilbert-style axioms, which correspond to the Gentzen-style structural rules of exchange, weakening, and contraction of LJ (so the resulting logics arise by removing just some of the three structural rules from LJ), and the law of double negation (which corresponds to starting from the calculus LK for classical logic instead of LJ for intuitionistic logic):

(E)	$\varphi \,\&\, \psi \searrow \psi \,\&\, \varphi$
(W)	$(\varphi \searrow \overline{1}) \wedge (\overline{0} \searrow \varphi)$
(C)	$\varphi \searrow \varphi \,\&\, \varphi$
$(\neg\neg)$	$((\overline{0} \swarrow \varphi) \searrow\!\!\!\nwarrow (\varphi \searrow \overline{0})) \wedge (((\varphi \searrow \overline{0}) \searrow \overline{0}) \searrow \varphi)$.

The algebraic properties of pointed residuated lattices corresponding to these axioms are, respectively: commutativity, boundedness and integrality,[51] square-increasingness (also know as superidempotence), and the 'classicality' of residual negation[52] (see Definitions 1.3.1, 2.1.5, and 2.1.6).

[49] The rules for disjunction affected by the absence of contraction operate with two or more formulae on the right-hand side of the sequent, which is forbidden in the calculus LJ. The split of disjunction thus does not occur in contraction-free logics based on LJ, and only occurs in contraction-free logics based on the calculus LK for classical logic (or equivalently, if the double negation law is added to LJ). The corresponding rules for $\wedge$, on the other hand, operate on the left-hand side, upon which no restriction is imposed in LJ; the split of conjunction thus occurs in all contraction-free logics of residuated lattices, including all t-norm and uninorm fuzzy logics (except for Gödel logic, which is contractive).

[50] In the context of substructural logics, the residual conjunction $\&$ and its dual disjunction $\oplus$ are often called *fusion* and *fission,* respectively, or alternatively, *multiplicative, group, parallel,* or *intensional* conjunction and disjunction. The connectives $\wedge$ and $\vee$ are often called *additive, lattice, comparative,* or *extensional* conjunction and disjunction. The names 'strong' and 'weak' conjunction or disjunction, common in t-norm fuzzy logics, are not suitable in the absence of weakening, since $\&$ is not generally stronger than $\wedge$ without weakening.

[51] The first conjunct of (W) corresponds algebraically to the integrality of a pointed residuated lattice and proof-theoretically to the rule of weakening on the left-hand side of sequents; similarly the second corresponds to boundedness and the weakening on the right (restricted to a single formula in LJ). Often (see, e.g., [68]), the conjuncts are discussed separately and denoted by 'i' and 'o' in the subscripts indicating extensions of FL.

[52] The first conjunct of $(\neg\neg)$ corresponds algebraically to the cyclicity of residual negation (i.e., $\overline{0} \mathbin{/} x = x \setminus \overline{0}$), and the second to its involutiveness (which in general needs to be formulated as $\overline{0} \mathbin{/} (x \setminus \overline{0}) = x$ and $(\overline{0} \mathbin{/} x) \setminus \overline{0} = x$, but in the presence of the first condition can be formulated in our way). Note that in commutative substructural logics, $(\neg\neg)$ can be replaced by $((\varphi \to \overline{0}) \to \overline{0}) \to \varphi$.

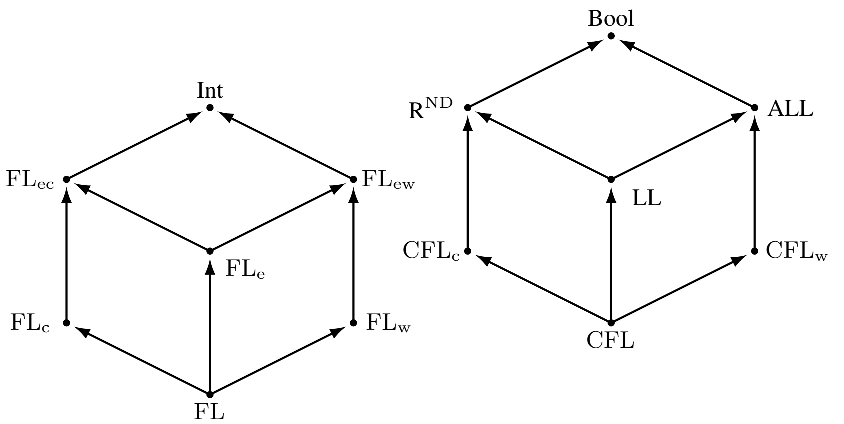

Figure 9. Basic intuitionistic and classical substructural logics

The logic $\mathrm{CFL} = \mathrm{FL} + (\neg\neg)$ is called the *classical full Lambek calculus.* The name refers to the fact that proof-theoretically it can be defined by removing the three structural rules from the calculus LK of classical logic. Extending FL or CFL by a subset of the axioms $\{(\mathrm{E}), (\mathrm{W}), (\mathrm{C})\}$ yields the basic (intuitionistic or classical, resp.) substructural logics, systematically denoted by subscripting FL resp. CFL by corresponding lowercase letters. The logic $\mathrm{FL_{ew}}$ is also known as Höhle's *monoidal logic* [107]; $\mathrm{CFL_e}$ as Girard's (multiplicative–additive) *linear logic* [72] (without exponentials and additive constants), LL; $\mathrm{CFL_{ew}}$ as *affine* linear logic [173], or *Grishin's logic* [80]; $\mathrm{FL_{e(w)}}$ as *intuitionistic (affine)* linear logic I(A)LL; and $\mathrm{CFL_{ec}}$ as Meyer's relevance logic $\mathrm{R^{ND}}$ (or "R *minus distribution*") or LR (for "*lattice* R") [48]. Since (E) is provable in $\mathrm{FL_{cw}}$, the logics $\mathrm{FL_{cw}} = \mathrm{FL_{ecw}}$ coincide with *intuitionistic logic* Int and $\mathrm{CFL_{cw}} = \mathrm{CFL_{ecw}}$ with *classical logic* Bool; all other logics $\mathrm{FL_x}$ and $\mathrm{CFL_x}$ are mutually different. The relationships between the 14 basic substructural logics are depicted in Figure 9.

Among the logics of residuated lattices, fuzzy logics introduced in Section 2 are distinguished by the property of semilinearity, i.e., completeness w.r.t. a class of linearly ordered residuated lattices. The main scope of mathematical fuzzy logic thus can be delimited as the study of *intuitionistic substructural semilinear logics,* or the logics of linearly ordered residuated lattices.[53]

Indeed, many important fuzzy logics arise as semilinear extensions of basic substructural logics, i.e., as the logics of linearly ordered $\mathrm{FL_x}$- or $\mathrm{CFL_x}$-algebras [54, 133]. For instance, the logic MTL turns out to be the logic of linear $\mathrm{FL_{ew}}$-algebras; UL of linear $\mathrm{FL_e}$-algebras; $\mathrm{psMTL^r}$ of linear $\mathrm{FL_w}$-algebras; IMTL of linear $\mathrm{CFL_{ew}}$-algebras;

[53] The name *deductive fuzzy logics* was proposed for this class of logics in [10], based on Ono's formulation in [151] suggesting that residuation gives logics a 'deductive face'. The connection of Łukasiewicz logic to other prominent substructural logics has first been pointed out in [23], and fuzzy logics have been firmly established as members of the family of substructural logics in [54, 153]. A formal delimitation of the class of fuzzy logics by the semilinearity property was proposed and advocated in [14]; cf. Section 3.3.

and G of linear $\mathrm{FL_{cw}}$-algebras (i.e., linear Heyting algebras). The restriction of the algebraic semantics to linear algebras will systematically be denoted by the superscript ℓ. We thus have the following identities:

$$
\begin{array}{ll}
\mathrm{FL}^{\ell}_{\mathrm{w}} = \mathrm{psMTL}^{\mathrm{r}} & \\
\mathrm{FL}^{\ell}_{\mathrm{e}} = \mathrm{UL} & \mathrm{CFL}^{\ell}_{\mathrm{e}} = \mathrm{IUL} \\
\mathrm{FL}^{\ell}_{\mathrm{ec}} = \mathrm{UML} & \\
\mathrm{FL}^{\ell}_{\mathrm{ew}} = \mathrm{MTL} & \mathrm{CFL}^{\ell}_{\mathrm{ew}} = \mathrm{IMTL} \\
\mathrm{FL}^{\ell}_{\mathrm{cw}} = \mathrm{G} & \mathrm{CFL}^{\ell}_{\mathrm{cw}} = \mathrm{Bool}.
\end{array}
$$

Other important fuzzy logics arise as extensions of these 'fundamental' fuzzy logics by special axioms or rules (e.g., $\mathrm{SMTL} = \mathrm{FL}^{\ell}_{\mathrm{ew}} + (\mathrm{S})$, $\mathrm{IUML} = \mathrm{CFL_{ec}} + (\overline{0} \leftrightarrow \overline{1})$, etc.), expansions by additional connectives with appropriate axioms and rules (e.g., $\mathrm{Ł\Pi}$, logics with $\triangle$ or $\sim$ etc.), fragments discarding some connectives (e.g., hoop fuzzy logics), logics defined by a combination of these methods, and similar modifications. Moreover, several fuzzy logics that have not yet been thoroughly investigated are obtained in this way, e.g., FL^{ℓ} or $\mathrm{FL}^{\ell}_{\mathrm{c}}$. Also the well-known substructural logic RM (*relevance with mingle*) is itself semilinear (i.e., complete w.r.t. linearly ordered algebras), and so belongs to the family of fuzzy logics in this sense.

Most prototypical fuzzy logics (including all t-norm fuzzy logics) are semilinear extensions of $\mathrm{FL_{ew}}$. The position of these fuzzy logics among other substructural logics is indicated in Figure 10. For axiomatic extensions of $\mathrm{FL_{ew}}$, semilinearity is equivalent to the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$. In axiomatic extensions of $\mathrm{FL_e}$ (which include uninorm logics), semilinearity is equivalent to $((\varphi \to \psi) \wedge \overline{1}) \vee ((\psi \to \varphi) \wedge \overline{1})$. More details on the relationship between fuzzy and substructural logics can be found in Section 4.3 and Chapters II, III, and IV.

3.2 Core and $\triangle$-core fuzzy logics

The two classes of fuzzy logics we are going to introduce in this subsection are not very broad from the general perspective of the whole logical landscape: in fact, they do not even cover the majority of fuzzy logics introduced in the previous section. Nevertheless, they do cover the most studied ones: the absolute majority of papers on mathematical fuzzy logic actually study logics from these two classes, and the study of other fuzzy logics has started only recently. The classes of logics were introduced in [98] in order to provide a common framework to the study of first-order fuzzy logics; later they played a similar rôle in the general study of completeness of fuzzy logics w.r.t. distinguished semantics in [35] (see Sections 5 and 4.1). The rough idea is to define a class of logics that share most desirable properties with MTL, and which could be delimited in a simple syntactic way.

As we have seen in the previous section, we need some flexibility as regards both propositional languages and logics. Therefore, for the sake of reference and in order to fix terminology in a way convenient for this section, we shall start with some standard general definitions and conventions. (For a detailed treatment of the general theory of logical calculi see, e.g., [42, 179].)

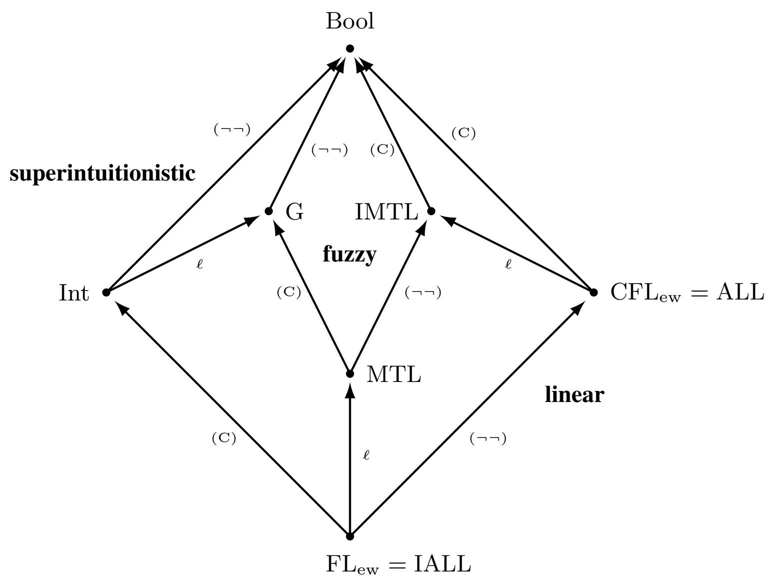

Figure 10. The position of t-norm fuzzy logics among substructural logics

DEFINITION 3.2.1. *A (propositional)* language *is a pair* $\mathcal{L} = \langle Conn_{\mathcal{L}}, Ar_{\mathcal{L}}\rangle$*, where* $Conn_{\mathcal{L}}$ *is a well-ordered countable set of (propositional)* connectives *and* $Ar_{\mathcal{L}}$ *is a function assigning a natural number to each element of* $Conn_{\mathcal{L}}$*. The number* $Ar_{\mathcal{L}}(c)$ *is called the* arity *of* $c \in Conn_{\mathcal{L}}$*. We shall write* $\langle c, n\rangle \in \mathcal{L}$ *as a shorthand for* $c \in Conn_{\mathcal{L}}$ *and* $Ar_{\mathcal{L}}(c) = n$*. Nullary connectives are also called* truth-constants.

The set $Form_{\mathcal{L}}$ *of (propositional)* formulae *in the language* $\mathcal{L}$ *over the fixed denumerable set* Var *of (propositional)* variables *is the smallest set containing* Var*, the truth constants of* $\mathcal{L}$*, and closed under the connectives from* $\mathcal{L}$ *(that is,* $c(\varphi_1, \dots, \varphi_n) \in Form_{\mathcal{L}}$ *whenever* $\varphi_1, \dots, \varphi_n \in Form_{\mathcal{L}}$ *and* $\langle c, n\rangle \in \mathcal{L}$*).*

An $\mathcal{L}$-substitution *is a mapping* $\sigma\colon Form_{\mathcal{L}} \to Form_{\mathcal{L}}$ *commuting with the connectives of* $\mathcal{L}$ *(i.e.,* $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ *for each* $\langle c, n\rangle \in \mathcal{L}$*).*

By a logic *(in the language* $\mathcal{L}$*) we mean a substitution-invariant Tarski consequence relation over* $\mathcal{L}$*; i.e., a relation* $\vdash_{\mathrm{L}} \subseteq \mathcal{P}(Form_{\mathcal{L}}) \times Form_{\mathcal{L}}$ *that satisfies, for each* $\Gamma, \Delta \subseteq Form_{\mathcal{L}}$*, each* $\varphi, \psi \in Form_{\mathcal{L}}$*, and each* $\mathcal{L}$*-substitution* σ*, the following conditions:*

1. *If* $\varphi \in \Gamma$*, then* $\Gamma \vdash_{\mathrm{L}} \varphi$*.*

2. *If* $\Gamma \vdash_{\mathrm{L}} \psi$ *for each* $\psi \in \Delta$*, and* $\Delta \vdash_{\mathrm{L}} \varphi$*, then* $\Gamma \vdash_{\mathrm{L}} \varphi$*.*

3. *If* $\Gamma \vdash_{\mathrm{L}} \varphi$*, then* $\sigma(\Gamma) \vdash_{\mathrm{L}} \sigma(\varphi)$*.*

We shall often use just L *as a synonym for* $\vdash_{\mathrm{L}}$*. A logic* L *is* finitary *if for each* $\Gamma \subseteq Form_{\mathcal{L}}$ *and* $\varphi \in Form_{\mathcal{L}}$ *such that* $\Gamma \vdash_{\mathrm{L}} \varphi$ *there exists a finite* $\Gamma' \subseteq \Gamma$ *such that* $\Gamma' \vdash_{\mathrm{L}} \varphi$*.*

Note that the cardinality restrictions on the sets $\mathcal{L}$ and Var are assumed just for simplicity. Clearly for each finitary[54] logic in the sense of this definition there is a Hilbert-style calculus (cf. Definition 1.2.1) such that the relation of provability (cf. Definition 1.2.2) in this calculus coincides with the logic; all such calculi will be called *axiomatic systems* (or *presentations*) of the logic in question.

EXAMPLE 3.2.2. All logics of (sets of) continuous t-norms (see Definition 1.1.19) are finitary logics in the sense of Definition 3.2.1. The *infinitary* logics of (sets of) continuous t-norms (i.e., the semantical consequence relations $\models_K$ of Definition 1.1.15) are also logics in the sense of this definition, but they are *not* finitary (with the exception of $K = \{*_{\mathrm{G}}\}$, see Theorem 1.1.18). Also all logics introduced in Section 2 are examples of finitary logics.

DEFINITION 3.2.3. *We say that a logic* L' *in the language* $\mathcal{L}'$ *is an* expansion *of a logic* L *in the language* $\mathcal{L} \subseteq \mathcal{L}'$ *if* $\Gamma \vdash_{\mathrm{L}} \varphi$ *implies* $\Gamma \vdash_{\mathrm{L}'} \varphi$ *for each* $\Gamma \cup \{\varphi\} \subseteq Form_{\mathcal{L}}$. *We say that the expansion is* conservative *if the converse implication holds as well. We say that the expansion is* axiomatic *if* L' *can be axiomatized by adding some axioms (but no rules) to some axiomatic system of* L. *We use the term* extension *instead of expansion if* $\mathcal{L} = \mathcal{L}'$.

Now we are able to define the core and $\triangle$-core fuzzy logics.

DEFINITION 3.2.4. *A finitary logic* L *in a language* $\mathcal{L}$ *is a* core fuzzy logic, *if:*

1. L *expands* MTL.
2. *For all* $\mathcal{L}$*-formulae* φ, ψ, χ *the following holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \chi', \qquad \text{(Cong)}$$

where χ' *is a formula resulting from* χ *by replacing some occurrences of its subformula* φ *by a formula* ψ.

3. L *has the Local Deduction Theorem, i.e., for each set of* $\mathcal{L}$*-formulae* $T \cup \{\varphi, \psi\}$ *holds:*

$$T, \varphi \vdash_{\mathrm{L}} \psi \textit{ iff there is } n \in \mathsf{N} \textit{ such that } T \vdash_{\mathrm{L}} \varphi^n \to \psi. \qquad \text{(LDT)}$$

EXAMPLE 3.2.5. The following logics introduced in the previous sections are core fuzzy logics: BL, MTL, IMTL, $\Pi\mathrm{MTL}$, NM, WNM, WCMTL, SBL, Łukasiewicz, product, and Gödel logic; the n-valued variants of these logics; extensions of these logics by the axiom (C_n) or (S_n); and some logics in expanded languages: the logic $\mathrm{P}Ł$, Rational Pavelka logic, and expansions of all mentioned logics by truth constants.

Note that classical logic is a core fuzzy logic as well, but intuitionistic logic is not. Many important fuzzy logics, including, e.g., the logics $\mathrm{MTL}_\triangle$ and $Ł\Pi$, however, fall outside the class of core fuzzy logics. Therefore we introduce a second, analogously defined, class.

[54]The definition of a Hilbert-style calculus can be modified to cover infinitary logics as well, but we shall not need it in this chapter.

DEFINITION 3.2.6. *A finitary logic* L *in a language* $\mathcal{L}$ *is a* $\triangle$-core fuzzy logic, *if:*

1. L *expands* $\mathrm{MTL}_\triangle$.
2. *For all* $\mathcal{L}$*-formulae* φ, ψ, χ *the following holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \chi', \qquad \text{(Cong)}$$

   where χ' *is a formula resulting from* χ *by replacing some occurrences of its subformula* φ *by a formula* ψ.

3. L *satisfies the* $\triangle$*-Deduction Theorem, i.e., for each set of* $\mathcal{L}$*-formulae* $T \cup \{\varphi, \psi\}$ *holds:*

$$T, \varphi \vdash_{\mathrm{L}} \psi \quad \textit{iff} \quad T \vdash_{\mathrm{L}} \triangle\varphi \to \psi. \qquad (\mathrm{DT}_\triangle)$$

EXAMPLE 3.2.7. The following logics introduced in the previous sections are $\triangle$-core fuzzy logics: extensions of all core fuzzy logics by $\triangle$; some fuzzy logics with additional involutive negation ($\mathrm{SMTL}_\sim$, $\mathrm{SBL}_\sim$, $\Pi_\sim$, $\mathrm{G}_\sim$); and the logics ŁΠ and $Ł\Pi\frac{1}{2}$.

The only consistent logic which is both core and $\triangle$-core is classical logic. Therefore all $\triangle$-core logics are examples of logics which are not core fuzzy logics. Of course all logics weaker than MTL are neither core nor $\triangle$-core (including, e.g., fragments of MTL, the uninorm logic UL, the non-commutative fuzzy logic $\mathrm{psMTL^r}$, etc.). However, there are natural examples of such logics even among expansions of MTL:[55]

EXAMPLE 3.2.8. The following expansions of MTL introduced in the previous sections are neither core nor $\triangle$-core fuzzy logics (because they neither enjoy LDT nor expand $\mathrm{MTL}_\triangle$): some fuzzy logics with additional involutive negation ($\mathrm{MTL}_\sim$, $\mathrm{BL}_\sim$, $Ł_\sim$); and the logic $\mathrm{PŁ}'$.

The following theorem gives an alternative definition of ($\triangle$-)core fuzzy logics. (It is a direct consequence of [34, Corollary 8 and Theorem 6].)

THEOREM 3.2.9. *Let* L *be an expansion of* MTL *(respectively, of* $\mathrm{MTL}_\triangle$*) that satisfies the condition* (Cong). *Then* L *is a core (resp.,* $\triangle$*-core) fuzzy logic if and only if it is an axiomatic expansion of* MTL *(*$\mathrm{MTL}_\triangle$*, resp.).*

The notion of L-algebra can be generally defined for ($\triangle$-)core fuzzy logics as follows:

DEFINITION 3.2.10. *Let* L *be a core fuzzy logic and let* C *be the set of connectives of* L *that are not present in* MTL. *An* L-algebra *is a structure* $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}, (c)_{c\in C}\rangle$ *such that* $\langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}\rangle$ *is an* MTL*-algebra, and for every axiom* φ *of* L*, the identity* $e(\varphi) = \overline{1}$ *holds under all* $\boldsymbol{A}$*-evaluations* e.

Analogously we define L*-algebras for* $\triangle$*-core fuzzy logics. An* L*-algebra is called an* L-chain *if its* MTL*-reduct is an* MTL*-chain. The class of all* L*-algebras is denoted by* $\mathbb{L}$.

[55] We have not seen (and will not see) any logic that does not satisfy (Cong).

The following theorem collects the basic properties of ($\triangle$-)core fuzzy logics. We can see that these logics share many important logical and algebraic properties with the logics BL and MTL.

THEOREM 3.2.11 ([35, 98]). *Let* L *be a ($\triangle$-)core fuzzy logic. Then:*

1. $\mathbb{L}$ *is a variety of algebras.*
2. *The lattice of finitary extensions of* L *is dually isomorphic to the lattice of subquasivarieties of* $\mathbb{L}$*.*
3. *The lattice of axiomatic extensions of* L *is dually isomorphic to the lattice of subvarieties of* $\mathbb{L}$*.*
4. L *satisfies the* proof by cases property PCP *(cf. Theorem 1.2.11): for each theory* Γ *and formulae* φ, ψ, χ*,*
$$\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi \textit{ whenever } \Gamma, \varphi \vdash_{\mathrm{L}} \chi \textit{ and } \Gamma, \psi \vdash_{\mathrm{L}} \chi.$$
5. L *satisfies the* semilinearity property SLP *(cf. Theorem 1.2.11): for each theory* Γ *and formulae* φ, ψ, χ*,*
$$\Gamma \vdash_{\mathrm{L}} \chi \textit{ whenever } \Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \textit{ and } \Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi.$$
6. *For each theory* Γ *and each formula* φ *such that* $\Gamma \nvdash_{\mathrm{L}} \chi$ *there is a linear (or equivalently, prime) theory* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{L}} \chi$ *(cf. Theorem 1.2.13).*
7. *Every* L*-algebra is representable as a subdirect product of* L*-chains (cf. Theorem 1.3.11).*
8. *The class of finitely subdirectly irreducible* L*-algebras coincides with the class of* L*-chains (cf. Theorem 1.3.13).*
9. *The following conditions (cf. Theorems 1.3.7 and 1.3.12) are equivalent for every theory* Γ *and a formula* φ*:*
   - $\Gamma \vdash_{\mathrm{L}} \varphi$.
   - $e(\varphi) = \overline{1}$ *for each* L*-algebra* $\boldsymbol{A}$ *and any* $\boldsymbol{A}$*-model* e *of* Γ.
   - $e(\varphi) = \overline{1}$ *for each* L-chain $\boldsymbol{A}$ *and any* $\boldsymbol{A}$*-model* e *of* Γ.

Using the axioms of $\triangle$, we can determine the semantics of $\triangle$ in L-chains:

THEOREM 3.2.12. *Let* L *be a* $\triangle$*-core fuzzy logic and* $\boldsymbol{A}$ *an* L*-chain. Then* $\triangle^{\boldsymbol{A}} x = \overline{1}^{\boldsymbol{A}}$ *if* $x = \overline{1}^{\boldsymbol{A}}$*, and* $\triangle^{\boldsymbol{A}} x = \overline{0}^{\boldsymbol{A}}$ *otherwise.*

For each core fuzzy logic L we can define the corresponding $\triangle$-fuzzy logic $\mathrm{L}_\triangle$ resulting from L in the same way as $\mathrm{MTL}_\triangle$ from MTL. The following two results are straightforward corollaries of the previous two theorems.

THEOREM 3.2.13. *For every core fuzzy logic* L*, the logic* $\mathrm{L}_{\triangle}$ *is a conservative expansion of* L.

THEOREM 3.2.14. *Let* L *be a finitary expansion of* $\mathrm{MTL}_{\triangle}$ *satisfying* (Cong). *Then* L *is a* $\triangle$*-core fuzzy logic if and only if* L *is strongly complete w.r.t.* L*-chains (see the last condition in Theorem 3.2.11).*

3.3 Fuzzy logics as algebraically implicative semilinear logics

The class of core fuzzy logics is quite broad, but it still does not cover all fuzzy logics mentioned in the Section 2, mainly because these logics are weaker than MTL. As we have seen in the previous subsection, there is a large family of such logics that is extensively studied in the literature. Here we shall briefly introduce fundamentals of an abstract theory of propositional fuzzy logics; for a detailed exposition and references see Chapter II of this Handbook.

DEFINITION 3.3.1 ([34, 159]). *A logic* L *in the language* $\mathcal{L}$ *is a* weakly implicative logic *if* $\mathcal{L}$ *contains a binary connective* $\to$ *such that:*

- $\vdash_{\mathrm{L}} \varphi \to \varphi$
- $\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$
- $\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$
- $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \to c(\chi_1, \dots \chi_i, \psi, \dots, \chi_n),$
  for every $\langle c, n\rangle \in \mathcal{L}$ *and* $i < n$.

A weakly implicative logic is Rasiowa-implicative *if*

- $\varphi \vdash_{\mathrm{L}} \psi \to \varphi$.

Note that the last condition in the definition of weakly implicative logics gives us that for each set of formulae $T \cup \{\varphi, \psi, \chi\}$ it holds:

$$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} \chi(\varphi) \to \chi(\psi). \tag{Cong}$$

EXAMPLE 3.3.2. All logics mentioned so far in this chapter are weakly implicative, including all substructural logics. (In non-commutative logics, the rôle of $\to$ can be played by both $\backslash$ and $/$.)

The logic $\mathrm{FL_w}$ and all its extensions, including all logics of (sets of) (left-)continuous t-norms, as well as all ($\triangle$)-core fuzzy logics are Rasiowa-implicative, and so are all fragments of these logics that contain implication (see Section 2.3).

It can be shown that Rasiowa-implicative logics share an important common feature of core fuzzy logics: namely that in their natural algebraic semantics there is always just a single element which is designated (i.e., is regarded as 'fully true' in the definition of the logic's consequence relation), and this element can be defined as the one satisfying the equation $x = \overline{1}$. This is no longer true in UL (which is clearly not Rasiowa-implicative), where the designated truth values are those bigger than $\overline{1}$; nevertheless, we can still define these truth values as those satisfying the equation $x \wedge \overline{1} = \overline{1}$. To encompass also logics like UL together with their 'natural' algebraic semantics we define a broader class of *algebraically implicative logics* where the designated truth values are equationally definable. Let us note here that finitary algebraically implicative logics are exactly those weakly implicative logics which are *algebraizable* in the sense of Blok and Pigozzi [18].

Let us now formalize this condition and express it in a purely syntactic way.[56] Observe that the logics and their algebraic semantics we have studied throughout this chapter have the following two properties: (i) if the logic proves $\varphi \vdash \psi$ and φ is designated, then so is ψ, and (ii) $x \to y$ and $y \to x$ are both designated iff $x = y$.

DEFINITION 3.3.3. *A weakly implicative logic* L *is* algebraically implicative *if there is a pair of formulae* $\alpha_{\mathrm{L}}(p), \beta_{\mathrm{L}}(p)$ *of single variable* p *such that for each formula* χ *holds:*

- $\chi \vdash_{\mathrm{L}} \alpha_{\mathrm{L}}(\chi) \to \beta_{\mathrm{L}}(\chi)$
- $\chi \vdash_{\mathrm{L}} \beta_{\mathrm{L}}(\chi) \to \alpha_{\mathrm{L}}(\chi)$
- $\alpha_{\mathrm{L}}(\chi) \to \beta_{\mathrm{L}}(\chi), \beta_{\mathrm{L}}(\chi) \to \alpha_{\mathrm{L}}(\chi) \vdash_{\mathrm{L}} \chi$.

EXAMPLE 3.3.4. Every Rasiowa-implicative logic is algebraically implicative, via the pair $\alpha_{\mathrm{L}}(p) = p$ and $\beta_{\mathrm{L}}(p) = p \to p$. (In fact, any theorem of L with at most one variable can play the rôle of β_{L}—e.g., the truth constant $\overline{1}$).

The full Lambek logic FL and all its extensions (including, e.g., the uninorm logic UL) are algebraically implicative, via the pair $\alpha_{\mathrm{L}}(p) = p \wedge \overline{1}$ and $\beta_{\mathrm{L}}(p) = \overline{1}$ (or $\alpha_{\mathrm{L}}(p) = p \vee \overline{1}$ and $\beta_{\mathrm{L}}(p) = p$).

DEFINITION 3.3.5 ($\mathcal{L}$-algebras). *Let $\mathcal{L}$ be a propositional language. An $\mathcal{L}$*-algebra *is an algebra* $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \rangle_{\langle c,n \rangle \in \mathcal{L}} \rangle$ *with the signature* $\langle n \rangle_{\langle c,n \rangle \in \mathcal{L}}$.

An $\boldsymbol{A}$-evaluation *is a mapping* $e\colon \mathit{Form}_{\mathcal{L}} \to A$ *that commutes with the connectives from $\mathcal{L}$ (i.e.,* $e(c_{\boldsymbol{A}}(\varphi_1, \dots, \varphi_n)) = c_{\boldsymbol{A}}(e(\varphi_1), \dots, e(\varphi_n))$ *for each* $\langle c, n \rangle \in \mathcal{L}$.

The next definition embodies the slogan that $\alpha_{\mathrm{L}}(p) = \beta_{\mathrm{L}}(p)$ defines the designated (fully true) truth values (cf. also Example 3.3.4).

DEFINITION 3.3.6 (Designated elements and models). *Let* L *be an algebraically implicative logic and* $\boldsymbol{A}$ *an $\mathcal{L}$-algebra. We define the set* $D^{\boldsymbol{A}}$ *of* designated elements *in* $\boldsymbol{A}$ *as:*

$$D^{\boldsymbol{A}} = \{x \mid \alpha_{\mathrm{L}}(x) = \beta_{\mathrm{L}}(x)\}.$$

We say that an $\boldsymbol{A}$*-evaluation* e *is an* $\boldsymbol{A}$-model *of a theory* Γ *if* $e(\psi) \in D^{\boldsymbol{A}}$ *for each* $\psi \in \Gamma$.

Note that $e(\psi) \in D^{\boldsymbol{A}}$ iff $e(\alpha_{\mathrm{L}}(\psi)) = e(\beta_{\mathrm{L}}(\psi))$. In a more general setting there is an abstract notion of *logical matrix* consisting of an algebra with a set of designated elements. In algebraically implicative logics we assume that the designated set is uniquely determined by an equation in the algebra; in the general setting, the situation is more complex and we leave its study to Chapter II.

[56] We provide a simplified account, assuming definability by a single pair of formulae. For the 'proper' definition see [41] or Chapter II.

DEFINITION 3.3.7 (Algebraic semantics). *Let* L *be an algebraically implicative logic and* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra. We say that* $\boldsymbol{A}$ *is an* L-algebra *if for each theory* Γ*, formula* φ*, and elements* $x, y \in A$ *it holds that:*

- *If* $\Gamma \vdash_{\mathrm{L}} \varphi$ *then any* $\boldsymbol{A}$*-model of* Γ *is also an* $\boldsymbol{A}$*-model of* φ*.*
- *If* $x \to^{\boldsymbol{A}} y \in D^{\boldsymbol{A}}$ *and* $y \to^{\boldsymbol{A}} x \in D^{\boldsymbol{A}}$*, then* $x = y$*.*

The class of all L*-algebras will be denoted by* $\mathbb{L}$*.*

We could easily show that our general definition of $\mathbb{L}$ for logics studied in this chapter coincides with the particular explicit definitions provided while defining those logics in the previous sections. It can be shown that it is sufficient to check the first condition for axioms and deduction rules of some (any) presentation of L. Let us now sample some abstract variants of theorems shown for particular logics in the previous sections. We start with results valid for all algebraically implicative logics, fuzzy or not.

THEOREM 3.3.8 ([18, 41]). *Let* L *be an algebraically implicative logic. Then:*

1. *Every extension of* L *is an algebraically implicative logic.*
2. *The following conditions are equivalent for every theory* Γ *and formula* φ*:*
   - $\Gamma \vdash_{\mathrm{L}} \varphi$*.*
   - *For each* L*-algebra* $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-model* e *of* Γ*,* e *is an* $\boldsymbol{A}$*-model of* φ*.*
3. *If* L *is finitary, then* $\mathbb{L}$ *is a quasivariety of algebras.*
4. *If* L *is finitary, then the lattice of finitary extensions of* L *is dually isomorphic to the lattice of subquasivarieties of* $\mathbb{L}$*.*
5. *The lattice of axiomatic extensions of* L *is dually isomorphic to the lattice of relative subvarieties of* $\mathbb{L}$*.*
6. *The relation* $\leq^{\boldsymbol{A}}$ *defined as* $x \leq^{\boldsymbol{A}} y$ *iff* $x \to^{\boldsymbol{A}} y \in D^{\boldsymbol{A}}$ *(i.e.,* x *is less than or equal to* y *whenever the implication* $x \to^{\boldsymbol{A}} y$ *is fully true) is an ordering on* A*.*

Note that the antisymmetry of $\leq^{\boldsymbol{A}}$ follows from the second condition in the definition of L-algebras. Also note that the majority of algebras related to logics introduced in the previous sections were ordered (usually lattice-ordered) and that this order coincides with the just defined order $\leq^{\boldsymbol{A}}$. We say that an L-algebra is *linearly ordered* (or that it is an L-*chain*) if $\leq^{\boldsymbol{A}}$ is a total order.

Let us now focus on fuzzy logics. Observe all logics that we have called 'fuzzy' in this chapter are complete w.r.t. their linearly ordered algebras.[57] Following the tradition of Universal Algebra to call a class of algebras 'semi-X' if its subdirectly irreducible members have the property X (see the next theorem), we shall call such logics *semilinear.*

[57] In [14] it is argued that all and only such logics should be called 'fuzzy logics', since this property is shared by a vast majority of logics studied in the literature under this name and does not apply to most logics that are generally not labeled as 'fuzzy' (e.g., intuitionistic logic); here we opt for a more neutral stance.

DEFINITION 3.3.9 (Semilinear logics, [41]). *An algebraically implicative logic* L *is* semilinear *if the following conditions are equivalent for each theory* Γ *and each formula* φ*:*

- $\Gamma \vdash_{\mathrm{L}} \varphi$.
- *For each* L-chain $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-model* e *of* Γ *holds that* e *is an* $\boldsymbol{A}$*-model of* φ.

A theory T *is* linear *in* L *if for each pair of formulae* φ, ψ *holds* $T \vdash_{\mathrm{L}} \varphi \to \psi$ *or* $T \vdash_{\mathrm{L}} \psi \to \varphi$*. In logic with* $\vee$ *in the language we also define: a theory* T *is* prime *in* L *if for each pair of formulae* φ, ψ *holds* $T \vdash_{\mathrm{L}} \varphi \vee \psi$ *implies* $T \vdash_{\mathrm{L}} \varphi$ *or* $T \vdash_{\mathrm{L}} \psi$*.*

The next central theorem shows that many properties proved separately in the literature on mathematical fuzzy logic are in fact instances of a general theorem.

THEOREM 3.3.10 ([41]). *Let* L *be a finitary algebraically implicative logic. Then the following conditions are equivalent:*

- L *is semilinear.*
- L *has* SLP*, i.e., for each theory* Γ *and formulae* φ, ψ, χ *holds:*

$$\textit{if } \Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \textit{ and } \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi, \textit{ then } \Gamma \vdash_{\mathrm{L}} \chi.$$

- *For every theory* Γ *and every formula* φ *such that* $\Gamma \nvdash_{\mathrm{L}} \varphi$*, there is a linear theory* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{L}} \varphi$.
- *Each* L*-algebra is a subdirect product of* L*-chains.*
- L*-chains are exactly the relatively finitely subdirectly irreducible* L*-algebras.*[58]

If the language of L *contains a connective* $\vee$ *such that*

$$\varphi \to \psi, \varphi \vee \psi \vdash_{\mathrm{L}} \psi \textit{ and } \varphi \to \psi, \psi \vee \varphi \vdash_{\mathrm{L}} \psi,$$

we can equivalently add:[59]

- L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and has* PCP*, i.e., for each theory* Γ *and formulae* φ, ψ, χ *holds:*

$$\textit{if } \Gamma, \varphi \vdash_{\mathrm{L}} \chi \textit{ and } \Gamma, \psi \vdash_{\mathrm{L}} \chi, \textit{ then } \Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi.$$

- L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$*, and for every theory* Γ *and every formula* φ *such that* $\Gamma \nvdash_{\mathrm{L}} \varphi$*, there is a prime theory* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{L}} \varphi$.

[58] I.e., they cannot be decomposed into a non-trivial subdirect product of L-*algebras*. The restriction to L-algebras holds automatically if $\mathbb{L}$ is a variety, because in that case each component of a subdirect product is a homomorphic image of an L-algebra, and therefore an L-algebra.

[59] The validity of the two rules required of $\vee$ is only necessary for proving that semilinearity implies the following three notions; the converse direction holds generally.

- L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$*, and for every finite theory* Γ*, a formula* φ *such that* $\Gamma \vdash_{\mathrm{L}} \varphi$*, and a propositional variable* p *not occurring in* Γ *and* φ *we have:* $\{\psi \vee p \mid \psi \in \Gamma\} \vdash_{\mathrm{L}} \varphi \vee p$.

One of the problems frequently studied in the literature is how, given a substructural logic L, to axiomatize the logic given by L-chains (see Section 3.2 in Chapter II).

THEOREM 3.3.11 ([41]). *Let* L *be a finitary algebraically implicative logic. Let furthermore the logic* L^{ℓ} *be defined as:* $\Gamma \vdash_{\mathrm{L}^{\ell}} \varphi$ *iff for each* L*-chain and each* $\boldsymbol{A}$*-model* e *of* Γ *holds that* e *is an* $\boldsymbol{A}$*-model of* φ*. Then* L^{ℓ} *is the least semilinear logic extending* L.

The next theorem shows that if the language of L contains $\vee$, we can easily find an axiomatization of the logic L^{ℓ}.

THEOREM 3.3.12 ([41]). *Let* L *be a finitary algebraically implicative logic in a language containing* $\vee$*, and let* $\mathcal{A}$ *be one of its axiomatic systems. Assume further that* $\varphi \to \psi, \varphi \vee \psi \vdash_{\mathrm{L}} \psi$ *and* $\varphi \to \psi, \psi \vee \varphi \vdash_{\mathrm{L}} \psi$.

Then the logic L^{ℓ} *is axiomatized by* $\mathcal{A}$ *plus the axiom* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *plus the rules of the form* $\{\psi \vee p \mid \psi \in T\} \vdash \varphi \vee p$*, for each rule* $T \vdash \varphi$ *from* $\mathcal{A}$*, where* p *is a variable that does not occur in* T *and* φ*.*

4 Metamathematics of propositional fuzzy logics

In this section we survey the basic metamathematical properties of fuzzy logics. We restrict ourselves to ($\triangle$-)core fuzzy logics (see Section 3.2), even though most of the definitions can be formulated and most of the results proven in weaker logics (e.g., non-commutative, non-integral, with restricted language, etc.).

The first subsection deals with (different) forms of completeness theorems with respect to some distinguished semantics. Note that we have already seen real-chain and standard semantics; besides those, we will also study hyperreal-, rational-, and finite-chain semantics of fuzzy logics. The results in this subsection are of two kinds: (i) general, i.e., for arbitrary semantics and/or arbitrary ($\triangle$-)core fuzzy logic; and (ii) particular, for concrete semantics and/or ($\triangle$-)core fuzzy logic. The general results are from [35]; see Chapter II for generalization of these results to a much wider context. Known particular results are also summarized in [35]; see the corresponding chapters of this Handbook for proofs of some of these results and detailed references.

The second subsection deals with functional representation, i.e., the question as to which real-valued functions are expressible by formulae of prominent ($\triangle$-)core fuzzy logic. This question is connected to a prominent algebraic question regarding the description of the free algebra in a given variety. We will not go into details here; see Chapter IX for a detailed exposition, proofs, and references.

The third subsection briefly introduces the proof theory of certain fuzzy logics, by means of (hyper)-sequent Gentzen systems. The whole Chapter III of this Handbook is dedicated to elaboration of this important aspect of mathematical fuzzy logic. The final subsection studies fuzzy logics from the viewpoint of computational complexity. In particular, the complexity of important sets of formulae (tautologies, satisfiable formulae) given by prominent ($\triangle$-)core fuzzy logic is determined in the subsection. Again, the whole Chapter X is dedicated to this topic; an interested reader can find detailed exposition, proofs, and references there.

4.1 Completeness theorems

Let us fix a ($\triangle$-)core logic L. For a class $\mathbb{K}$ of L-chains we define the semantical consequence w.r.t. $\mathbb{K}$ analogously to the case of continuous t-norms (cf. Definition 1.1.15):

$$\Gamma \models_{\mathbb{K}} \varphi \quad \text{iff} \quad e(\varphi) = 1 \text{ for each } \boldsymbol{A} \in \mathbb{K} \text{ and each } \boldsymbol{A}\text{-model } e \text{ of } \Gamma.$$

DEFINITION 4.1.1 (Completeness properties). *Let $\mathbb{K}$ be a class of* L*-chains. We say that* L *has the property of:*

- Strong $\mathbb{K}$-completeness, $\mathrm{S}\mathbb{K}\mathrm{C}$ *for short, if for every set of formulae* $\Gamma \cup \{\varphi\}$ *holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *if and only if* $\Gamma \models_{\mathbb{K}} \varphi$*.*
- Finite strong $\mathbb{K}$-completeness, $\mathrm{FS}\mathbb{K}\mathrm{C}$ *for short, if for every* finite *set of formulae* $\Gamma \cup \{\varphi\}$ *holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *if and only if* $\Gamma \models_{\mathbb{K}} \varphi$*.*
- (Weak) $\mathbb{K}$-completeness, $\mathbb{K}\mathrm{C}$ *for short, if for every formula* φ *holds:* $\vdash_{\mathrm{L}} \varphi$ *if and only if* $\models_{\mathbb{K}} \varphi$*.*

Now we give three theorems characterizing the just defined properties in a purely algebraic fashion. First we need one important definition.

DEFINITION 4.1.2 (Partial embeddability). *Given two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same language $\mathcal{L}$, we say that a finite subset X of $\boldsymbol{A}$ is* partially embeddable *into $\boldsymbol{B}$ if there is a one-to-one mapping $f\colon X \to B$ such that for each $\langle c, n\rangle \in \mathcal{L}$ and each $a_1, \ldots, a_n \in X$ satisfying $c^{\boldsymbol{A}}(a_1, \ldots, a_n) \in X$, $f(c^{\boldsymbol{A}}(a_1, \ldots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n))$.*

A class $\mathbb{K}$ of algebras is partially embeddable *into a class $\mathbb{K}'$ if every finite subset of every member of $\mathbb{K}$ is partially embeddable into a member of $\mathbb{K}'$.*

THEOREM 4.1.3. *Let $\mathbb{K}$ be a class of* L*-chains. Then:*

1. L *has* $\mathbb{K}\mathrm{C}$ *if and only if* $\mathbb{L} = \mathbf{V}(\mathbb{K})$*, where* $\mathbf{V}(\mathbb{K})$ *is the variety generated by* $\mathbb{K}$*.*
2. L *has* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *if and only if* $\mathbb{L} = \mathbf{Q}(\mathbb{K})$*, where* $\mathbf{Q}(\mathbb{K})$ *is the quasi-variety generated by* $\mathbb{K}$*.*
3. L *has* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if and only if* $\mathbb{L} = \mathbf{ISP}_{\sigma\text{-f}}(\mathbb{K})$*, where* $\mathbf{ISP}_{\sigma\text{-f}}$ *denotes the closure under isomorphic images, subalgebras, and the operator of reduced products over countably complete filters.*

THEOREM 4.1.4. *Let $\mathbb{K}$ be a class of* L*-chains. Then the following conditions are equivalent:*

1. L *has* $\mathrm{S}\mathbb{K}\mathrm{C}$*.*
2. *Every countable* L*-chain is embeddable into a member of* $\mathbb{K}$*.*
3. *Every countable subdirectly irreducible* L*-chain is embeddable into a member of* $\mathbb{K}$*.*

THEOREM 4.1.5. *Let* $\mathbb{K}$ *be a class of* L*-chains. Then the following conditions are equivalent:*

1. L *has* $\mathrm{FS}\mathbb{K}\mathrm{C}$.
2. *Every* L*-chain is embeddable into an ultraproduct of members of* $\mathbb{K}$.

If furthermore the language of L *is* finite, *we can equivalently add:*

3. *Every* L*-chain is partially embeddable into* $\mathbb{K}$.
4. *Every countable subdirectly irreducible* L*-chain is partially embeddable into* $\mathbb{K}$.

The condition 4 is a sufficient condition for the $\mathrm{FS}\mathbb{K}\mathrm{C}$ of L, but is not necessary, as shown in [35, Example 3.10]. Next we shall give some relationships between strong and finite strong $\mathbb{K}$-completeness:

THEOREM 4.1.6. *Let* $\mathbb{K}$ *be a class of* L*-chains. Assume that* L *has* $\mathrm{FS}\mathbb{K}\mathrm{C}$. *Then:*

1. L *has* $\mathrm{S}\mathbb{K}\mathrm{C}$ *iff* $\models_K$ *is finitary.*
2. L *has* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if* $\mathbb{K}$ *is closed under ultraproducts.*
3. L *has* $\mathrm{S}\mathbf{P}_U(\mathbb{K})\mathrm{C}$*, where* $\mathbf{P}_U$ *denotes the closure under ultraproducts.*

We can also prove that a $\triangle$-core fuzzy logic has $\mathbb{K}\mathrm{C}$ if and only if it has $\mathrm{FS}\mathbb{K}\mathrm{C}$. Also for a core fuzzy logic L it can be shown that: L has the $\mathrm{S}\mathbb{K}\mathrm{C}$ (resp. $\mathrm{FS}\mathbb{K}\mathrm{C}$) with respect to a class of L-chains $\mathbb{K}$ if and only if $\mathrm{L}_\triangle$ has the $\mathrm{S}\mathbb{K}_\triangle\mathrm{C}$ (resp. $\mathrm{FS}\mathbb{K}_\triangle\mathrm{C}$), where $\mathbb{K}_\triangle$ is the class of $\triangle$-expansions of chains in $\mathbb{K}$.

Now we move to particular distinguished semantics. Let us fix a ($\triangle$-)core logic L. We define the following classes of L-chains:

- *Real-valued* L-chains, $\mathcal{R}$: those whose lattice reduct is $[0,1]$ with the usual order.
- *Rational-valued* L-chains, $\mathcal{Q}$: those whose lattice reduct is $[0,1]^{\mathbb{Q}}$ with the usual order.
- *Hyperreal-valued* L-chains, $\mathcal{R}^\star$: those whose lattice reduct is any ultrapower of $[0,1]$ with the usual order.
- *Finite-valued* L-chains, $\mathcal{F}$: those chains which are finite.

Recall that by conventions of Section 2, *standard* L-chains are usually a (conventionally distinguished) subclass of real-valued L-chains (often all of them, cf. Convention 2.0.2). Note also that the class of hyperreal-valued L-chains $\mathcal{R}^\star$ contains the class of real-valued L-chains $\mathcal{R}$ (just take the ultrapower over a principal ultrafilter). We could furthermore consider *strict hyperreal-valued* L-chains, which arise by removing the real-valued ones from $\mathcal{R}^\star$; however, it can be shown that all the three completeness properties remain unchanged by this modification. In fact, even the rational-valued and hyperreal-valued completeness properties coincide: all known relationships of these properties are depicted in Figure 11.

$$\begin{array}{ccccc}
\mathrm{S}\mathcal{R}^{\star}\mathrm{C} & \rightleftarrows & \mathrm{FS}\mathcal{R}^{\star}\mathrm{C} & \longrightarrow & \mathcal{R}^{\star}\mathrm{C} \\
\downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\
\mathrm{S}\mathcal{Q}\mathrm{C} & \rightleftarrows & \mathrm{FS}\mathcal{Q}\mathrm{C} & \longrightarrow & \mathcal{Q}\mathrm{C} \\
\not\downarrow\uparrow & & \not\downarrow\uparrow & & \not\downarrow\uparrow \\
\mathrm{S}\mathcal{R}\mathrm{C} & \underset{\not\leftarrow}{\rightarrow} & \mathrm{FS}\mathcal{R}\mathrm{C} & \longrightarrow & \mathcal{R}\mathrm{C}
\end{array}$$

Figure 11. Real-chain, rational-chain, and hyperreal-chain completeness properties

The completeness w.r.t. finite chains has already been mentioned in Section 2.4.3, where we have defined the n-valued extension of a given fuzzy logic by means of the axiom $(\leq n)$. This axiom in fact characterizes the strong finite-chain completeness, as the following can be proved to be equivalent:

1. L enjoys $\mathrm{S}\mathcal{F}\mathrm{C}$.
2. All L-chains are finite.
3. There is a natural number n such that the length of each L-chain is less or equal than n.
4. There is a natural number n such that $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

The weaker forms of completeness w.r.t. finite chains correspond to some well-known algebraic properties, namely:

- L enjoys $\mathcal{F}\mathrm{C}$ if and only if the variety of L-algebras enjoys FMP (the finite model property).
- L enjoys $\mathrm{FS}\mathcal{F}\mathrm{C}$ if and only if the variety of L-algebras enjoys SFMP (the strong finite model property). Moreover, if the language of L is finite, these properties are also equivalent to FEP (the finite embeddability property) for the variety of L-algebras.

The known completeness results for fuzzy logics introduced in this chapter can be summarized as follows:

- The following logics enjoy $\mathrm{S}\mathcal{R}\mathrm{C}$ (and thus also $\mathrm{S}\mathcal{Q}\mathrm{C}$ and $\mathrm{S}\mathcal{R}^{\star}\mathrm{C}$): MTL, IMTL, SMTL, G, WNM, NM, C_nMTL, and C_nIMTL.
- The following logics enjoy $\mathrm{FS}\mathcal{R}\mathrm{C}$ (and thus also $\mathrm{S}\mathcal{Q}\mathrm{C}$ and $\mathrm{S}\mathcal{R}^{\star}\mathrm{C}$), but not $\mathrm{S}\mathcal{R}\mathrm{C}$: WCMTL, ΠMTL, BL, SBL, Ł, and Π.
- The logic $\Pi^{\star}$ introduced in [45] is an example of logic enjoying $\mathrm{S}\mathcal{Q}\mathrm{C}$ but not even $\mathcal{R}\mathrm{C}$. $\Pi^{\star}$ is the axiomatic extension of BL by the axiom

$$\neg(\varphi \wedge \neg\varphi) \wedge ((\varphi \to \varphi \,\&\, \varphi) \to \neg\varphi \vee \varphi).$$

- The only logics enjoying $\mathrm{S}\mathcal{F}\mathrm{C}$ introduced in this chapter are the finite-valued logics G_n and Ł_n.
- The following logics enjoy $\mathrm{FS}\mathcal{F}\mathrm{C}$, but not $\mathrm{S}\mathcal{F}\mathrm{C}$: MTL, IMTL, SMTL, WNM, C_nMTL, C_nIMTL, BL, SBL, Ł, G, and NM.
- The following logics do not enjoy $\mathcal{F}\mathrm{C}$: WCMTL, ΠMTL, and Π.

4.2 Functional representation

It is well known that all Boolean functions (i.e., functions from $\{0,1\}^n$ to $\{0,1\}$) are expressible by formulae of classical logic. If we replace the set $\{0,1\}$ by $[0,1]$, we are in the realm of fuzzy logic and a natural question is which functions from $[0,1]^n$ to $[0,1]$ are expressible by a formula of a particular fuzzy logic. Clearly, due to the cardinality reasons we cannot expect the analog of the classical results (there are uncountably many functions and only countably many formulae).

We shall restrict ourselves to ($\triangle$-)core fuzzy logics that enjoy standard completeness w.r.t. a unique standard algebra. For this section let L be one of the following logics: Ł, G, Π, RPL, PŁ$'$, their extensions by $\triangle$, and the logics ŁΠ and ŁΠ$\frac{1}{2}$. By $[0,1]_{\mathrm{L}}$ we denote the corresponding standard algebra.

DEFINITION 4.2.1. *Let f be a function $f\colon [0,1]^n \to [0,1]$ and $\varphi(v_1,\dots,v_n)$ a formula of a logic* L*. We say that the function f is* represented *by the formula φ in* L *(or that φ is a* representation *of f) if $e(\varphi) = f(e(v_1), e(v_2), \dots, e(v_n))$ for each $[0,1]_{\mathrm{L}}$-evaluation e.*

The functional representation *of the logic* L *is the class of all functions from any power of $[0,1]$ into $[0,1]$ that are representable by some formula of* L.

The goal of this subsection is to summarize known characterizations of functional representation of aforementioned logics.[60] See Chapter IX of this Handbook for detailed exposition of this area, including more results, proofs, and references.

The first theorem of this kind is McNaughton's celebrated result:

THEOREM 4.2.2 ([131, 144]). *The functional representation of Łukasiewicz logic is the class of McNaughton functions, i.e., continuous piecewise linear functions with integer coefficients.*

Moving to the logic RPL would change little—only the absolute coefficients of the linear functions could now be rational. The description for product logic is more complex. First let us define:

DEFINITION 4.2.3. *Let n be a natural number. Then a set $S \subseteq [0,1]^n$ is a* region of positivity *if there is $M \subseteq \{i \mid 1 \leq i \leq n\}$ such that $S = \{\langle x_1,\dots,x_n\rangle \in [0,1]^n \mid x_i > 0$ iff $i \in M\}$.*

THEOREM 4.2.4 ([36]). *The functional representation of product logic is the class of functions which, restricted to any region of positivity are either identically equal to 0 or continuous piecewise monomial.*

[60] One can observe that functional representation is not (strictly speaking) a property of a logic, but of an algebra; standard completeness allows us to perform this slight abuse of language.

Logic	Contin.	Domains	Pieces
Ł	yes	linear	linear functions with integer coefficients
$Ł_{\triangle}$	no	linear	linear functions with integer coefficients
RPL	yes	linear	linear integer coefficients and a rational shift
PŁ′	yes	?	? [62]
$PŁ'_{\triangle}$	no	all	polynomials with integer coefficients
ŁΠ	no	all	fractions of polynomials with integer coefficients
$Ł\Pi\frac{1}{2}$	no	all	fractions of polyn. w. int. coeff., $f[\{0,1\}^n] \subseteq \{0,1\}$

Table 4. Functional representations of prominent fuzzy logics

For a (rather complex) characterization of the functional representation of Gödel logic we refer the reader to Chapter IX.

DEFINITION 4.2.5. *A subset S of $[0,1]^n$ is* Q-semialgebraic *if it is a Boolean combination of sets of the form*

$$\{\langle x_1, \dots, x_n\rangle \in [0,1]^n \mid P(\langle x_1, \dots, x_n\rangle) > 0\}$$

for polynomials P with integer coefficients. If all of the polynomials are linear, then S is linear Q-semialgebraic.

It can be easily shown that the domain of each linear piece in a McNaughton function is linear Q-semialgebraic. In Theorem 4.2.2, the latter property was ensured by the continuity; however, if the non-continuous Baaz Delta operation is added, the condition has to be explicitly stated.

THEOREM 4.2.6. *The functional representation of $Ł_{\triangle}$ is the set of piecewise linear functions with integer coefficients, with the pieces having linear* Q*-semialgebraic domains.*[61]

As we have seen, the functions expressible in expansions of Łukasiewicz logic are piecewise functions determined by three parameters: (i) the requirement or non-requirement of continuity, (ii) the form of functions describing the pieces, and (iii) the description of domains of the pieces. Table 4 summarizes known results (mainly from [141]). The column 'Contin.' indicates whether the continuity of the functions is required or not. The column 'Domains' indicates whether the domains are all Q-semialgebraic sets, or just the linear ones.

[61] In more detail, the functions $f\colon [0,1]^n \to [0,1]$ for which there is a finite partition of $[0,1]^n$ such that each block of the partition is a linear Q-semi-algebraic set and f restricted to each block is a linear function with integer coefficients.

[62] The well-known Pierce–Birkhoff conjecture [120] is equivalent to the supposition that the functional representation of PŁ′ is the set of continuous piecewise polynomial functions with integer coefficients.

4.3 Proof theory

Proof theory is an important branch of mathematical logic (see [22] for a detailed and representative introduction). It deals with analysis of proofs (in a given logic) and the study of calculi designed to satisfy certain convenient properties. Such calculi (e.g., the Gentzen-style sequent calculi) are in many respects better behaved (e.g., as regards the complexity of proofs) than the Hilbert-style calculi employed in this chapter so far. Unlike general Hilbert-style calculi, which trivially exist for all logics (understood as consequence relations), the construction of calculi satisfying some required properties can be a non-trivial task, in many cases even demonstrably impossible.

Proof theory for fuzzy logics (with the exception of Gödel logic, whose position as both fuzzy and intermediate logic makes it a special case) has seen a rapid development in recent years. The field has been started by Metcalfe [132] in the early 2000's; a milestone is the monograph [136]. In this Handbook, the proof theory of fuzzy logics is treated in detail in Chapter III. Here we present just a basic idea and one motivational example.

Recall that *Gentzen-style calculi* have two kinds of rules:

- *Operational rules* for introduction of propositional connectives (into formulae)
- *Structural rules* for manipulation with premises and conclusions (as whole formulae).

The rules operate on *sequents,* which are pairs of sequences of formulae, usually written as $\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_m$. Capital Greek letters (esp. $\Gamma, \Delta, \Pi, \Sigma$) will represent sequences of formulae (including the empty sequence) in sequent schemata. The rule (R) indicating the possibility of deriving a sequent S from sequents $S_1, \dots, S_k$ is written in the following form:

$$(\mathrm{R})\ \frac{S_1 \quad \dots \quad S_k}{S}.$$

A *sequent calculus* is a set of (schematic) rules. A *derivation* in the calculus is a (finite) tree with nodes evaluated by sequents which are derived by the rules of the calculus from their immediate predecessors. We say that the sequent S is derivable from $S_1, \dots, S_n$ in the calculus if there is a derivation with leaves evaluated by $S_1, \dots, S_n$ and the root by S; if moreover $n = 0$, we say that S is provable in the calculus.

Recall the full Lambek logic with exchange and weakening, $\mathrm{FL_{ew}}$, and its 'classical' (i.e., involutive) version $\mathrm{CFL_{ew}}$, introduced in Section 3.1. The following example gives Gentzen-style calculi for these logics.

EXAMPLE 4.3.1. The calculus $\mathrm{GCFL_{ew}}$ has the following *operational rules* for the connectives $\wedge$, &, $\vee$, $\oplus$, $\to$, $\overline{0}$, and $\overline{1}$:

$$(\overline{0}\,\mathrm{L})\ \frac{}{\overline{0}, \Gamma \Rightarrow \Pi} \qquad\qquad (\overline{1}\,\mathrm{R})\ \frac{}{\Gamma \Rightarrow \Pi, \overline{1}}$$

$$(\wedge\,\mathrm{L})\ \frac{C, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ \textit{for } C \in \{\varphi, \psi\} \qquad (\wedge\,\mathrm{R})\ \frac{\Gamma \Rightarrow \Pi, \varphi \qquad \Gamma \Rightarrow \Pi, \psi}{\Gamma \Rightarrow \Pi, \varphi \wedge \psi}$$

(& L) $\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \mathbin{\&} \psi, \Gamma \Rightarrow \Delta}$ (& R) $\dfrac{\Gamma_1 \Rightarrow \Pi_1, \varphi \qquad \Gamma_2 \Rightarrow \Pi_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Pi_1, \Pi_2, \varphi \mathbin{\&} \psi}$

($\vee$ L) $\dfrac{\varphi, \Gamma \Rightarrow \Pi \qquad \psi, \Gamma \Rightarrow \Pi}{\varphi \vee \psi, \Gamma \Rightarrow \Pi}$ ($\vee$ R) $\dfrac{\Gamma \Rightarrow \Pi, C}{\Gamma \Rightarrow \Pi, \varphi \vee \psi}$ *for* $C \in \{\varphi, \psi\}$

($\oplus$ L) $\dfrac{\varphi, \Gamma_1 \Rightarrow \Pi_1 \qquad \psi, \Gamma_2 \Rightarrow \Pi_2}{\varphi \oplus \psi, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1, \Pi_2}$ ($\oplus$ R) $\dfrac{\Gamma \Rightarrow \Pi, \varphi, \psi}{\Gamma \Rightarrow \Pi, \varphi \oplus \psi}$

($\to$ L) $\dfrac{\Gamma \Rightarrow \Pi, \varphi \qquad \psi, \Delta \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Delta \Rightarrow \Pi, \Sigma}$ ($\to$ R) $\dfrac{\varphi, \Gamma \Rightarrow \Pi, \psi}{\Gamma \Rightarrow \Pi, \varphi \to \psi}$

Moreover, the calculus $\mathrm{GCFL_{ew}}$ has the following *structural rules:*

(Ax) $\dfrac{}{\varphi \Rightarrow \varphi}$ (Cut) $\dfrac{\Gamma \Rightarrow \Pi, \varphi \qquad \varphi, \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Pi, \Sigma}$

(E-L) $\dfrac{\Gamma, \varphi, \psi, \Delta \Rightarrow \Pi}{\Gamma, \psi, \varphi, \Delta \Rightarrow \Pi}$ (E-R) $\dfrac{\Gamma \Rightarrow \Pi, \varphi, \psi, \Sigma}{\Gamma \Rightarrow \Pi, \psi, \varphi, \Sigma}$

(W-L) $\dfrac{\Gamma \Rightarrow \Pi}{\varphi, \Gamma \Rightarrow \Pi}$ (W-R) $\dfrac{\Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi, \varphi}$

The abbreviated labels of the structural rules stand for *Axiom, Exchange,* and *Weakening,* (left or right). Note that the structural rules of *Contraction*

(C-L) $\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Pi}{\varphi, \Gamma \Rightarrow \Pi}$ (C-R) $\dfrac{\Gamma \Rightarrow \Pi, \varphi, \varphi}{\Gamma \Rightarrow \Pi, \varphi}$

are missing as $\mathrm{CFL_{ew}}$ is a substructural logic. By removing further structural rules of *Exchange* and *Weakening* we would obtain calculi for the weaker logics $\mathrm{CFL_{(e)(w)}}$.

The calculus $\mathrm{GFL_{ew}}$ has the same rules as $\mathrm{GCFL_{ew}}$, with the restriction that the length of the right-hand sequence of any sequent in a derivation is at most 1. (Consequently of the structural rules, only (E-L), (W-L), and (C-L), and the rule (W-R) for sequents of the form $\Gamma \Rightarrow$ occur in $\mathrm{GFL_{ew}}$.)

$\mathrm{GCFL_{ew}}$ is a calculus for the logic $\mathrm{CFL_{ew}}$ in the following sense: a sequent $\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_m$ is provable in $\mathrm{GCFL_{ew}}$ iff its interpretation $\varphi_1 \mathbin{\&} \cdots \mathbin{\&} \varphi_n \to \psi_1 \oplus \cdots \oplus \psi_m$ is a theorem of $\mathrm{CFL_{ew}}$. Analogously, $\mathrm{GFL_{ew}}$ is a calculus for $\mathrm{FL_{ew}}$.[63]

One of the central topics in proof-theory is the question of redundance of the (Cut) rule in a given Gentzen-style calculus. Note that all remaining rules enjoy the so-called *subformula property:* i.e., the formulae in premises of rules are subformulae of those in the conclusion (thus, roughly speaking, any 'backward' step in the proof-tree reduces the complexity of formulae occurring in the labeling sequents in a very transparent way, which often provides a decision procedure for a given logic). Cut-free calculi enjoying subformula property are often called *analytic*.

[63] For $m = 0$, i.e., the empty right-hand side of the sequent, we use the interpretation $\varphi_1 \mathbin{\&} \cdots \mathbin{\&} \varphi_n \to \overline{0}$; analogously for $n = 0$ we use the interpretation $\overline{1} \to \psi_1 \oplus \cdots \oplus \psi_m$. Note that in the case of MTL each sequent has at most one formula on the right-hand side, thus the connective $\oplus$ is not needed in its interpretation.

THEOREM 4.3.2 (Cut elimination). *The calculi* $\mathrm{GFL_{ew}}$ *and* $\mathrm{GCFL_{ew}}$ *enjoy* cut elimination; *i.e., removing the rule* (Cut) *does not change the set of provable sequents.*

A natural question is whether we can 'strengthen' the two Gentzen-style calculi (preferably preserving the cut elimination property) to obtain proof systems for the semilinear versions of the logics $\mathrm{FL_{ew}}$, and $\mathrm{CFL_{ew}}$, i.e., the logics MTL and IMTL. In Hilbert-style calculi, the transition is provided by adding the prelinearity axiom, $(\varphi \to \psi) \vee (\psi \to \varphi)$. However, for Gentzen-style calculi with the cut elimination property (and the interpretation of sequents as above), this question was answered negatively by Ciabattoni, Galatos, and Terui:

THEOREM 4.3.3 ([28]). *If the sequents* $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$ *are interpreted as* $\varphi_1 \mathbin{\&} \cdots \mathbin{\&} \varphi_n \to \psi_1 \oplus \cdots \oplus \psi_m$*, then there is no Gentzen system for* IMTL *extending* $\mathrm{GCFL_{ew}}$ *with structural rules (analogously for* MTL*).*

One way to overcome this problem is to change the syntactical framework from sequents to the so-called *hypersequents.* Hypersequents were originally introduced by Avron as a proof-theoretic framework for the Relevance-Mingle logic RM [3]. Later the hypersequent calculi were generalized to Gödel logic [4] and other fuzzy logics [66, 133–136]. As an example of a hypersequent calculi for fuzzy logics, we present here a hypersequent Gentzen-style calculus for the logic IMTL (originally introduced in [6]).

A hypersequent is a finite multiset of sequents written as:

$$\Pi_1 \Rightarrow \Sigma_1 \mid \Pi_2 \Rightarrow \Sigma_2 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n.$$

A hypersequent version of the sequent rule (R) has the form

$$\text{(H-R)}\ \frac{\mathcal{H} \mid S_1 \quad \ldots \quad \mathcal{H} \mid S_k}{\mathcal{H} \mid S},$$

where $S_1, \ldots, S_k/S$ is an instance of (R), and $\mathcal{H}$ is a hypersequent variable.

DEFINITION 4.3.4. *The hypersequent calculus* GIMTL *consists of the hypersequent versions of all rules of* $\mathrm{GCFL_{ew}}$*, plus the following rules:*

$$(\mathrm{EW})\ \frac{\mathcal{H}}{\mathcal{H} \mid \mathcal{G}} \qquad (\mathrm{EC})\ \frac{\mathcal{H} \mid \mathcal{G} \mid \mathcal{G}}{\mathcal{H} \mid \mathcal{G}} \qquad (\mathrm{Com})\ \frac{\mathcal{H} \mid \Gamma_1, \Pi_1 \Rightarrow \Sigma_1, \Delta_1 \quad \mathcal{H} \mid \Gamma_2, \Pi_2 \Rightarrow \Sigma_2, \Delta_2}{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}.$$

The abbreviated labels of these rules stand for External Weakening, External Contraction, *and* Communication.

THEOREM 4.3.5. *For a sequent* $S = \varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$ *we set*

$$I(S) = \varphi_1 \mathbin{\&} \cdots \mathbin{\&} \varphi_n \to \psi_1 \oplus \cdots \oplus \psi_m;$$

for a hypersequent $\mathcal{H} = S_1 \mid \cdots \mid S_n$ *we set*

$$I(\mathcal{H}) = I(S_1) \vee I(S_2) \vee \ldots \vee I(S_n).$$

Then the hypersequent $\mathcal{H}$ *is derivable in* GIMTL *if and only if* $I(\mathcal{H})$ *is a theorem of* IMTL*. Furthermore, the* (Cut) *rule is eliminable from* GIMTL.

4.4 Computational complexity

In this section we deal with computational complexity issues in ($\triangle$-)core fuzzy logics. Unlike in classical logic, the sets of tautologies and satisfiable formulae do not in fuzzy logics determine each other. This is caused by the fact that $e(\neg\varphi) \neq 1$ does not imply $e(\varphi) = 1$, but only $e(\varphi) > 0$. Consequently, in addition to the sets of tautologies and 1-satisfiable formulae, also the sets of *positive* tautologies and *positively* satisfiable set of formulae are studied. For simplicity, in this section we consider only the first four notions and only with respect to the *standard* semantics. Furthermore we fix a ($\triangle$-)core logic enjoying the finite strong standard completeness (thus in particular we have that its standard tautologies coincide with its theorems). Chapter X studies these notions relativized to other important classes of algebras and for logics other than ($\triangle$-)core, and also tackles other complexity problems encountered in mathematical fuzzy logic (e.g., the complexity of the provability relation of a propositional fuzzy logic, or even of the universal fragment of the first-order theory of particular classes of algebras).

DEFINITION 4.4.1. *Let* L *be a ($\triangle$-)core fuzzy logic enjoying the finite strong standard completeness. Then we define the following sets of formulae:*

- $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L})$ *if there is a standard* L*-algebra* $\boldsymbol{A}$ *and an* $\boldsymbol{A}$*-evaluation* e *such that* $e(\varphi) > \overline{0}$.
- $\varphi \in \mathrm{SAT}(\mathrm{L})$ *if there is a standard* L*-algebra* $\boldsymbol{A}$ *and an* $\boldsymbol{A}$*-evaluation* e *such that* $e(\varphi) = \overline{1}$.
- $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$ *if for each standard* L*-algebra* $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-evaluation* e *holds* $e(\varphi) > \overline{0}$.
- $\varphi \in \mathrm{TAUT}(\mathrm{L})$ *if for each standard* L*-algebra* $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-evaluation* e *holds* $e(\varphi) = \overline{1}$.

Although these sets of formulae are not as tightly related as their classical analogs, some interrelations can still be proved (especially for stronger fuzzy logics):

LEMMA 4.4.2. *In general we can prove:*

$$\begin{array}{lll} \varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L}) & \textit{iff} & \neg\varphi \notin \mathrm{SAT}(\mathrm{L}) \\ \varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L}) & \textit{iff} & \neg\varphi \notin \mathrm{TAUT}(\mathrm{L}). \end{array}$$

If the logic L *expands* IMTL*, we can also prove the 'converse':*[64]

$$\begin{array}{lll} \varphi \in \mathrm{SAT}(\mathrm{L}) & \textit{iff} & \neg\varphi \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L}) \\ \varphi \in \mathrm{TAUT}(\mathrm{L}) & \textit{iff} & \neg\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L}). \end{array}$$

If the logic L *extends* SMTL*, we can also prove:*

$$\begin{array}{lllll} \varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L}) & \textit{iff} & \varphi \in \mathrm{SAT}(\mathrm{L}) & \textit{iff} & \varphi \textit{ is classically satisfiable} \\ \varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L}) & \textit{iff} & \neg\neg\varphi \in \mathrm{TAUT}(\mathrm{L}) & \textit{iff} & \varphi \textit{ is classical tautology} \\ \varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L}) & \textit{iff} & \neg\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L}) & \textit{iff} & \varphi \textit{ is classical tautology.} \end{array}$$

[64] It can also be proved in expansions of $\mathrm{MTL}_{\sim}$, for $\sim$ instead of $\neg$.

Logic	$\mathrm{SAT}(\mathrm{L})$, $\mathrm{SAT}_{\mathrm{pos}}(\mathrm{L})$	$\mathrm{TAUT}(\mathrm{L})$, $\mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$	reference
BL	**NP**-complete	**coNP**-complete	[8]
Ł	**NP**-complete	**coNP**-complete	[143]
G	**NP**-complete	**coNP**-complete	[7]
Π	**NP**-complete	**coNP**-complete	[7]
SMTL, ΠMTL	**NP**-complete	decidable, **coNP**-hard	[109, 114]
MTL, IMTL	**NP**-hard	decidable, **coNP**-hard	[19, 114]
ŁΠ, $Ł\Pi\frac{1}{2}$	**PSPACE**	**PSPACE**	[103]

Table 5. Standard computational complexity of prominent fuzzy logics

Finally, if the logic L *is* $\triangle$*-core, then we can also prove:*

$$\varphi \in \mathrm{SAT}(\mathrm{L}) \quad \textit{iff} \quad \triangle\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L})$$
$$\varphi \in \mathrm{TAUT}(\mathrm{L}) \quad \textit{iff} \quad \triangle\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L}).$$

Despite the more complex definitions involved, the computational complexity of these problems for prominent fuzzy logics does not differ much from their classical counterparts: see Table 5 for a selection of known and unknown results on standard computational complexity.

5 Predicate fuzzy logics

In this section we survey basic facts about predicate fuzzy logics. We restrict ourselves to ($\triangle$-)core fuzzy logics (see Section 3.2), even though most of the definitions and theorems can be formulated and proven in weaker logics—e.g., non-commutative, non-integral, with restricted language, etc.—as well: for more general formulations and results see Chapter II. The text of this subsection is loosely based on the survey paper [37]; basic information with full proofs for the logics of continuous t-norms is found in the monograph [83].

5.1 Syntax

In the following let L be a fixed ($\triangle$-)core fuzzy logic in a propositional language $\mathcal{L}$. The language of the first-order fuzzy logic is defined in the same way as in classical first-order logic. In order to fix notation and terminology we give an explicit definition:

DEFINITION 5.1.1. *A* predicate language $\mathcal{P}$ *is a triple* $\langle Pred_{\mathcal{P}}, Func_{\mathcal{P}}, Ar_{\mathcal{P}} \rangle$*, where* $Pred_{\mathcal{P}}$ *is a non-empty set of* predicate symbols, $Func_{\mathcal{P}}$ *is a set (disjoint with* $Pred_{\mathcal{P}}$*) of* function symbols, *and* $Ar_{\mathcal{P}}$ *is the* arity function, *assigning to each predicate or function symbol a natural number called the* arity *of the symbol. The function symbols* F *with* $Ar_{\mathcal{P}}(F) = 0$ *are called* object *or* individual constants. *The predicates symbols* P *for which* $Ar_{\mathcal{P}}(P) = 0$ *are called* truth constants.[65]

[65] The rôles of nullary predicates of $\mathcal{P}$ and nullary connectives of $\mathcal{L}$ are analogous, even though the values of the former are only fixed under a given interpretation of the predicate language, while the values of the latter are fixed under all such interpretations. The ambiguity of the term *truth constant* (see Definition 3.2.1 and Section 2.2.3) is thus a harmless abuse of language.

$\mathcal{P}$-terms and (atomic) $\mathcal{P}$-formulae of a given predicate language are defined as in classical logic (note that the notion of formula also depends on propositional connectives in $\mathcal{L}$). A $\mathcal{P}$-*theory* is a set of $\mathcal{P}$-formulae. The notions of free occurrence of a variable, substitutability, open formula, and closed formula (or, synonymously, *sentence*) are defined in the same way as in classical logic. Unlike in classical logic, in fuzzy logics without involutive negation the quantifiers $\forall$ and $\exists$ are not mutually definable, so the primitive language of $\mathrm{L}\forall$ has to contain both of them.

There are several variants of a first-order extension of a propositional fuzzy logic L that can be defined. Here we shall introduce the first-order logics $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$ (of models over general resp. linear algebras); later we shall extend the family by considering the logics $\mathrm{L}\forall^{\mathrm{w}}$ and $\mathrm{L}\forall^{\mathrm{s}}$ of, respectively, witnessed and standard models. The axiomatic systems of the logics $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$ are defined as follows:

DEFINITION 5.1.2. *Let* L *be a* ($\triangle$-)*core fuzzy logic and* $\mathcal{P}$ *a first-order language. The logic* $\mathrm{L}\forall^{\mathrm{m}}$ *has the following axioms:*

(P)	Instances of the axioms of L (with $\mathcal{P}$-formulae substituted for propositional variables)
$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$, where the $\mathcal{P}$-term t is substitutable for x in φ
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$, where the $\mathcal{P}$-term t is substitutable for x in φ
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ.

The deduction rules of $\mathrm{L}\forall^{\mathrm{m}}$ *are those of* L *and the rule of* generalization:

(Gen) From φ infer $(\forall x)\varphi$.

The logic $\mathrm{L}\forall$ *is the extension of* $\mathrm{L}\forall^{\mathrm{m}}$ *by the axiom:*

$(\forall 3)$ $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ.

The notions of proof and provability are in first-order fuzzy logics defined in the same way as in first-order classical logic. The fact that the formula φ *is provable in* $\mathrm{L}\forall^{\mathrm{m}}$ *from a theory* T *will be denoted by* $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$*, and analogously for* $\mathrm{L}\forall$*; in a fixed context we can write just* $T \vdash \varphi$*.*

A general theory of first-order non-classical logics was first given by H. Rasiowa in [159]. Her first-order extension of a given Rasiowa-implicative logic (see Definition 3.3.1) was axiomatized analogously to $\mathrm{L}\forall^{\mathrm{m}}$, only the axioms $(\forall 2)$ and $(\exists 2)$ were replaced by the corresponding rules. It can be shown that in the context of ($\triangle$)-core fuzzy logic these two axiomatizations coincide (for the proof of an even more general formulation of this claim see Chapter II). The superscript 'm' stands for 'minimal', as $\mathrm{L}\forall^{\mathrm{m}}$ is, in a sense, the weakest first-order extension of L: as will be seen in Section 5.2, $\mathrm{L}\forall^{\mathrm{m}}$ is sound and complete w.r.t. first-order models built over *arbitrary* L-algebras.

However, the axioms of $\mathrm{L}\forall^{\mathrm{m}}$ are not strong enough to ensure the completeness w.r.t. first-order models over *linear* L-algebras—i.e., the linear completeness theorem, common to all fuzzy logics. This is why it is needed to add the axiom $(\forall 3)$, which is

Logic	$\exists$-defin.	$(\forall 3)$-elim.
G, Π, WNM (and weaker logics)	No	No
IMTL, NM	Yes	No
$G_\sim$, $SBL_\sim$	Yes	?
Ł (and stronger logics)	Yes	Yes

Table 6. $\exists$-definability and $(\forall 3)$-eliminability in prominent first-order fuzzy logics $L\forall$

valid in all models over linear L-algebras (though not generally in models over arbitrary L-algebras) and ensures the linear completeness theorem for the resulting logic $L\forall$.[66] This makes $L\forall$ the 'natural' first-order extension of a given $(\triangle)$-core *fuzzy* logic L. Consequently, this first-order logic is denoted as $L\forall$ with no superscript, though its more systematic denotation would be $L\forall^\ell$. Finally let us note that in the context of mathematical fuzzy logic, the logics $L\forall^{\mathrm{m}}$ were rediscovered by P. Hájek in [84], denoted there by $L\forall^-$.

The following two notions distinguish logics for which the axiomatic systems of $L\forall^{\mathrm{m}}$ and $L\forall$ can be simplified.

DEFINITION 5.1.3. *Let* L *be a* ($\triangle$-)*core fuzzy logic. We say that:*

- *The logic* $L\forall$ *has* $(\forall 3)$-eliminability *if axiom* $(\forall 3)$ *is redundant, i.e., if* $L\forall^{\mathrm{m}} = L\forall$.
- *The logic* $L\forall$ *has* $\exists$-definability *if there is a (definable) unary connective* $\sim$ *in the language of* L *such that* $\vdash_{L\forall} (\exists x)\varphi \leftrightarrow {\sim}(\forall x){\sim}\varphi$.

If a unary connective $\sim$ such that $\varphi \to \psi \vdash_{\mathrm{L}} {\sim}\psi \to {\sim}\varphi$ and $\vdash_{\mathrm{L}} \varphi \leftrightarrow {\sim}{\sim}\varphi$ (i.e., $\sim$ is an involutive negation) is definable in the language of a ($\triangle$-)core logic L, then $L\forall$ has $\exists$-definability. If $L\forall$ has $\exists$-definability, then the axioms $(\exists 1)$ and $(\exists 2)$ are redundant. Table 6 contains known facts on $\exists$-definability and $(\forall 3)$-eliminability in particular t-norm based logics.

Let us list some important theorems that are provable in all logics $L\forall^{\mathrm{m}}$. For their proofs in MTL or BL see [53, 83]; their proofs in a weaker setting can be found in Chapter II.

THEOREM 5.1.4. *Let* L *be a* ($\triangle$-)*core fuzzy logic and* $\mathcal{P}$ *a predicate language. Let* φ, ψ, χ *be* $\mathcal{P}$*-formulae,* x *a variable not free in* χ, *and* x' *a variable not occurring in* φ. *The following* $\mathcal{P}$*-formulae are then theorems of* $L\forall^{\mathrm{m}}$:

(T∀1) $\chi \leftrightarrow (\forall x)\chi$

(T∀2) $(\exists x)\chi \leftrightarrow \chi$

(T∀3) $(\forall x)\varphi(x) \leftrightarrow (\forall x')\varphi(x')$

(T∀4) $(\exists x)\varphi(x) \leftrightarrow (\exists x')\varphi(x')$

[66] The fact was first observed for Gödel logic by Horn in [115].

(T∀5) $(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$

(T∀6) $(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$

(T∀7) $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$

(T∀8) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$

(T∀9) $(\chi \to (\forall x)\varphi) \leftrightarrow (\forall x)(\chi \to \varphi)$

(T∀10) $((\exists x)\varphi \to \chi) \leftrightarrow (\forall x)(\varphi \to \chi)$

(T∀11) $(\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$

(T∀12) $(\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$

(T∀13) $(\forall x)\varphi \wedge (\forall x)\psi \leftrightarrow (\forall x)(\varphi \wedge \psi)$

(T∀14) $(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$

(T∀15) $(\forall x)\varphi \vee \chi \to (\forall x)(\varphi \vee \chi)$

(T∀16) $(\exists x)(\varphi \wedge \chi) \to (\exists x)\varphi \wedge \chi$

(T∀17) $(\exists x)(\varphi \mathbin{\&} \chi) \leftrightarrow (\exists x)\varphi \mathbin{\&} \chi$

(T∀18) $(\exists x)(\varphi^n) \leftrightarrow ((\exists x)\varphi)^n$

(T∀19) $(\exists x)\varphi \to \neg(\forall x)\neg\varphi$

(T∀20) $\neg(\exists x)\varphi \leftrightarrow (\forall x)\neg\varphi$.

The implication converse to (T∀11) *is provable in* $\text{Ł}\forall^{\text{m}}$ *and* $\Pi\forall$*, but not in* $\text{G}\forall$*. The implications converse to* (T∀12) *and* (T∀19) *are provable in* $\text{Ł}\forall^{\text{m}}$*, but neither in* $\text{G}\forall$ *nor* $\Pi\forall$*.*

Finally, the formula $(\exists x)\varphi \wedge \chi \to (\exists x)(\varphi \wedge \chi)$ *is a theorem of* $\text{L}\forall$ *for any* L*.*

Some syntactic metatheorems known from propositional fuzzy logics hold analogously for first-order fuzzy logics:[67]

THEOREM 5.1.5. *Let* $\mathcal{P}$ *be a predicate language.*

1. *Let* L *be a* $\triangle$*-core fuzzy logic,* χ *a* $\mathcal{P}$*-formula and* χ' *a* $\mathcal{P}$*-formula obtained from* χ *by replacing some occurrences* φ *by* ψ*. Then*

$$\varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \chi'.$$

(The intersubstitutivity *for* $\text{L}\forall^{\text{m}}$ *and* $\text{L}\forall$*)*

2. *Let* T *be a theory,* $\varphi(x)$ *a* $\mathcal{P}$*-formula, and* c *a constant not occurring in* $T \cup \{\varphi\}$*. Then* $T \vdash \varphi(c)$ *iff* $T \vdash \varphi(x)$*.* *(The* constants theorem *for* $\text{L}\forall^{\text{m}}$ *and* $\text{L}\forall$*)*

3. *Let* L *be a core fuzzy logic. Then for each theory* T *and sentences* φ, ψ*, the following holds:* $T, \varphi \vdash \psi$ *iff there is natural* n *such that* $T \vdash \varphi^n \to \psi$*.*
(The local deduction theorem *for* $\text{L}\forall^{\text{m}}$ *and* $\text{L}\forall$*)*

[67] For details see Chapter II or [98].

4. *Let* L *be a* $\triangle$*-core fuzzy logic. Then for each* $\mathcal{P}$*-theory* T *and* $\mathcal{P}$*-sentences* φ, ψ*, the following holds:* $T, \varphi \vdash \psi$ *iff* $T \vdash \triangle\varphi \to \psi$*.*
   (The $\triangle$-deduction theorem *for* $\mathrm{L}\forall^{\mathrm{m}}$ *and* $\mathrm{L}\forall$*)*

5. *Consequently, let* L *be (*$\triangle$*-)core fuzzy logic,* T *a* $\mathcal{P}$*-theory, and* φ, ψ, χ $\mathcal{P}$*-sentences. Then the following metatheorems hold for* $\mathrm{L}\forall$*:*

   (a) If $T, \varphi \vdash \chi$ *and* $T, \psi \vdash \chi$*, then* $T, \varphi \vee \psi \vdash \chi$*. (The* proof by cases *property)*

   (b) If $T, \varphi \to \psi \vdash \chi$ *and* $T, \psi \to \varphi \vdash \chi$*, then* $T \vdash \chi$*.*
   (The semilinearity *property)*

The following theorem demonstrates that in ($\triangle$-)core fuzzy logics one can conservatively introduce Skolem functions in a similar manner as in classical logic. (The definition of conservative extension in first-order fuzzy logics is analogous to the classical definition.)

THEOREM 5.1.6. *Let* L *be a core fuzzy logic,* $\mathcal{P}$ *a predicate language,* $\varphi(y, x_1, \ldots, x_n)$ *a* $\mathcal{P}$*-formula,* T *a* $\mathcal{P}$*-theory such that* $T \vdash_{\mathrm{L}\forall} (\exists y)\varphi(y, x_1, \ldots, x_n)$*, and* F_φ *a function symbol of arity* n *not present in* $\mathcal{P}$*. Then the* $(\mathcal{P} \cup \{F_\varphi\})$*-theory*

$$T' = T \cup \{\varphi(F_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)\}$$

is a conservative extension of T *over* $\mathrm{L}\forall$.

The analogous theorem holds, with one small modification, for $\triangle$-core fuzzy logics.

THEOREM 5.1.7. *Let* L *be a* $\triangle$*-core fuzzy logic,* $\mathcal{P}$ *a predicate language,* T *a* $\mathcal{P}$*-theory,* $\varphi(y, x_1, \ldots, x_n)$ *a* $\mathcal{P}$*-formula such that* $T \vdash_{\mathrm{L}\forall} (\exists y)\triangle\varphi(y, x_1, \ldots, x_n)$*, and* F_φ *a function symbol of arity* n *not present in* $\mathcal{P}$*. Then the* $(\mathcal{P} \cup \{F_\varphi\})$*-theory*

$$T' = T \cup \{\varphi(F_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)\}$$

is a conservative extension of T *over* $\mathrm{L}\forall$.

5.2 Semantics

In this subsection, we shall introduce the general and linear semantics of predicate fuzzy logics, corresponding to the axiomatic systems $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$, as well as the logics $\mathrm{L}\forall^{\mathrm{s}}$ and $\mathrm{L}\forall^{\mathrm{w}}$, corresponding to the (more restrictive) semantics of standard and witnessed predicate models.

5.2.1 Basic definitions and completeness theorems

Now we shall look at the semantics of first-order fuzzy logics. To simplify the formulation of upcoming definitions let us fix: a ($\triangle$-)core fuzzy logic L in a propositional language $\mathcal{L}$, a predicate language $\mathcal{P} = (\mathit{Pred}, \mathit{Func}, \mathit{Ar})$, and an L-algebra $\boldsymbol{B}$.

DEFINITION 5.2.1. *A* $\boldsymbol{B}$-structure $\mathbf{M}$ *for the predicate language* $\mathcal{P}$ *has the form:* $\mathbf{M} = \langle M, (P_{\mathbf{M}})_{P \in \mathit{Pred}}, (F_{\mathbf{M}})_{F \in \mathit{Func}} \rangle$*, where* M *is a non-empty domain; for each* n*-ary predicate symbol* $P \in \mathit{Pred}$*,* $P_{\mathbf{M}}$ *is an* n*-ary fuzzy relation on* M*, i.e., a function*

$M^n \to \boldsymbol{B}$ *(identified with an element of* $\boldsymbol{B}$ *if* $n = 0$*); for each* n*-ary function symbol* $F \in$ *Func,* $F_{\mathbf{M}}$ *is a function* $M^n \to M$ *(identified with an element of* M *if* $n = 0$*).*

Let $\mathbf{M}$ *be a* $\boldsymbol{B}$*-structure for* $\mathcal{P}$*. An* $\mathbf{M}$-evaluation *of the object variables is a mapping* v *which assigns an element from* M *to each object variable. Let* v *be an* $\mathbf{M}$*-evaluation,* x *a variable, and* $a \in M$*. Then by* $\mathrm{v}[x{\mapsto}a]$ *we denote the* $\mathbf{M}$*-evaluation such that* $\mathrm{v}[x{\mapsto}a](x) = a$ *and* $\mathrm{v}[x{\mapsto}a](y) = \mathrm{v}(y)$ *for each object variable* y *different from* x*.*

Let $\mathbf{M}$ *be a* $\boldsymbol{B}$*-structure for* $\mathcal{P}$ *and* v *an* $\mathbf{M}$*-evaluation. We define the* values *of terms and the* truth values *of formulae in* $\mathbf{M}$ *for an evaluation* v *recursively as follows:*

$$\begin{aligned}
\|x\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \mathrm{v}(x) \\
\|F(t_1,\dots,t_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \text{for } F \in \mathit{Func} \\
\|P(t_1,\dots,t_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \text{for } P \in \mathit{Pred} \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= c_{\boldsymbol{B}}(\|\varphi_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|\varphi_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \text{for } c \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \inf\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\} \\
\|(\exists x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \sup\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\}.
\end{aligned}$$

If the infimum or supremum does not exist, we take the truth value of the quantified formula as undefined. We say that the $\boldsymbol{B}$*-structure* $\mathbf{M}$ *is* safe *if* $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}$ *is defined for each* $\mathcal{P}$*-formula* φ *and each* $\mathbf{M}$*-evaluation* v*.*

We shall write:

- $\|\varphi(a_1,\dots,a_n)\|^{\langle \boldsymbol{B},\mathbf{M}\rangle}$ *for* $\|\varphi(x_1,\dots,x_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}$ *if* $\mathrm{v}(x_i) = a_i$ *for all* $i \le n$*.*
- $\langle \boldsymbol{B},\mathbf{M}\rangle \models \varphi[\mathrm{v}]$ *if* $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} = \overline{1}$*.*
- $\langle \boldsymbol{B},\mathbf{M}\rangle \models \varphi$ *if* $\langle \boldsymbol{B},\mathbf{M}\rangle \models \varphi[\mathrm{v}]$ *for each* $\mathbf{M}$*-evaluation* v*. When* $\boldsymbol{B}$ *is known from the context, we write* $\mathbf{M} \models \varphi$ *only.*
- $\boldsymbol{B} \models \varphi$ *if* $\langle \boldsymbol{B},\mathbf{M}\rangle \models \varphi$ *for each safe* $\boldsymbol{B}$*-structure* $\mathbf{M}$ *(we also say that* φ *is a* $\boldsymbol{B}$-tautology*).*

Let $\mathbf{M}$ *be a safe* $\boldsymbol{B}$*-structure for* $\mathcal{P}$ *and* T *a* $\mathcal{P}$*-theory. Then* $\mathbf{M}$ *is called a* $\boldsymbol{B}$-model of T *if* $\langle \boldsymbol{B},\mathbf{M}\rangle \models \varphi$ *for each* $\varphi \in T$*.*

Observe that models are safe structures (by definition). As obviously each safe $\boldsymbol{B}$-structure is a $\boldsymbol{B}$-model of the empty theory, we shall use the term *model* for both models and safe structures in the rest of the text. By a slight abuse of language we use the term model also for the pair $\langle \boldsymbol{B},\mathbf{M}\rangle$. We say that $\langle \boldsymbol{B},\mathbf{M}\rangle$ is an ℓ*-model* (of T) whenever $\boldsymbol{B}$ is linearly ordered. Thus by the phrase "for each (ℓ-)model $\langle \boldsymbol{B},\mathbf{M}\rangle$ (of T)" we mean "for each (linear) L-algebra $\boldsymbol{B}$ and each safe $\boldsymbol{B}$-model $\mathbf{M}$ (of T)". The syntax and semantics of first-order fuzzy logics are bound together by the following completeness theorems.[68]

[68] For the proofs of Theorems 5.2.2 and 5.2.3 see [98]. Instances of Theorem 5.2.3 for various ($\triangle$-)core fuzzy logics were originally proved separately (usually for countable predicate languages only): the proofs for prominent core fuzzy logics can be found in [53, 83].

THEOREM 5.2.2 (General strong completeness for $\mathrm{L}\forall^{\mathrm{m}}$). *Let* L *be a* ($\triangle$-)*core fuzzy logic,* $\mathcal{P}$ *a predicate language,* T *a* $\mathcal{P}$*-theory, and* φ *a* $\mathcal{P}$*-formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$.
- $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *of the theory* T.

THEOREM 5.2.3 (Linear strong completeness for $\mathrm{L}\forall$). *Let* L *be a* ($\triangle$-)*core fuzzy logic,* $\mathcal{P}$ *a predicate language,* T *a* $\mathcal{P}$*-theory, and* φ *a* $\mathcal{P}$*-formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each* ℓ*-model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *of the theory* T.

5.2.2 Standard semantics

Besides the general and linear semantics, also the intended (or "standard") $[0,1]$-semantics can be considered for first-order fuzzy logics.

DEFINITION 5.2.4. *We shall say that* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *is an* s-model *(of* T*) if* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *is a model (of* T*) and* $\boldsymbol{B}$ *is a standard* L*-algebra.*[69]

Let L *be a* ($\triangle$-)*core fuzzy logic. If the equivalence*

$$T \vdash \varphi \quad \textit{iff} \quad \langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi \textit{ for each } s\textit{-model } \langle \boldsymbol{B}, \mathbf{M}\rangle \textit{ of the theory } T$$

holds for:

- $T = \emptyset$*, we say that* $\mathrm{L}\forall$ *enjoys* (weak) standard completeness
- *All finite theories* T*, we say that* $\mathrm{L}\forall$ *enjoys* finite strong standard completeness
- *All theories* T*, we say that* $\mathrm{L}\forall$ *enjoys* strong standard completeness.

The next theorem summarizes known results on the standard completeness of prominent ($\triangle$-)core fuzzy logics. Their proofs are scattered in the literature; see [53, 83] for the proofs for the main core fuzzy logics, and the survey paper [35] for more information and detailed references.

THEOREM 5.2.5. *The logics* $\mathrm{L}\forall$ *are strong standard complete if* L *is any of the following logics:* MTL, SMTL, IMTL, WNM, NM, G *(all of them with or without* $\triangle$*), and* $\mathrm{G}_{\sim}$.

The logics $\mathrm{L}\forall$ *are not even weakly standard complete if* L *is any of the following logics:* BL, SBL, $\mathrm{Ł}$, Π, $\Pi\mathrm{MTL}$, WCMTL[70] *(all of them with or without* $\triangle$*),* $\mathrm{SBL}_{\sim}$*, and* $\mathrm{Ł}\Pi$.

[69] For the notion of standard L-algebra see Definition 1.1.19, Convention 2.0.2, and some exceptions to this convention in Section 2.

[70] There is a general result in [140] that any logic between WCMTL and Π is not standard complete.

If the logic L$\forall$ does not enjoy standard completeness, the *standard first-order logic* L$\forall^{\mathrm{s}}$ of s-models over L can be introduced. It can be shown, for instance, that the standard Łukasiewicz first-order logic Ł$\forall^{\mathrm{s}}$ is not finitary, but can be axiomatized by means of an infinitary rule (see [106]).

Let us mention Pavelka logic RPL in this context.[71] Clearly RPL is a core fuzzy logic, so its predicate version RPL$\forall$ is strongly complete w.r.t. all safe interpretations over all RPL-chains. Furthermore, the logic RPL$\forall$ extends Łukasiewicz predicate fuzzy logic Ł$\forall$ conservatively [102]. Now let us turn to its standard semantics: consider only models over the standard RPL-algebra (i.e., s-models). Then the definitions of provability degree and truth degree can be introduced in the same manner as in the propositional case: let T be a theory and φ a formula; the provability degree $|\varphi|_T$ is the supremum of all $r \in \mathrm{Q}$ such that T proves $\bar{r} \to \varphi$ (i.e., that φ is at least r-true); and the truth degree $\|\varphi\|_T$ is the infimum of the truth degrees of φ in all (standard) models of T. Then the Pavelka-style completeness holds true: let T be a theory over RPL$\forall$ and φ a formula; then $|\varphi|_T = \|\varphi\|_T$. Notice, however, that the Pavelka-style completeness of the logic RPL$\forall$ *does not* entail its standard completeness: we only know that a standard tautology φ has the provability degree 1, i.e., that for each $r < 1$ we can prove $\bar{r} \to \varphi$.

A usual method to prove the failure of standard completeness of a given logic is to show that the set of its standard tautologies is not recursively enumerable, and therefore it cannot coincide with the set of its theorems. Determining the position in the arithmetical hierarchy (see e.g. [162]) of prominent sets of formulae (such as the tautologies of a given logic) is an important field of study in mathematical logic. Here we just briefly mention a few results related to fuzzy logics: for a full treatment of the arithmetical complexity of fuzzy logics see Chapter XI; some important papers on the topic are [85, 89, 90, 138, 139].

First let us introduce some prominent sets of formulae given by a fuzzy logic L:

DEFINITION 5.2.6. *Let φ be a sentence of* L$\forall$. *We say that φ is*

- *A* general *(resp.,* standard*)* tautology *of* L$\forall$ *if* $\|\varphi\|^{\langle \boldsymbol{B},\mathbf{M}\rangle} = \bar{1}$ *for each ℓ- (resp., s-) model $\langle \boldsymbol{B},\mathbf{M}\rangle$.*
- Generally *(resp.,* standardly*)* satisfiable *in* L$\forall$ *if* $\|\varphi\|^{\langle \boldsymbol{B},\mathbf{M}\rangle} = \bar{1}$ *for some ℓ- (resp., s-) model $\langle \boldsymbol{B},\mathbf{M}\rangle$.*

The sets of general and standard tautologies and generally and standardly satisfiable sentences will be denoted, respectively, by genTAUT, stTAUT, genSAT, *and* stSAT.

For illustration, let us state the results for four predicate logics: BL$\forall$, Ł$\forall$, G$\forall$, and $\Pi\forall$. For each of them, the set of general tautologies is $\mathbf{\Sigma_1}$-complete (thus they are recursively axiomatizable, but undecidable) and the set of generally satisfiable formulae is $\mathbf{\Pi_1}$-complete. For the arithmetical complexity of their standard semantics see Table 7 (where "-c" stands for "-complete" and "NA" for "non-arithmetical"). It can be seen that as far as standard semantics is concerned, the four logics differ drastically with respect to the degree of undecidability.

[71] See Section 2.2.3 and [83, 154].

	G∀	Ł∀	Π∀	BL∀
stTAUT	$\mathbf{\Sigma_1}$-c	$\mathbf{\Pi_2}$-c	NA	NA
stSAT	$\mathbf{\Pi_1}$-c	$\mathbf{\Pi_1}$-c	NA	NA

Table 7. Arithmetical complexity of standard semantics

5.2.3 Witnessed models

Recall from Definition 5.2.1 that the truth degree of an existentially quantified formula is defined as the supremum of the truth degrees of its instances. The supremum may, though, be larger than the truth value of any instance. If, nevertheless, the supremum of the truth values *is* achieved by some instance (i.e., is the *maximum* of the truth degrees of instances), we call the quantified formula *witnessed* (and similarly for universally quantified formulae):

DEFINITION 5.2.7. *A formula $(\exists x)\varphi$ with free variables $y_1, \dots, y_n$ is* witnessed *in $\langle \boldsymbol{B}, \mathbf{M}\rangle$ if for each evaluation $a_1, \dots, a_n \in M$ of $y_1, \dots, y_n$ there is an element $b \in M$ such that*

$$\|(\exists x)\varphi(x, a_1, \dots, a_n)\|^{\langle \boldsymbol{B}, \mathbf{M}\rangle} = \|\varphi(b, a_1, \dots, a_n)\|^{\langle \boldsymbol{B}, \mathbf{M}\rangle},$$

and similarly for $(\forall x)\varphi$. We call a model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ witnessed *if each formula beginning with a quantifier is witnessed in $\langle \boldsymbol{B}, \mathbf{M}\rangle$.*

Let L be a core fuzzy logic. We define the logic $\mathrm{L}\forall^{\mathrm{w}}$ as the extension of L∀ by the axiom schemata:[72]

(C∀) $(\exists x)(\varphi(x) \to (\forall y)\varphi(y))$

(C∃) $(\exists x)((\exists y)\varphi(y) \to \varphi(x))$.

THEOREM 5.2.8 (Witnessed completeness). *Let* L∀ *be a core predicate fuzzy logic, $\mathcal{P}$ a predicate language, T a theory, and φ a formula. Then $T \vdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$ iff $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ for each witnessed ℓ-model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ of the theory T.*

Note that the only first-order logic $\mathrm{L}_*\forall$ of a continuous t-norm $*$ that proves (C∃) and (C∀) is Łukasiewicz logic, which is thus strongly complete with respect to witnessed models. For product logic, we have the following:

DEFINITION 5.2.9. *Call a formula $(\forall x)\varphi$ with free variables $y_1, \dots, y_n$* weakly witnessed *in $\langle \boldsymbol{B}, \mathbf{M}\rangle$ if for each evaluation $a_1, \dots, a_n \in M$ of $y_1, \dots, y_n$ such that*

$$\|(\forall x)\varphi(x, a_1, \dots, a_n)\|^{\langle \boldsymbol{B}, \mathbf{M}\rangle} \neq \overline{0}$$

there is an element $b \in M$ such that

$$\|(\forall x)\varphi(x, a_1, \dots, a_n)\|^{\langle \boldsymbol{B}, \mathbf{M}\rangle} = \|\varphi(b, a_1, \dots, a_n)\|^{\langle \boldsymbol{B}, \mathbf{M}\rangle}.$$

[72] References for witnessed semantics of fuzzy logic are [94–96, 98].

Call a model $\langle \boldsymbol{B}, \mathbf{M} \rangle$ weakly witnessed *if each formula beginning with an existential quantifier is witnessed and each formula beginning with a universal quantifier is weakly witnessed in* $\langle \boldsymbol{B}, \mathbf{M} \rangle$.

The analog of Theorem 5.2.8 can be proved for $\Pi\forall$ with respect to weakly witnessed models [127].

5.3 The identity and sorts of objects

Further possible extensions of first-order fuzzy logics regard the identity (or equality) and sorts of objects.

5.3.1 The identity predicate

The identity (or equality) of objects can be included in the logical vocabulary of any first-order fuzzy logic as a new primitive binary predicate $=$, with the intended semantics of the crisp identity on the domain of a first-order model $\mathbf{M}$; i.e., $\|x = y\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} = \overline{1}$ iff $\mathrm{v}(x) = \mathrm{v}(y)$ and $\|x = y\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} = \overline{0}$ iff $\mathrm{v}(x) \neq \mathrm{v}(y)$. This intended behavior of $=$, however, clearly cannot be ensured by any set of first-order axioms.[73] In any extension of first-order MTL, the following axioms approximate the intended behavior of $=$ by allowing only those non-identical objects that are logically indistinguishable in the given predicate language (i.e., that share all properties φ expressible in the predicate language in all models to the same degrees) to be fully identical:

$(\mathrm{Refl}_=)$	$x = x$
(LP)	$x = y \to (\varphi(x) \to \varphi(y))$
$(\mathrm{Crisp}_=)$	$x = y \vee \neg(x = y)$,

for any φ (which can contain other free variables besides x) and y substitutable for x in $\varphi(x)$. The axiom $(\mathrm{Refl}_=)$ expresses the fact that any object is identical with itself, and the axiom schema (LP) expresses one direction of the Leibniz principle of identity, namely the fact that identical objects have the same properties. The axiom $(\mathrm{Crisp}_=)$ of the crispness of $=$ is redundant in the extensions of $\mathrm{MTL}_\triangle\forall^{\mathrm{m}}$ or $\text{Ł}\forall^{\mathrm{m}}$, where it is provable from the schema (LP). For any ($\triangle$-)core fuzzy logic L the first-order fuzzy logic $\mathrm{L}\forall$ extended by the axioms $(\mathrm{Refl}_=)$, (LP), and $(\mathrm{Crisp}_=)$ will be called "$\mathrm{L}\forall$ with identity" or $\mathrm{L}\forall_=$ (and similarly for the logics $\mathrm{L}\forall^{\mathrm{m}}_=$, $\mathrm{L}\forall^{\mathrm{s}}_=$, and $\mathrm{L}\forall^{\mathrm{w}}_=$).

Even though in $\mathrm{L}\forall^{\cdots}_=$, the predicate $=$ need not be interpreted as the crisp identity, it is always a crisp equivalence relation, as $\mathrm{L}\forall^{\mathrm{m}}_=$ proves its symmetry and transitivity. Since in any model $\mathbf{M}$ of $\mathrm{L}\forall^{\cdots}_=$ all formulae φ are by (LP) congruent with $=$, factorizing $\mathbf{M}$ by this crisp equivalence relation yields a model $\mathbf{M}'$ that satisfies the same formulae as $\mathbf{M}$ to the same degrees and in which $=$ has the intended semantics (i.e., is interpreted as the crisp identity of objects).

The identity predicate enables, i.a., elimination of function symbols (including the Skolem functions, see Theorems 5.1.6 and 5.1.7) and (crisp) sorts of variables (intro-

[73] Like in classical first-order logic, this is because for each model $\mathbf{M}$ in which $=$ is interpreted as the crisp identity of objects we can easily construct a model $\mathbf{M}'$ satisfying the same formulae, in which $=$ does not coincide with the identity of objects. (Just replicate any element a of $\mathbf{M}$ and define all predicates and functors identically on the new copies as they were on a.)

duced later in this section) in ($\triangle$-)core fuzzy logics in the same way as in classical logic. For fuzzy logics with identity see [37].

5.3.2 Sorts of objects

To enable a smooth work in axiomatic theories over first-order fuzzy logics, similar to the practice of classical mathematics, it is expedient to introduce sorts of objects in the predicate language. We shall only consider crisp sorts of objects, as quantification restricted to fuzzy domains is rather problematic in contraction-free fuzzy logics. On the other hand, we shall allow for subsumption of sorts, formalizing the situation that all objects of one sort are also objects of another sort (e.g., the fact that all natural numbers are also rational numbers, which in turn are also real numbers). This kind of sorts for predicate fuzzy logics was introduced in [12].

Let us fix some weakly implicative fuzzy logic L. We shall present a multi-sorted variant of the logic L$\forall$, but mutatis mutandis the definitions apply to L$\forall^{\mathrm{m}}$, L$\forall^{\mathrm{s}}$, L$\forall^{\mathrm{w}}$, or any combination thereof. We shall only give the differences from the single-sorted case, not repeating the definitions that are the same for multi- and single-sorted L$\forall$.

DEFINITION 5.3.1. *A* multi-sorted predicate language *is a quintuple*

$$\mathcal{P} = \langle Sort, \preceq, Pred, Func, Ar\rangle,$$

where $Sort$ is a non-empty set of sorts, *$\preceq$ is an ordering on $Sort$ (indicating the subsumption of sorts), $Pred$ is a non-empty set of predicate symbols, $Func$ is a set (disjoint with $Pred$) of function symbols, and Ar is the arity function that assigns to each predicate symbol a finite sequence of elements of $Sort$ and to each function symbol a non-empty finite sequence of elements of $Sort$.*

The length $|Ar(P)|$ of the sequence $Ar(P)$ is called the arity of the predicate symbol *P and the number $|Ar(F)| - 1$ is called the* arity of the function symbol *F. If $Ar(F) = \langle s\rangle$, then the function F is called an* individual constant of sort *s. If $s_1 \preceq s_2$, we say that the sort s_2* subsumes *the sort s_1.*

For each $s \in Sort$ there is an infinite countable set of individual variables of sort s, denoted by x^s, y^s, … The notion of $\mathcal{P}$-term *is defined recursively as follows:*

- *All individual variables of each sort $s \in Sort$ are $\mathcal{P}$-terms of sort s.*
- *If $t_1, \dots, t_n$ are $\mathcal{P}$-terms of the respective sorts $s_1, \dots, s_n \in Sort$ and F is a function symbol such that $Ar(F) = \langle s'_1, \dots, s'_n, s_{n+1}\rangle$, where $s_i \preceq s'_i$ for all $i \le n$, then $F(t_1, \dots, t_n)$ is a $\mathcal{P}$-term of sort s_{n+1}.*
- *Only the expressions that arise by a finite iteration of the previous two rules are $\mathcal{P}$-terms.*

If $t_1, \dots, t_n$ are $\mathcal{P}$-terms of respective sorts $s_1, \dots, s_n$ and P a predicate symbol such that $Ar(P) = \langle s'_1, \dots, s'_n\rangle$, where $s_i \preceq s'_i$ for all $i \le n$, then $P(t_1, \dots, t_n)$ is an atomic $\mathcal{P}$-formula.

The notions of $\mathcal{P}$-formula, $\mathcal{P}$-theory, bound and free variable, and $\mathcal{P}$-sentence are defined analogously to the single-sorted case. A term t of sort s is substitutable *for the variable $x^{\bar{s}}$ in a formula $\varphi(x^{\bar{s}}, \dots)$ if $s \preceq \bar{s}$ and no variable occurring in t is bounded in $\varphi(t, \dots)$.*

In multi-sorted first-order logics with the identity predicate we furthermore assume that for each sort $s \in Sort$ there is a predicate symbol $=_s \in Pred$ with $Ar(=_s) = \langle s, s\rangle$. (The predicate $=_s$ is usually denoted just by $=$, as s is determined by the sort of its arguments.)

Let us fix a multi-sorted predicate language $\mathcal{P} = \langle Sort, \preceq, Pred, Func, Ar\rangle$. The semantics of multi-sorted first-order fuzzy logic for this language is defined as follows:

DEFINITION 5.3.2. *A* $\boldsymbol{B}$-structure $\mathbf{M}$ *for the predicate language* $\mathcal{P}$ *has the form* $\mathbf{M} = \langle (M_s)_{s\in Sort}, (P_{\mathbf{M}})_{P\in Pred}, (F_{\mathbf{M}})_{F\in Func}\rangle$, *where:*

- *For each $s \in Sort$, M_s is a non-empty set and $M_s \subseteq M_{s'}$ iff $s \preceq s'$;*
- *For each predicate symbol $P \in Pred$ such that $Ar(P) = \langle s_1, \ldots, s_n\rangle$, $P_{\mathbf{M}}$ is a function $\left(\prod_{i=1}^n M_{s_i}\right) \to \boldsymbol{B}$ (identified with an element of $\boldsymbol{B}$ if $n = 0$); and*
- *For each function symbol $F \in Func$ such that $Ar(F) = \langle s_1, \ldots, s_n, s_{n+1}\rangle$, $F_{\mathbf{M}}$ is a function $\left(\prod_{i=1}^n M_{s_i}\right) \to M_{s_{n+1}}$ (identified with an element of M_{s_1} if $n = 0$).*
- *In multi-sorted first-order logics with identity it is furthermore assumed that if $s \preceq s'$, then the realizations of $=_s$ and $=_{s'}$ coincide on the domain of s.*

An $\mathbf{M}$-evaluation *of the object variables in a $\boldsymbol{B}$-structure $\mathbf{M}$ is a mapping that assigns to each variable of sort s an element from M_s. The definitions of the value of a term, the truth value of a formula, a safe structure, a $\boldsymbol{B}$-tautology, and a $\boldsymbol{B}$-model of a theory are the same as in the single-sorted case.*

Notice that the domains of the sorts of variables are assumed to be *crisp* sets. This requirement is analogous to single-sorted models, whose domains are also assumed to be crisp.[74] Due to this analogy, most metamathematical properties of single-sorted first-order fuzzy logics directly translate for multi-sorted languages as well. In particular, the axioms, and consequently the provable formulae, of first-order fuzzy logics for multi-sorted languages are the same as in single-sorted first-order fuzzy logics (see Definition 5.1.2), only with the appropriate restrictions on the sorts of terms and variables. Also many metatheorems on first-order fuzzy logics (such as the completeness Theorems 5.2.2 and 5.2.3) can be straightforwardly generalized for multi-sorted languages. Sorts of variables can also be conservatively introduced (by crisp formulae) and eliminated (in logics with $=$) in first-order fuzzy logics in the manner fully similar to that in classical first-order logic (cf. [97]).

5.4 Axiomatic theories over fuzzy logic

First-order fuzzy logics are strong enough to support non-trivial axiomatic mathematical theories. We shall briefly survey the development of some kinds of arithmetic and set theory over fuzzy logic. The reader is assumed to know basic facts about classical Peano arithmetic PA and Zermelo–Fraenkel set theory ZF.

[74] One reason for this design choice is the fact that quantification over a fuzzy domain has rather problematic properties (unless the membership degrees of elements in the domain are idempotent w.r.t. conjunction).

5.4.1 Arithmetic over fuzzy logics

Clearly, over classical logic one cannot extend PA by a truth predicate Tr and the dequotation axiom schema $\varphi \leftrightarrow \mathrm{Tr}(\overline{\varphi})$, where $\overline{\varphi}$ is a name (the Gödel number) of the sentence φ. This is due to the Russell paradox: using the Gödel diagonalization method we could construct a formula φ such that $\varphi \leftrightarrow \neg\mathrm{Tr}(\overline{\varphi})$, hence satisfying $\mathrm{Tr}(\overline{\varphi}) \leftrightarrow \neg\mathrm{Tr}(\overline{\varphi})$, which is contradictory over classical logic: a sentence cannot be equivalent to its own negation. But it need not be contradictory over Łukasiewicz logic: the sentence may have the truth value one half.

An arithmetical theory with a truth predicate and the dequotation schema has indeed been introduced and shown to be consistent over Łukasiewicz logic.[75] The theory PATr over Ł$\forall$ (or RPL$\forall$) contains the predicate of equality and the functional predicates of successor, addition, and multiplication, all of them postulated crisp; a (fuzzy) unary predicate Tr; the axioms of Robinson's arithmetic; the schema of induction for all formulae (including those containing the predicate Tr) formulated as a deduction rule; and the dequotation schema as above. Roughly speaking, PATr is thus an extension of crisp PA by a fuzzy truth predicate Tr. The following are the main results on the theory:

THEOREM 5.4.1 ([101, 160, 181]).

1. *The theory* PATr *is consistent: there is a crisp (classical) model of* PA *expandable to a* $[0,1]_Ł$*-model of* PATr.
2. *The standard crisp model of* PA *(the structure of natural numbers) cannot be expanded to a model of* PATr*—the theory is* ω*-inconsistent.*
3. *The extension of* PATr *with the axioms saying that (arithmetized)* Tr *commutes with connectives is inconsistent.*

Also a weak variant FQ^- of Robinson's arithmetic Q was studied over first-order fuzzy logic [81, 93]. In FQ^-, the ternary predicates of addition and multiplication are not assumed to be total functions. The underlying logic is fuzzy (BL$\forall$ or similar). A variant of Gödel's first incompleteness theorem can be proved even for this weak theory: each axiomatizable consistent extension of FQ^- is incomplete; also the essential undecidability of FQ^- can be proved.

5.4.2 Axiomatic fuzzy set theory

Now let us turn to set theory. First, there is a fuzzy set theory in the style of Zermelo–Fraenkel set theory. Recall that the universe of the classical Zermelo–Fraenkel set theory ZF is the union of the hierarchy of sets constructed from the empty set by the transfinitely iterated operation of power set. We shall present here an axiomatic system FST of ZF-style fuzzy set theory over the logic $\mathrm{BL}_\triangle\forall$ due to Hájek and Haniková introduced in [100].[76]

[75] By Hájek, Paris, and Shepherdson in [101]; for alternative proofs of Theorem 5.4.1(2) see [160] and [181].

[76] Hájek and Haniková's work follows previous versions of axiomatic fuzzy set theory over a variant and an expansion of Gödel logic by Takeuti and Titani [170, 171], which in turn follows the works on intuitionistic ZF-style set theory [79, 156]. ZF-style universes of fuzzy sets over Łukasiewicz logic had been semantically studied by Klaua and Gottwald [75, 78, 121, 122]. A comprehensive survey of various approaches to axiomatic fuzzy set theory is Gottwald's [77].

DEFINITION 5.4.2. FST *is a first-order theory over the logic* $\mathrm{BL}_{\triangle}\forall$ *with (crisp) equality* $=$ *and the binary predicate* $\in$*. Below are listed the axioms of the theory* FST*, for any formula* φ *in which* z *is not free and any formula* ψ *(both* φ *and* ψ *can contain additional free variables besides those explicitly written in the axioms). The formulation of the axioms employs the predicates* Crisp *(crispness of sets) and* $\subseteq$ *(fuzzy subsethood), defined as* $\mathrm{Crisp}(x) \equiv (\forall q)\triangle(q \in x \vee q \notin x)$*, where* $q \notin x$ *abbreviates* $\neg(q \in x)$*, and* $x \subseteq y \equiv (\forall q)(q \in x \to q \in y)$*, and the operations of union* $\cup$*, singleton* $\{x\}$*, and the empty set* $\emptyset$*, which can be introduced as the Skolem functions (see Theorem 5.1.7) of the corresponding axioms.*

(extensionality)	$x = y \leftrightarrow (\triangle(x \subseteq y) \mathbin{\&} \triangle(y \subseteq x))$
(empty set)	$(\exists x)\triangle(\forall y)\neg(y \in x)$
(pair)	$(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$
(union)	$(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (\exists z)(u \in y \mathbin{\&} y \in x))$
(weak power)	$(\exists z)\triangle(\forall u)(u \in z \leftrightarrow \triangle(u \subseteq x))$
(infinity)	$(\exists z)\triangle(\emptyset \in z \mathbin{\&} (\forall x)(x \in z \to (x \cup \{x\} \in z)))$
(separation)	$(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u \in x \mathbin{\&} \varphi(u,x)))$
(collection)	$(\exists z)\triangle[(\forall u)(u \in x \to (\exists v)\varphi(u,v)) \to$
	$(\forall u)(u \in x \to (\exists v)(v \in z \mathbin{\&} \varphi(u,v)))]$
($\in$-induction)	$\triangle((\forall y)(y \in x \to \psi(y)) \to \varphi(x)) \to \triangle(\forall x)\psi(x)$
(support)	$(\exists z)(\mathrm{Crisp}(z) \mathbin{\&} \triangle(x \subseteq z))$.

In FST, one can define the predicate HCrisp of hereditary crispness in a straightforward way. Let HC be the class of all hereditarily crisp sets. It can be shown that FST proves all axioms of ZF with all quantifiers restricted to HC. The class HC is thus an *inner model* of ZF (over classical logic) in FST (over $\mathrm{BL}_{\triangle}\forall$). In classical ZFC, each completely ordered BL-chain determines a natural model of FST, containing with each fuzzy set all its fuzzy subsets. For more details on FST see [100].

In fuzzy logic it is moreover possible to develop set theory of a different style, namely a fuzzy version of Cantor's naïve set theory with unrestricted comprehension. Recall that in classical logic, unrestricted comprehension is contradictory by Russell's paradox: if $x = \{y \mid y \notin y\}$ then $x \in x \leftrightarrow x \notin x$. However, it is known that for the derivation of a contradiction from the latter formula in classical or intuitionistic logic, the rule of contraction (see Section 3.1) is necessary: in some contraction-free logics (namely, BCK and certain variants of linear logic), the unrestricted comprehension schema is known to be consistent [80, 155, 172]. It has been conjectured by Skolem [168, 169] that the full comprehension schema is consistent over Łukasiewicz logic. (Notice that the formula $x \in x \leftrightarrow x \notin x$ can be satisfied in $[0,1]_{\text{Ł}}$ if the truth value of $x \in x$ is one half.) Partial consistency results (with comprehension restricted to formulae of limited quantifier complexity) are due to Skolem, Chang, and Fenstad [25, 60, 169].

A set theory over Ł∀ with full comprehension (called *Cantor–Łukasiewicz set theory* CŁ)[77] can be defined as follows: the language contains a single binary predicate $\in$ and the comprehension terms $\{x \mid \varphi(x)\}$ for each formula φ (possibly containing other free variables besides x); the only extralogical axioms are the instances of the comprehension schema

$$u \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(u)$$

for each formula φ. If consistent (see below), the theory shows markedly different behavior from ZF-style fuzzy set theories. For instance, the assumption that the extensional and intensional (or Leibniz) equalities

$$x =_{\mathrm{e}} y \equiv (\forall z)(z \in x \leftrightarrow z \in y)$$
$$x = y \equiv (\forall z)(x \in z \leftrightarrow y \in z)$$

coincide is contradictory in CŁ. Various results (esp. on natural numbers, some of them negative) have been proved in CŁ [92, 182]. However, no natural classical model of CŁ has been found and the consistency status of the theory is (as of 2011) unknown.[78] Nevertheless, even if CŁ turns out to be contradictory, the full comprehension schema has been proved to be consistent at least over IMTL (Terui 2011, pers. comm.), in which many theorems of CŁ still hold and which shows similar metamathematical features as CŁ.

5.5 Higher-order fuzzy logic

Two systems of higher-order extension of first-order fuzzy logic have been developed: fuzzy Church-style simple type theory FTT and Henkin-style higher-order fuzzy logic, also known as Fuzzy Class Theory FCT.

5.5.1 Fuzzy type theory

Fuzzy type theory FTT was originally (in [146]) defined over the logic $\mathrm{IMTL}_\triangle$; later it has been generalized to several other logics, including $\text{Ł}_\triangle$, $\mathrm{BL}_\triangle$, and ŁΠ [147]. Being a Church-style type theory, it is not formulated in the language of first-order fuzzy logic, but in the typed λ-calculus (for a standard reference see [9]). We shall therefore provide only a brief sketch of the theory here, and refer the interested reader to [146] for the full description.

The system of types in FTT is defined over two primitive types, ε (for elements) and o (for truth degrees), in the usual way (for each types α, β there is a complex type $(\beta\alpha)$ for crisp functions from the domain of α to the domain of β). The language of FTT contains an infinite countable set of variables x_α for each type α and the special constants $\mathrm{C}_{(oo)o}$, D_{oo}, and $\mathrm{E}_{(o\alpha)\alpha}$ for each type α. Formulae (i.e., λ-terms) of type α are defined as usual in typed λ-calculus.

[77] In [92] and subsequent papers, the theory was denoted by CŁ_0, while CŁ denoted a contradictory extension of CŁ_0. A corresponding theory over standard first-order Łukasiewicz logic $\text{Ł}\forall^{\mathrm{s}}$ is denoted by H in [178, 182].

[78] In 2010, Terui (pers. comm.) found what appears to be a serious gap in White's [178] proof of the relative consistency w.r.t. ZFC.

The semantics of FTT interprets the type ε as a (non-empty) set M_ε (of atomic objects), the type o as a complete linearly ordered IMTL$_\triangle$-algebra M_o, and each complex type $\beta\alpha$ as a subset of the set $M_\beta^{M_\alpha}$ of all (crisp) functions from M_α to M_β. The constants $\mathrm{C}_{(oo)o}$, D_{oo}, and $\mathrm{E}_{(oo)o}$ are interpreted, respectively, as $\wedge$, $\triangle$, and $\leftrightarrow$ in the IMTL$_\triangle$-algebra M_o, the constant $\mathrm{E}_{(o\varepsilon)\varepsilon}$ as a (primitive) fuzzy equality[79] on M_ε, and the constants $\mathrm{E}_{(o(\beta\alpha))(\beta\alpha)}$ as fuzzy equalities $E_{\beta\alpha}$ defined recursively as $E_{\beta\alpha}(f,g) = \bigwedge_{x\in M_\alpha} E_\beta(f(x),g(x))$ for each $f,g \in M_{\beta\alpha}$. The functions $f \in M_{\beta\alpha}$ are furthermore required to be *weakly extensional* w.r.t. E_α and E_β, i.e., if $E_\alpha(x,y) = 1$ then $E_\beta(f(x),f(y)) = 1$ for all $x,y \in M_\alpha$. The interpretation of λ-terms is defined as usual in typed λ-calculus. A general model is an interpretation such that each formula is assigned a value (which is a non-trivial condition since $M_{\beta\alpha}$ can be a proper subset of $M_\beta^{M_\alpha}$).

The axiomatic system of FTT is given by 16 formulae of type o and two inference rules (of intersubstitutivity of equivalent formulae and $\triangle$-necessitation). The notions of provability, theory, and model are then defined analogously as in first-order (fuzzy) logic. FTT extends the first-order logic IMTL$_\triangle$, is non-trivial (i.e., has non-bivalent models), and is sound w.r.t. general models sketched above. It enjoys the $\triangle$-deduction theorem and the following form of completeness: a theory over FTT is consistent iff it has a general model.

FTT can serve as a background framework for more specialized axiomatic theories (formulated as sets of FTT-formulae of type o). It has mostly been used for modeling linguistic phenomena connected with the agenda of fuzzy logic in broader sense, such as generalized quantifiers [149], evaluative linguistic expressions [148]. This endeavor includes the development of necessary fuzzy-mathematical background—e.g., fuzzy measures and integrals [49].

5.5.2 Fuzzy Class Theory

Fuzzy Class Theory FCT was originally introduced in [12] over the logics ŁΠ and ŁΠ$\frac{1}{2}$, but can in fact be defined over any extension of MTL$_\triangle$ (see, e.g., [11]). It can be characterized as Henkin-style higher-order fuzzy logic, or Russell-style simple fuzzy type theory.

Let L be a $\triangle$-core fuzzy logic. Fuzzy Class Theory (FCT) over L, or Henkin-style higher-order fuzzy logic L, is a theory in multi-sorted first-order fuzzy logic L with identity. The language of FCT has sorts of variables for:

- *Atomic objects* (lowercase letters $x, y, \dots$)
- *Fuzzy classes* of atomic objects (uppercase letters $A, B, \dots$)
- *Fuzzy classes of fuzzy classes* of atomic objects (calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$)
- Etc., in general *fuzzy classes of the n-th order* ($X^{(n)}, Y^{(n)}, \dots$).

[79] I.e., a function $E\colon M_\varepsilon^2 \to M_o$ such that (i) $Exx = 1$, (ii) $Exy = Eyx$, (iii) $Exy * Eyz \leq Exz$, and (iv) $Exy = 1$ iff $x = y$, for all $x, y, z \in M_o$. Such functions are also known as *separated* (or *unimodal*) fuzzy *similarities*.

Each of these sorts subsumes sorts for tuples of all arities $k \in \mathsf{N}$. Variables of order n and arity k will be denoted by $X^{(n,k)}$, $Y^{(n,k)}$, ...

Besides the (crisp) identity predicate $=$ on each sort, the language of FCT contains:

- The *membership predicate* $\in$ between objects of sorts $^{(n)}$ and $^{(n+1,1)}$
- *Class terms* $\{x \mid \varphi\}$ of sort $^{(n+1,1)}$, for any variable x of sort $^{(n)}$ and any formula φ
- Terms $\langle x_1, \dots, x_k \rangle$ of sort $^{(n,k)}$ for *k-tuples* of individuals $x_1, \dots, x_k$ of sort $^{(n,1)}$.

FCT has the following axioms over multi-sorted first-order logic L, for all formulae $\varphi(x, \dots)$, variables of all compatible sorts, and all $k \in \mathsf{N}$:

- The *tuple-identity* axioms: $\langle x_1, \dots, x_k \rangle = \langle y_1, \dots, y_k \rangle \to x_1 = y_1 \ \& \ \dots \ \& \ x_k = y_k$
- The *comprehension* axioms: $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$
- The *extensionality* axioms: $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$.

The axioms of extensionality express the stipulation that fuzzy sets (of any order) are determined by their membership functions. The axioms of comprehension ensure that each property φ definable in the language of the theory delimits a fuzzy set $\{x \mid \varphi\}$ of the appropriate order.[80]

The *intended models* of FCT are determined by a (linear) L-algebra $\boldsymbol{L}$ and a crisp universe of discourse U. The variables of sort $^{(n,k)}$ range over the set $U_{n,k} = U_{n,1}^k$ and the universal variables of sort $^{(n)}$ over $U_n = \bigcup_{k \in \mathsf{N}} U_{n,k}$, where $U_{0,1} = U$ and $U_{n+1,1} = \boldsymbol{L}^{U_n}$ for each $n \in \mathsf{N}$. For $x \in U_n$ and $A \in U_{n+1,1}$, the value of membership predicate $x \in A$ is defined simply as the functional value $A(x)$; the tuples and identity are interpreted in the obvious way. An intended model of FCT thus consists of fuzzy sets of all finite orders and arities, represented by $\boldsymbol{L}$-valued membership functions. FCT is sound w.r.t. its intended models, and can thus be viewed as an (incomplete, but sufficiently strong) axiomatization of Zadeh's notion of fuzzy set (of all orders and arities).[81]

Since the theory of intended models of FCT is not recursively axiomatizable (for the same reasons as in classical higher-order fuzzy logic, namely the definability of true arithmetic), FCT is only complete w.r.t. its *general models,* in which $U_{n+1,1}$ can be a proper subset of $\boldsymbol{L}^{U_n}$ (closed under all definable operations) and U_n a proper superset of $\bigcup_{k \in \mathsf{N}} U_{n,k}$. Thus, not all fuzzy subsets of U_n (i.e., functions $U_n \to \boldsymbol{L}$) may be represented as a fuzzy class (i.e., an element of $U_{n+1,1}$) in a general model of FCT;[82] nevertheless, the axioms of comprehension ensure that at least all fuzzy subsets of U_n definable in the language of the theory are represented by elements of $U_{n+1,1}$.

[80] Alternatively, FCT can be formulated with the comprehension axioms $(\exists A)\triangle(x \in A \leftrightarrow \varphi(x))$. The (eliminable) comprehension terms then arise as Skolem functions (see Section 5.3) of these axioms.

[81] Intended models over standard L-algebras are therefore called *Zadeh models* in [12] and subsequent papers.

[82] It can be observed that if the underlying logic L is weaker than classical logic, then FCT also has crisp models, isomorphic to general models of classical Henkin-style higher-order logic. If necessary, an additional axiom such as $\neg(\forall A)(x \in A \vee \neg(x \in A))$ can be adopted to ensure the fuzziness of all its models.

The axiomatization of fuzzy sets in FCT turns out to be strong enough to formalize all usual notions of fuzzy set theory and other disciplines of fuzzy mathematics. By means of $\triangle$, all notions definable in classical higher-order logic can in fact be represented in FCT extended by appropriate definitions [12, 39]; all usual structures of classical mathematics (e.g., natural or real numbers) are thus available in the framework of FCT. Since moreover the language of FCT is similar to that of classical mathematics, FCT was proposed (in [13]) for a foundational theory for *logic-based fuzzy mathematics*. Several disciplines of fuzzy mathematics have thereafter been developed in the foundational framework of FCT, including elementary theory of fuzzy sets [12], fuzzy relations [11, 15], fuzzy topology [16, 125], fuzzy interval arithmetic [111], etc.

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BIBLIOGRAPHY

[1] Romà J. Adillon, Àngel García-Cerdaña, and Ventura Verdú. On three implication-less fragments of t-norm based fuzzy logics. *Fuzzy Sets and Systems*, 158(23):2575–2590, 2007.

[2] Paolo Aglianò, Isabel Maria André Ferreirim, and Franco Montagna. Basic hoops: An algebraic study of continuous t-norms. *Studia Logica*, 87(1):73–98, 2007.

[3] Arnon Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52(4):939–951, 1987.

[4] Arnon Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4(3–4):225–248, 1991.

[5] Arnon Avron and Iddo Lev. Non-deterministic multiple-valued structures. *Journal of Logic and Computation*, 15(3):241–261, 2005.

[6] Matthias Baaz, Agata Ciabattoni, and Franco Montagna. Analytic calculi for monoidal t-norm based logic. *Fundamenta Informaticae*, 59(4):315–332, 2004.

[7] Matthias Baaz, Petr Hájek, Jan Krajíček, and David Švejda. Embedding logics into product logic. *Studia Logica*, 61(1):35–47, 1998.

[8] Matthias Baaz, Petr Hájek, Franco Montagna, and Helmut Veith. Complexity of t-tautologies. *Annals of Pure and Applied Logic*, 113(1–3):3–11, 2002.

[9] Henk P. Barendregt. *The Lambda Calculus, Its Syntax and Semantics*, volume 103 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1984.

[10] Libor Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159(10):1153–1164, 2008.

[11] Libor Běhounek, Ulrich Bodenhofer, and Petr Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159(14):1729–1772, 2008.

[12] Libor Běhounek and Petr Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[13] Libor Běhounek and Petr Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[14] Libor Běhounek and Petr Cintula. Fuzzy logics as the logics of chains. *Fuzzy Sets and Systems*, 157(5):604–610, 2006.

[15] Libor Běhounek and Martina Daňková. Relational compositions in Fuzzy Class Theory. *Fuzzy Sets and Systems*, 160(8):1005–1036, 2009.

[16] Libor Běhounek and Tomáš Kroupa. Topology in Fuzzy Class Theory: Basic notions. In Patricia Melin, Oscar Castillo, Lluis T. Aguilar, Janusz Kacprzyk, and Witold Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 513–522. Springer, Berlin, Heidelberg, 2007.

[17] Willem J. Blok and Isabel Maria André Ferreirim. On the structure of hoops. *Algebra Universalis*, 43(2–3):233–257, 2000.

[18] Willem J. Blok and Don L. Pigozzi. *Algebraizable Logics*, volume 396 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, RI, 1989. Freely downloadable from `http://orion.math.iastate.edu/dpigozzi/`.

[19] Willem J. Blok and Clint J. van Alten. The finite embeddability property for residuated lattices, pocrims and BCK-algebras. *Algebra Universalis*, 48(3):253–271, 2002.

[20] Félix Bou, Francesc Esteva, Josep Maria Font, Àngel Gil, Lluís Godo, Antoni Torrens, and Ventura Verdú. Logics preserving degrees of truth from varieties of residuated lattices. *Journal of Logic and Computation*, 19(6):1031–1069, 2009.

[21] J.R. Büchi and T.M. Owens. Complemented monoids and hoops. Unpublished manuscript.

[22] Sam Buss, editor. *Handbook of Proof Theory*. Kluwer, 1998.

[23] Ettore Casari. Comparative logics and Abelian ℓ-groups. In *Logic Colloquium '88 (Padova, 1988)*, volume 127 of *Studies in Logic and the Foundations of Mathematics*, pages 161–190. North-Holland, Amsterdam, 1989.

[24] Chen Chung Chang. A new proof of the completeness of the Łukasiewicz axioms. *Transactions of the American Mathematical Society*, 93(1):74–80, 1959.

[25] Chen Chung Chang. The axiom of comprehension in infinite valued logic. *Mathematica Scandinavica*, 13:9–30, 1963.

[26] Karel Chvalovský. On the independence of axioms in BL and MTL. To appear in *Fuzzy Sets and Systems,* doi:10.1016/j.fss.2011.10.018.

[27] Agata Ciabattoni, Francesc Esteva, and Lluís Godo. T-norm based logics with n-contraction. *Neural Network World*, 12(5):453–460, 2002.

[28] Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. From axioms to analytic rules in nonclassical logics. In *Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science (LICS)*, pages 229–240, 2008.

[29] Roberto Cignoli, Francesc Esteva, Lluís Godo, and Antoni Torrens. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4(2):106–112, 2000.

[30] Petr Cintula. An alternative approach to the ŁΠ logic. *Neural Network World*, 11(6):561–572, 2001.

[31] Petr Cintula. The ŁΠ and ŁΠ$\frac{1}{2}$ propositional and predicate logics. *Fuzzy Sets and Systems*, 124(3): 289–302, 2001.

[32] Petr Cintula. Advances in the ŁΠ and ŁΠ$\frac{1}{2}$ logics. *Archive for Mathematical Logic*, 42(5):449–468, 2003.

[33] Petr Cintula. *From Fuzzy Logic to Fuzzy Mathematics*. PhD thesis, Czech Technical University, Faculty of Nuclear Sciences and Physical Engineering, Prague, 2005.

[34] Petr Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[35] Petr Cintula, Francesc Esteva, Joan Gispert, Lluís Godo, Franco Montagna, and Carles Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies. *Annals of Pure and Applied Logic*, 160(1):53–81, 2009.

[36] Petr Cintula and Brunella Gerla. Semi-normal forms and functional representation of product fuzzy logic. *Fuzzy Sets and Systems*, 143(1):89–110, 2004.

[37] Petr Cintula and Petr Hájek. Triangular norm predicate fuzzy logics. *Fuzzy Sets and Systems*, 161(3): 311–346, 2010.

[38] Petr Cintula, Petr Hájek, and Rostislav Horčík. Formal systems of fuzzy logic and their fragments. *Annals of Pure and Applied Logic*, 150(1–3):40–65, 2007.

[39] Petr Cintula and Rostislav Horčík. Fuzzy Class Theory: Some advanced topics. In Martin Štěpnička, Vilém Novák, and Ulrich Bodenhofer, editors, *New Dimensions in Fuzzy Logic and Related Technologies. Proceedings of the 5th EUSFLAT Conference*, volume I, pages 137–144, Ostrava, 2007. University of Ostrava.

[40] Petr Cintula, Erich Peter Klement, Radko Mesiar, and Mirko Navara. Residuated logics based on strict t-norms with an involutive negation. *Mathematical Logic Quarterly*, 52(3):269–282, 2006.

[41] Petr Cintula and Carles Noguera. Implicational (semilinear) logics I: A new hierarchy. *Archive for Mathematical Logic*, 49(4):417–446, 2010.

[42] Janusz Czelakowski. *Protoalgebraic Logics*, volume 10 of *Trends in Logic*. Kluwer, Dordrecht, 2001.

[43] Bernard De Baets. Idempotent uninorms. *European Journal of Operational Research*, 118(3):631–642, 1999.

[44] Antonio Di Nola and Anatolij Dvurečenskij. Product MV-algebras. *Multiple-Valued Logic*, 6(1–2): 193–215, 2001.

[45] Antonio Di Nola, Francesc Esteva, Pere Garcia, Lluís Godo, and Salvatore Sessa. Subvarieties of BL-algebras generated by single-component chains. *Archive for Mathematical Logic*, 41(7):673–685, 2002.

[46] Antonio Di Nola, George Georgescu, and Afrodita Iorgulescu. Pseudo-BL-algebras, part I–II. *Multiple-Valued Logic*, 8(5–6):673–714, 717–750, 2002.

[47] Michael Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 24(2):97–106, 1959.

[48] J. Michael Dunn and Greg Restall. Relevance logic. In Dov Gabbay and Franz Guenther, editors, *Handbook of Philosophical Logic*, volume 6, pages 1–136. Kluwer, second edition, 2002.

[49] Antonín Dvořák and Michal Holčapek. $\mathbf{L}$-fuzzy quantifiers of type $\langle 1 \rangle$ determined by fuzzy measures. *Fuzzy Sets and Systems*, 160(23):3425–3452, 2009.

[50] Francesc Esteva, Joan Gispert, Lluís Godo, and Franco Montagna. On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica*, 71(2):199–226, 2002.

[51] Francesc Esteva, Joan Gispert, Lluís Godo, and Carles Noguera. Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results. *Fuzzy Sets and Systems*, 158(6): 597–618, 2007.

[52] Francesc Esteva and Lluís Godo. Putting together Łukasiewicz and product logic. *Mathware and Soft Computing*, 6(2–3):219–234, 1999.

[53] Francesc Esteva and Lluís Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[54] Francesc Esteva, Lluís Godo, and Àngel García-Cerdaña. On the hierarchy of t-norm based residuated fuzzy logics. In Melvin Chris Fitting and Ewa Orlowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 251–272. Physica-Verlag, Heidelberg, 2003.

[55] Francesc Esteva, Lluís Godo, Petr Hájek, and Franco Montagna. Hoops and fuzzy logic. *Journal of Logic and Computation*, 13(4):532–555, 2003.

[56] Francesc Esteva, Lluís Godo, Petr Hájek, and Mirko Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[57] Francesc Esteva, Lluís Godo, and Franco Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[58] Francesc Esteva, Lluís Godo, and Franco Montagna. Axiomatization of any residuated fuzzy logic defined by a continuous t-norm. In Taner Bilgic, Bernard De Baets, and Okays Kaynak, editors, *Fuzzy Sets and Systems IFSA 2003*, volume 2715 of *Lecture Notes in Computer Science*, pages 172–179. Springer, Berlin/Heidelberg, 2003.

[59] Francesc Esteva, Lluís Godo, and Carles Noguera. Expanding the propositional logic of a t-norm with truth-constants: Completeness results for rational semantics. *Soft Computing*, 14(3):273–284, 2010.

[60] Jens Erik Fenstad. On the consistency of the axiom of comprehension in the Łukasiewicz infinite valued logic. *Mathematica Scandinavica*, 14:65–74, 1964.

[61] Isabel Maria André Ferreirim. *On Varieties and Quasivarieties of Hoops and Their Reducts*. PhD thesis, University of Illinois at Chicago, Chicago, 1992.

[62] Tommaso Flaminio and Enrico Marchioni. T-norm based logics with an independent involutive negation. *Fuzzy Sets and Systems*, 157(4):3125–3144, 2006.

[63] Paul Flondor, George Georgescu, and Afrodita Iorgulescu. Pseudo t-norms and pseudo-BL-algebras. *Soft Computing*, 5(5):355–371, 2001.

[64] János Fodor. Nilpotent minimum and related connectives for fuzzy logic. In *Proc. FUZZ-IEEE 1995*, pages 2077–2082, 1995.

[65] Josep Maria Font. Taking degrees of truth seriously. *Studia Logica*, 91(3):383–406, 2009.

[66] Dov M. Gabbay and George Metcalfe. Fuzzy logics based on $[0, 1)$-continuous uninorms. *Archive for Mathematical Logic*, 46(6):425–469, 2007.

[67] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[68] Nikolaos Galatos and Hiroakira Ono. Algebraization, parametrized local deduction theorem and interpolation for substructural logics over FL. *Studia Logica*, 83(1–3):279–308, 2006.

[69] George Georgescu and Afrodita Iorgulescu. Pseudo-MV algebras. *Multiple-Valued Logic*, 6:95–135, 2001.

[70] George Georgescu, Laurenţiu Leuştean, and Viorel Preoteasa. Pseudo-hoops. *Journal of Multiple-Valued Logic and Soft Computing*, 11(1–2):153–184, 2005.

[71] Giangiacomo Gerla. *Fuzzy Logic—Mathematical Tool for Approximate Reasoning*, volume 11 of *Trends in Logic*. Kluwer and Plenum Press, New York, 2001.

[72] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[73] Joan Gispert and Antoni Torrens. Axiomatic extensions of IMT3 logic. *Studia Logica*, 81(3):311–324, 2005.

[74] Kurt Gödel. Zum intuitionistischen Aussagenkalkül. *Anzeiger Akademie der Wissenschaften Wien*, 69:65–66, 1932.

[75] Siegfried Gottwald. Set theory for fuzzy sets of higher level. *Fuzzy Sets and Systems*, 2(2):125–151, 1979.

[76] Siegfried Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View*. Vieweg, Wiesbaden, 1993.

[77] Siegfried Gottwald. Universes of fuzzy sets and axiomatizations of fuzzy set theory. Part I: Model-based and axiomatic approaches. *Studia Logica*, 82(2):211–144, 2006.

[78] Siegfried Gottwald. An early approach toward graded identity and graded membership in set theory. *Fuzzy Sets and Systems*, 161(18):2369–2379, 2010.

[79] Robin J. Grayson. Heyting-valued models for intuitionistic set theory. In Michael P. Fourman, Christopher Mulvey, and Dana S. Scott, editors, *Application of Sheaves*, volume 743 of *Lecture Notes in Computer Science*, pages 402–414. Springer, Berlin, 1979.

[80] Vyacheslav Nikolaevich Grishin. Predicate and set-theoretic calculi based on logic without contractions. *Mathematics of the USSR—Izvestiya*, 18:41–59, 1982.

[81] Petr Hájek. Towards metamathematics of weak arithmetic over fuzzy logic. To appear in *Logic Journal of IGPL,* doi:10.1093/jigpal/jzp091.

[82] Petr Hájek. Basic fuzzy logic and BL-algebras. *Soft Computing*, 2(3):124–128, 1998.

[83] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.

[84] Petr Hájek. Function symbols in fuzzy logic. In *Proceedings of the East–West Fuzzy Colloquium*, pages 2–8, Zittau/Görlitz, 2000. IPM.

[85] Petr Hájek. Fuzzy logic and arithmetical hierarchy III. *Studia Logica*, 68(1):129–142, 2001.

[86] Petr Hájek. Observations on the monoidal t-norm logic. *Fuzzy Sets and Systems*, 132(1):107–112, 2002.

[87] Petr Hájek. Fuzzy logics with noncommutative conjunctions. *Journal of Logic and Computation*, 13(4): 469–479, 2003.

[88] Petr Hájek. Observations on non-commutative fuzzy logic. *Soft Computing*, 8(1):38–43, 2003.

[89] Petr Hájek. Fuzzy logic and arithmetical hierarchy IV. In Vincent F. Hendricks, Fabian Neuhaus, Stig Andur Pedersen, Uwe Scheffler, and Heinrich Wansing, editors, *First-Order Logic Revised*, pages 107–115. Logos Verlag, Berlin, 2004.

[90] Petr Hájek. Arithmetical complexity of fuzzy predicate logics—a survey. *Soft Computing*, 9(12): 935–941, 2005.

[91] Petr Hájek. Fleas and fuzzy logic. *Journal of Multiple-Valued Logic and Soft Computing*, 11(1–2): 137–152, 2005.

[92] Petr Hájek. On arithmetic in Cantor-Łukasiewicz fuzzy set theory. *Archive for Mathematical Logic*, 44(6):763–782, 2005.

[93] Petr Hájek. Mathematical fuzzy logic and natural numbers. *Fundamenta Informaticae*, 81:155–163, 2007.

[94] Petr Hájek. On witnessed models in fuzzy logic. *Mathematical Logic Quarterly*, 53(1):66–77, 2007.

[95] Petr Hájek. On witnessed models in fuzzy logic II. *Mathematical Logic Quarterly*, 53(6):610–615, 2007.

[96] Petr Hájek. On witnessed models in fuzzy logic III — witnessed Gödel logics. *Mathematical Logic Quarterly*, 56(2):171–174, 2010.

[97] Petr Hájek. Comments on interpretability and decidability in fuzzy logic. *Journal of Logic and Computation*, 21(5):823–828, 2011.

[98] Petr Hájek and Petr Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

[99] Petr Hájek, Lluís Godo, and Francesc Esteva. A complete many-valued logic with product conjunction. *Archive for Mathematical Logic*, 35(3):191–208, 1996.

[100] Petr Hájek and Zuzana Haniková. A development of set theory in fuzzy logic. In Melvin Chris Fitting and Ewa Orlowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 273–285. Physica-Verlag, Heidelberg, 2003.

[101] Petr Hájek, Jeff Paris, and John C. Shepherdson. The liar paradox and fuzzy logic. *Journal of Symbolic Logic*, 65(1):339–346, 2000.

[102] Petr Hájek, Jeff Paris, and John C. Shepherdson. Rational Pavelka logic is a conservative extension of Łukasiewicz logic. *Journal of Symbolic Logic*, 65(2):669–682, 2000.

[103] Petr Hájek and Sauro Tulipani. Complexity of fuzzy probability logics. *Fundamenta Informaticae*, 45(3):207–213, 2001.

[104] Zuzana Haniková. Varieties generated by standard BL-algebras. Submitted.

[105] Zuzana Haniková and Petr Savický. Distinguishing standard SBL-algebras with involutive negations by propositional formulas. *Mathematical Logic Quarterly*, 54(6):579–596, 2008.

[106] Louise Schmir Hay. Axiomatization of the infinite-valued predicate calculus. *Journal of Symbolic Logic*, 28(1):77–86, 1963.

[107] Ulrich Höhle. Commutative, residuated l-monoids. In Ulrich Höhle and Erich Peter Klement, editors, *Non-Classical Logics and Their Applications to Fuzzy Subsets*, pages 53–106. Kluwer, Dordrecht, 1995.

[108] Rostislav Horčík. Standard completeness theorem for ΠMTL. *Archive for Mathematical Logic*, 44(4): 413–424, 2005.

[109] Rostislav Horčík. Decidability of cancellative extension of monoidal t-norm based logic. *Logic Journal of the Interest Group of Pure and Applied Logic*, 14(6):827–843, 2006.

[110] Rostislav Horčík. On the failure of standard completeness in ΠMTL for infinite theories. *Fuzzy Sets and Systems*, 158(6):619–624, 2007.

[111] Rostislav Horčík. Solution of a system of linear equations with fuzzy numbers. *Fuzzy Sets and Systems*, 159(14):1788–1810, 2008.

[112] Rostislav Horčík and Petr Cintula. Product Łukasiewicz logic. *Archive for Mathematical Logic*, 43(4): 477–503, 2004.

[113] Rostislav Horčík, Carles Noguera, and Milan Petrík. On n-contractive fuzzy logics. *Mathematical Logic Quarterly*, 53(3):268–288, 2007.

[114] Rostislav Horčík and Kazushige Terui. Disjunction property and complexity of substructural logics. *Theoretical Computer Science*, 412(31):3992–4006, 2011.

[115] Alfred Horn. Logic with truth values in a linearly ordered Heyting algebras. *Journal of Symbolic Logic*, 34(3):395–408, 1969.

[116] Yasuyuki Imai and Kiyoshi Iséki. On axiomatic systems of propositional calculi XIV. *Proceedings Japan Academy*, 42(1):19–22, 1966.

[117] Sándor Jenei and Franco Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[118] Sándor Jenei and Franco Montagna. A proof of standard completeness for non-commutative monoidal t-norm logic. *Neural Network World*, 13(5):481–489, 2003.

[119] Peter Jipsen and Constantine Tsinakis. A survey of residuated lattices. In Jorge Martínez, editor, *Ordered Algebraic Structures*, pages 19–56. Kluwer Academic Publishers, Dordrecht, 2002.

[120] Heldermann Keimel. Some trends in lattice-ordered groups and rings. In Kirby A. Baker, Garrett Birkhoff, and Rudolf Wille, editors, *Lattice Theory and its Applications. In Celebration of Garrett Birkhoff's 80th Birthday.*, volume 23 of *Research and Exposition in Mathematics*, pages 131–161. Heldermann, Darmstadt, 1995.

[121] Dieter Klaua. Über einen Ansatz zur mehrwertigen Mengenlehre. *Monatsb. Deutsch. Akad. Wiss. Berlin*, 7:859–867, 1965.

[122] Dieter Klaua. Über einen zweiten Ansatz zur mehrwertigen Mengenlehre. *Monatsb. Deutsch. Akad. Wiss. Berlin*, 8:782–802, 1966.

[123] Erich Peter Klement, Radko Mesiar, and Endre Pap. *Triangular Norms*, volume 8 of *Trends in Logic*. Kluwer, Dordrecht, 2000.

[124] Tomasz Kowalski. Semisimplicity, EDPC and discriminator varieties of residuated lattices. *Studia Logica*, 77(2):255–265, 2004.

[125] Tomáš Kroupa. Filters in Fuzzy Class Theory. *Fuzzy Sets and Systems*, 159(14):1773–1787, 2008.

[126] Jan Kühr. Pseudo-BL algebras and PRl-monoids. *Mathematica Bohemica*, 128:199–208, 2003.

[127] Michael C. Laskowski and Shirin Malekpour. Provability in predicate product logic. *Archive for Mathematical Logic*, 46(5–6):365–378, 2007.

[128] Jan Łukasiewicz. O logice trójwartościowej (On three-valued logic). *Ruch filozoficzny*, 5:170–171, 1920.

[129] Jan Łukasiewicz and Alfred Tarski. Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, cl. III*, 23(iii):30–50, 1930.

[130] Enrico Marchioni and Franco Montagna. On triangular norms and uninorms definable in $Ł\Pi\frac{1}{2}$. *International Journal of Approximate Reasoning*, 47(2):179–201, 2008.

[131] Robert McNaughton. A theorem about infinite-valued sentential logic. *Journal of Symbolic Logic*, 16(1):1–13, 1951.

[132] George Metcalfe. *Proof Theory for Propositional Fuzzy Logics*. PhD thesis, Department of Computer Science, King's College, London, 2004.

[133] George Metcalfe and Franco Montagna. Substructural fuzzy logics. *Journal of Symbolic Logic*, 72(3):834–864, 2007.

[134] George Metcalfe, Nicola Olivetti, and Dov Gabbay. Analytic proof calculi for product logics. *Archive for Mathematical Logic*, 43(7):859–889, 2004.

[135] George Metcalfe, Nicola Olivetti, and Dov Gabbay. Sequent and hypersequent calculi for Abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.

[136] George Metcalfe, Nicola Olivetti, and Dov M. Gabbay. *Proof Theory for Fuzzy Logics*, volume 36 of *Applied Logic Series*. Springer, 2008.

[137] Franco Montagna. An algebraic approach to propositional fuzzy logic. *Journal of Logic, Language and Information*, 9(1):91–124, 2000.

[138] Franco Montagna. Three complexity problems in quantified fuzzy logic. *Studia Logica*, 68(1):143–152, 2001.

[139] Franco Montagna. On the predicate logics of continuous t-norm BL-algebras. *Archive for Mathematical Logic*, 44(1):97–114, 2005.

[140] Franco Montagna, Carles Noguera, and Rostislav Horčík. On weakly cancellative fuzzy logics. *Journal of Logic and Computation*, 16(4):423–450, 2006.

[141] Franco Montagna and Giovanni Panti. Adding structure to MV-algebras. *Journal of Pure and Applied Algebra*, 164(3):365–387, 2001.

[142] António Monteiro. Sur les algèbres de Heyting symétriques. *Portugaliae Mathematica*, 39(1–4):1–239, 1980.

[143] Daniele Mundici. Satisfiability in many-valued sentential logic is NP-complete. *Theoretical Computer Science*, 52(1–2):145–153, 1987.

[144] Daniele Mundici. A constructive proof of McNaughton's Theorem in infinite-valued logics. *Journal of Symbolic Logic*, 59(2):596–602, 1994.

[145] Carles Noguera. *Algebraic Study of Axiomatic Extensions of Triangular Norm Based Fuzzy Logics*, volume 27 of *Monografies de l'Institut d'Investigació en Intel·ligència Artificial*. Consell Superior d'Investigacions Científiques, Barcelona, 2007.

[146] Vilém Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149(2):235–273, 2004.

[147] Vilém Novák. Fuzzy type theory as higher order fuzzy logic. In *Proc. The Sixth International Conference on Intelligent Technologies*, pages 21–26, Bankgok, 2005.

[148] Vilém Novák. Fuzzy logic theory of evaluating expressions and comparative quantifiers. In *Proceedings of 11th IPMU Conference*, volume 2, pages 1572–1579, Paris, 2006. Edition EDK.

[149] Vilém Novák. A formal theory of intermediate quantifiers. *Fuzzy Sets and Systems*, 159(10):1229–1246, 2008.

[150] Vilém Novák, Irina Perfilieva, and Jiří Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer, Dordrecht, 2000.

[151] Hiroakira Ono. Substructural logics and residuated lattices—an introduction. In Vincent F. Hendricks and Jacek Malinowski, editors, *50 Years of Studia Logica*, volume 21 of *Trends in Logic*, pages 193–228. Kluwer, Dordrecht, 2003.

[152] Hiroakira Ono and Yuichi Komori. Logic without the contraction rule. *Journal of Symbolic Logic*, 50(1):169–201, 1985.

[153] Francesco Paoli. *Substructural Logics: A Primer*, volume 13 of *Trends in Logic*. Kluwer, Dordrecht, 2002.

[154] Jan Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.

[155] Uwe Petersen. Logic without contraction as based on inclusion and unrestricted abstraction. *Studia Logica*, 64(3):365–403, 2000.

[156] William C. Powell. Extending Gödel's negative interpretation to ZF. *Journal of Symbolic Logic*, 40(2):221–229, 1975.

[157] Graham Priest. *In Contradiction: A Study of the Transconsistent*. Oxford University Press, Oxford, second edition, 2006.

[158] Andreja Prijatelj. Bounded contraction and Gentzen-style formulation of Łukasiewicz logics. *Studia Logica*, 57(2–3):437–456, 1996.

[159] Helena Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.

[160] Greg Restall. Arithmetic and truth in Łukasiewicz's infinitely valued logic. *Logique et Analyse*, 36: 25–38, 1995.

[161] Greg Restall. *An Introduction to Substructural Logics*. Routledge, New York, 2000.

[162] Hartley Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967.

[163] Alan Rose. Formalisation du calcul propositionnel implicatif a $\aleph_0$ valeurs de Łukasiewicz. *Comptes Rendus Hebdomadaire des Séances de l'Académie des Sciences, Paris*, 243:1183–1185, 1956.

[164] Alan Rose and J. Barkley Rosser. Fragments of many-valued statement calculi. *Transactions of the American Mathematical Society*, 87(1):1–53, 1958.

[165] Richard Routley, Val Plumwood, Robert K. Meyer, and Ross T. Brady. *Relevant Logics and their Rivals. Part I*. Ridgeview Publishing Co., Atascadero, CA, 1982. The Basic Philosophical and Semantical Theory.

[166] Giovanni Sambin, Giulia Battilotti, and Claudia Faggian. Basic logic: Reflection, symmetry, visibility. *Journal of Symbolic Logic*, 65(3):979–1013, 2000.

[167] Petr Savický, Roberto Cignoli, Francesc Esteva, Lluís Godo, and Carles Noguera. On product logic with truth constants. *Journal of Logic and Computation*, 16(2):205–225, 2006.

[168] Thoralf Skolem. Bemerkungen zum Komprehensionsaxiom. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 3(1–5):1–17, 1957.

[169] Thoralf Skolem. A set theory based on a certain 3-valued logic. *Mathematica Scandinavica*, 8:71–80, 1960.

[170] Gaisi Takeuti and Satoko Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *Journal of Symbolic Logic*, 49(3):851–866, 1984.

[171] Gaisi Takeuti and Satoko Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32(1):1–32, 1992.

[172] Kazushige Terui. Light affine set theory: A naive set theory of polynomial time. *Studia Logica*, 77(1): 9–40, 2004.

[173] Anne S. Troelstra. *Lectures on Linear Logic*, volume 29 of *CSLI Lecture Notes*. Center For The Study Of Language And Information, Stanford, CA, 1992.

[174] Thomas Vetterlein. Partial algebras for Łukasiewicz logic and its extensions. *Archive for Mathematical Logic*, 44(7):913–933, 2005.

[175] Albert Visser. A propositional logic with explicit fixed points. *Studia Logica*, 40(2):155–175, 1981.

[176] San-Min Wang and Petr Cintula. Logics with disjunction and proof by cases. *Archive for Mathematical Logic*, 47(5):435–446, 2008.

[177] Sanmin Wang and Bin Zhao. HpsUL is not the logic of pseudo-uninorms and their residua. *Logic Journal of the Interest Group of Pure and Applied Logic*, 17(4):413–419, 2009.

[178] Richard B. White. The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz. *Journal of Philosophical Logic*, 8(1):509–534, 1979.

[179] Ryszard Wójcicki. *Theory of Logical Calculi*, volume 199 of *Synthese Library*. Kluwer Academic Publishers, Dordrecht/Boston/London, 1988.

[180] Ronald R. Yager and Alexander Rybalov. Uninorm aggregation operators. *Fuzzy Sets and Systems*, 80(1):111–120, 1996.

[181] Shunsuke Yatabe. A note on Hájek, Paris and Shepherdson's theorem. *Logic Journal of the Interest Group of Pure and Applied Logic*, 13(2):261–266, 2005.

[182] Shunsuke Yatabe. Distinguishing non-standard natural numbers in a set theory within Łukasiewicz logic. *Archive for Mathematical Logic*, 46(3–4):281–287, 2007.

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