A Gentle Introduction to Mathematical Fuzzy Logic

1. Motivation, history and two new logics

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Logic is the science that studies correct reasoning.

It is studied as part of Philosophy, Mathematics, and Computer Science.

From XIXth century, it has become a formal science that studies symbolic abstractions capturing the formal aspects of inference: symbolic logic or mathematical logic.

What is a correct reasoning?

Example 1.1

"If God exists, He must be good and omnipotent. If God was good and omnipotent, He would not allow human suffering. But, there is human suffering. Therefore, God does not exist."

Is this a correct reasoning?

What is a correct reasoning?

Formalization

Atomic parts:

- p: God exists
- q: God is good
- r: God is omnipotent
- s: There is human suffering

The form of the reasoning:

$$\begin{array}{l} p \to q \wedge r \\ q \wedge r \to \neg s \\ s \\ \hline \neg p \end{array}$$

Is this a correct reasoning?

Classical logic — syntax

We consider primitive connectives $\mathcal{L} = \{\to, \wedge, \vee, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Formulas are built from fixed countable set of atoms using the connectives

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

Classical logic — semantics

Bivalence Principle

Every proposition is either true or false.

Definition 1.2

A 2-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $\{0, 1\}$ such that:

- $e(\overline{0}) = 0$
- $e(\varphi \wedge \psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi) \\ 0 & \text{otherwise.} \end{cases}$

Correct reasoning in classical logic

Definition 1.3

A formula φ is a logical consequence of set of formulas Γ, (in classical logic), $\Gamma \models_{\mathbf{2}} \varphi$, if for every $\mathbf{2}$-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = 1.$$

Correct reasoning = logical consequence

Definition 1.4

Given $\psi_1, \ldots, \psi_n, \varphi \in Fm_{\mathcal{L}}$ we say that $\langle \psi_1, \ldots, \psi_n, \varphi \rangle$ is a correct reasoning if $\{\psi_1, \ldots, \psi_n\} \models_{\mathbf{2}} \varphi$. In this case, $\psi_1, \ldots, \psi_n$ are the premises of the reasoning and φ is the conclusion.

Correct reasoning in classical logic

Remark

$$\frac{\begin{array}{c}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{array}}{\varphi}$$

is a correct reasoning iff there is no interpretation making the premises true and the conclusion false.

Correct reasoning in classical logic

Example 1.5

Modus ponens:

$$\frac{\begin{array}{l} p \to q \\ p \end{array}}{q}$$

It is a correct reasoning (if $e(p \to q) = e(p) = 1$, then $e(q) = 1$).

Example 1.6

Abduction:

$$\frac{\begin{array}{l} p \to q \\ q \end{array}}{p}$$

It is NOT a correct reasoning (take: $e(p) = 0$ and $e(q) = 1$).

Correct reasoning in classical logic

Example 1.7

$$\begin{array}{l} p \to q \wedge r \\ q \wedge r \to \neg s \\ s \\ \hline \neg p \end{array}$$

Assume $e(p \to q \wedge r) = e(q \wedge r \to \neg s) = e(s) = 1$. Then $e(\neg s) = 0$ and so $e(q \wedge r) = 0$. Thus, we must have $e(p) = 0$, and therefore: $e(\neg p) = 1$.

It is a correct reasoning!

BUT, is this really a proof that God does not exist?

NO! We only know that if the premisses were true, then the conclusion would be true as well.

Structurality of ‘logical’ reasoning

Compare our ‘god example’ with other ones of the same structure:

“If God exists, He must be good and omnipotent. If God was good and omnipotent, He would not allow human suffering. But, there is human suffering. Therefore, God does not exist.”

“If our politicians were ideal, they would be inteligent and honest. If politicians were inteligent and honest, there would be no corruption. But, there is corruption. Therefore, our politicians are not ideal.”

“If X is the set of rationals, then it is denumerable and dense. If a set is denumerable and dense, then we can embed integers in it. But we cannot embed integers in X. Therefore, X is not the set of rationals.”

Is classical logic enough?

Because of the Bivalence Principle, in classical logic every predicate yields a perfect division between those objects it applies to, and those it does not. We call them *crisp*.

Examples: prime number, even number, monotonic function, continuous function, divisible group, ... (any mathematical predicate)

Therefore, classical logic is especially designed to capture the notion of correct reasoning in Mathematics.

Sorites paradox [Eubulides of Miletus, IV century BC]

A man who has no money is poor. If a poor man earns one euro, he remains poor. Therefore, a man who has one million euros is poor.

Formalization:

p_n: *A man who has exactly* n *euros is poor*

$$\begin{array}{l} p_0 \\ p_0 \to p_1 \\ p_1 \to p_2 \\ p_2 \to p_3 \\ \vdots \\ p_{999999} \to p_{1000000} \\ \hline p_{1000000} \end{array}$$

Sorites paradox [Eubulides of Miletus, IV century BC]

- There is no doubt that the premise p_0 is true.
- There is no doubt that the conclusion $p_{1000000}$ is false.
- For each i, the premise $p_i \to p_{i+1}$ seems to be true.
- The reasoning is logically correct (application of *modus ponens* one million times).
- We have a paradox!

Vagueness

The predicates that generate this kind of paradoxes are called *vague*.

Remark

A predicate is vague iff it has borderline cases, i.e. there are objects for which we cannot tell whether they fall under the scope of the predicate.

Example: Consider the predicate *tall*. Is a man measuring 1.78 meters tall?

Vagueness

- It is not a problem of ambiguity. Once we fix an unambiguous context, the problem remains.
- It is not a problem of uncertainty. Uncertainty typically appears when some relevant information is not known. Even if we assume that all relevant information is known, the problem remains.
- It cannot be solved by establishing a crisp definition of the predicate. The problem is: with the meaning that the predicate *tall* has in the natural language, whatever it might be, is a man measuring 1.78 meters tall?

Solutions in Analytical Philosophy

(1) Nihilist solution: *Vague predicates have no meaning*. If they would have, sorites paradox would lead to a contradiction.

(2) Epistemicist solution: *Vagueness is a problem of ignorance*. All predicates are crisp, but our epistemological constitution makes us unable to know the exact extension of a vague predicate. Some premise $p_i \to p_{i+1}$ is false.

(3) Supervaluationist solution: The meaning of vague predicate is the set of its precisifications (possible ways to make it crisp). *Truth is supertruth*, i.e. true under all precisifications. Some premise $p_i \to p_{i+1}$ is false.

Solutions in Analytical Philosophy

(4) Pragmatist solution: *Vague predicates do not have a univocal meaning*. A vague language is a set of crisp languages. For every utterance of a sentence involving a vague predicate, pragmatical conventions endow it with some particular crisp meaning. Some premise $p_i \to p_{i+1}$ is false.

(5) Degree-based solution: *Truth comes in degrees*. p_0 is completely true and $p_{1000000}$ is completely false. The premises $p_i \to p_{i+1}$ are very true, but not completely.

Logic as the language of computer science

Formal systems of mathematical logic are essential in many areas of computer science:

- formal verification (dynamic and temporal logics)
- artificial intelligence (epistemic and deontic logics)
- knowledge representation (epistemic and description logics)

$\vdots$

Their appreciation is due to their

- rigorous formal language
- deductive apparatus
- universality and portability
- the power gained from their mathematical background

$\vdots$

Graded notions

The logics mentioned before are usually tailored for the two-valued notions.

But many the notions or concepts in CS are naturally *graded*:

- graded notions (e.g. tall, old) and relations (e.g. much taller than, distant ancestor) in description logic
- degrees of prohibition in deontic logic
- the cost of knowledge in epistemic logic
- feasibility of computation in a dynamic logic

$\vdots$

Degrees of truth

Most attempts at categorizations of objects forces us to work with *degrees* of some quality.

These degrees are often not behaving as degrees of probability, but rather as *degrees of truth*.

Degrees of probability vs. degrees of truth: The latter requires of the truth-functionality of connectives.

That suggests formalization using suitable formal logical system.

Fuzzy logic *in the broad sense*

Truth values = real unit interval $[0, 1]$

Connectives: conjunction usually interpreted as $\min\{x, y\})$,
disjunction as $\max\{x, y\}$, and negation as $1 - x$.

It is a bunch of engineering methods

- which rely on the theory of fuzzy sets — Zadeh 1965
- are usually tailored to particular purposes
- sometimes are a major success at certain applications
- have no deduction and proof systems
- are difficult to extend and transfer into a different setting.

To sum it up: Fuzzy logic in the *broad sense'* lacks the ‘blessings’ that mathematical logics brings into computer science.

Fuzzy logic *in the narrow sense*

A bunch of formal theories, which

- are analogous to classical logic in its formal and deductive nature
- thus partake of the advantages of classical mathematical logic
- share also many of its methods and results
- have many important mathematical results of their own
- aim at establishing a deep and stable background for applications, in particular in computer science

A little bit of history: fuzzy logic (*in the broad sense*)

- Zadeh 1965
- Goguen 1967
- Mamdani 1974
- Bandler, Kohout 1980
- Pultr 1984
- Novák 1984
- Trillas, Valverde 1985
- Klir, Folger 1988

⋮

A little bit of history: many-valued logic

- Łukasiewicz 1920
- Łukasiewicz–Tarski 1930
- Gödel 1932
- Moisil 1940
- Rose–Rosser 1958
- Chang 1959
- Dummett 1959
- Belluce–Chang 1963
- Ragaz 1981
- Mundici 1987, 1993
- Gottwald 1988

⋮

A little bit of history: fuzzy logic (*in the narrow sense*)

- Pavelka 1979
- Pultr 1984
- Takeuti–Titani 1984, 1992
- Novák 1990
- Gottwald 1993
- Hájek–Esteva–Godo 1996
- Hájek: *Metamathematics of Fuzzy Logic.* Kluwer, 1998.

$$\Downarrow$$

Mathematical Fuzzy Logic

Two logicians and two logics

Gödel vs Łukasiewicz

Keeping the syntax

We consider primitive connectives $\mathcal{L} = \{\rightarrow, \wedge, \vee, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \rightarrow \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

Recall the semantics of classical logic

Definition 1.2

A $\mathbf{2}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $\{0, 1\}$ such that:

- $e(\overline{0}) = \overline{0}^{\mathbf{2}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\mathbf{2}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\mathbf{2}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\mathbf{2}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi) \\ 0 & \text{otherwise.} \end{cases}$

Definition 1.3

A formula φ is a logical consequence of set of formulas Γ, (in classical logic), $\Gamma \models_{\mathbf{2}} \varphi$, if for every $\mathbf{2}$-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = 1.$$

Changing the semantics

Definition 1.8

A $[0,1]_{\mathrm{G}}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$ such that:

- $e(\overline{0}) = \overline{0}^{[0,1]_{\mathrm{G}}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{\mathrm{G}}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{\mathrm{G}}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{\mathrm{G}}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi) \\ e(\psi) & \text{otherwise.} \end{cases}$

Definition 1.9

A formula φ is a logical consequence of set of formulas Γ, (in Gödel–Dummett logic), $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$, if for every $[0,1]_{\mathrm{G}}$-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

Changing the semantics

Some classical properties fail in $\models_{[0,1]_G}$:

- $\not\models_{[0,1]_G} \neg\neg\varphi \to \varphi$ $\qquad \neg\neg\frac{1}{2} \to \frac{1}{2} = 1 \to \frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_G} \varphi \vee \neg\varphi$ $\qquad \frac{1}{2} \vee \neg\frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_G} \neg(\neg\varphi \wedge \neg\psi) \to \varphi \vee \psi$ $\qquad \neg(\neg\frac{1}{2} \wedge \neg\frac{1}{2}) \to \frac{1}{2} \vee \frac{1}{2} = 1 \to \frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_G} ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
  $((\frac{1}{2} \to 0) \to 0) \to ((0 \to \frac{1}{2}) \to \frac{1}{2}) = 1 \to \frac{1}{2} = \frac{1}{2}$

A proof system for classical logic

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\vee$a)	$\varphi \to \varphi \vee \psi$	
($\vee$b)	$\psi \to \varphi \vee \psi$	
($\vee$c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Inference rule: *modus ponens* from $\varphi \to \psi$ and φ infer ψ.

A proof system for classical logic

Proof: a proof of a formula φ from a set of formulas Γ is a finite sequence of formulas $\langle \psi_1, \dots, \psi_n \rangle$ such that:

- $\psi_n = \varphi$
- for every $i \leq n$, either $\psi_i \in \Gamma$, or ψ_i is an instance of an axiom, or there are $j, k < i$ such that $\psi_k = \psi_j \to \psi_i$.

We write $\Gamma \vdash_{\mathrm{CL}} \varphi$ if there is a proof of φ from Γ.

The proof system is finitary: if $\Gamma \vdash_{\mathrm{CL}} \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{CL}} \varphi$.

Completeness theorem for classical logic

Theorem 1.10

For every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have:*

$$\Gamma \vdash_{\mathrm{CL}} \varphi \text{ if, and only if, } \Gamma \models_{\mathbf{2}} \varphi.$$

A proof system for Gödel–Dummett logic

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\vee$a)	$\varphi \to \varphi \vee \psi$	
($\vee$b)	$\psi \to \varphi \vee \psi$	
($\vee$c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{\mathrm{G}} \varphi$ if there is a proof of φ from Γ.

Completeness theorem for Gödel–Dummett logic

Theorem 1.11

For every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have:*

$$\Gamma \vdash_{\mathrm{G}} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_{\mathrm{G}}} \varphi.$$

A solution to *sorites* paradox?

- Consider variables $\{p_0, p_1, p_2, \ldots, p_{10^6}\}$ and define $\varepsilon = 10^{-6}$.
- Define a $[0,1]_{\mathrm{G}}$-evaluation e as $e(p_n) = 1 - n\varepsilon$.
- Note that $e(p_0) = 1$ and $e(p_{10^6}) = 0$, i.e. first premise is completely true, the conclusion is completely false.
- Furthermore $e(p_n \to p_{n+1}) = e(p_n) \to^{[0,1]_{\mathrm{G}}} e(p_{n+1}) = e(p_{n+1}) = 1 - n\varepsilon$.

It tends to 0 as well!

This semantics does not give a good interpretation of the *sorites* paradox, as it does not explain why the premises are seemingly true.

Changing the semantics again

Definition 1.12

A $[0,1]_{Ł}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$ such that:

- $e(\overline{0}) = \overline{0}^{[0,1]_{Ł}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{Ł}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{Ł}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{Ł}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi) \\ 1 - e(\varphi) + e(\psi) & \text{otherwise.} \end{cases}$

Definition 1.13

A formula φ is a logical consequence of set of formulas Γ, (in Łukasiewicz logic), $\Gamma \models_{[0,1]_{Ł}} \varphi$, if for every $[0,1]_{Ł}$-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

Changing the semantics again

Some classical properties fail in $\models [0,1]_{Ł}$:

- $\not\models_{[0,1]_{Ł}} \varphi \vee \neg\varphi$ $\qquad \frac{1}{2} \vee \neg\frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_{Ł}} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
  $(\frac{1}{2} \to (\frac{1}{2} \to 0)) \to (\frac{1}{2} \to 0) = 1 \to \frac{1}{2} = \frac{1}{2}$

BUT other classical properties hold, e.g.:

- $\models_{[0,1]_{Ł}} \neg\neg\varphi \to \varphi$
- $\models_{[0,1]_{Ł}} ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
- all De Morgan laws involving $\neg, \vee, \wedge$

Fuzzy Logic solution to sorites paradox

- Consider variables $\{p_0, p_1, p_2, \ldots, p_{10^6}\}$ and define $\varepsilon = 10^{-6}$.
- Define a $[0,1]_{Ł}$-evaluation e as $e(p_n) = 1 - n\varepsilon$.
- Note that $e(p_0) = 1$ and $e(p_{10^6}) = 0$, i.e. first premise is completely true, the conclusion is completely false.
- Furthermore $e(p_n \to p_{n+1}) = e(p_n) \to^{[0,1]_{Ł}} e(p_{n+1}) =$ $1 - e(p_n) + e(p_{n+1}) = 1 - \varepsilon$.

All premises have the same, almost completely true, truth value!

A proof system for Łukasiewicz logic

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\vee$a)	$\varphi \to \varphi \vee \psi$	
($\vee$b)	$\psi \to \varphi \vee \psi$	
($\vee$c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{Ł} \varphi$ if there is a proof of φ from Γ.

Completeness theorem for Łukasiewicz logic

Theorem 1.14

For every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have:*

$$\Gamma \vdash_{\text{Ł}} \varphi \textit{ if, and only if, } \Gamma \models_{[0,1]_{\text{Ł}}} \varphi.$$

Splitting of conjunction properties

In classical logic one can define conjunction in different ways:

$$\varphi \wedge \psi \quad \equiv_{\mathrm{CL}} \quad \neg(\varphi \to \neg\psi) \quad \equiv_{\mathrm{CL}} \quad \neg((\psi \to \varphi) \to \neg\psi)$$

In $[0,1]_{Ł}$:

$$\neg(\tfrac{1}{2} \to \neg\tfrac{1}{2}) = 0 \qquad \neg((\tfrac{1}{2} \to \tfrac{1}{2}) \to \neg\tfrac{1}{2}) = \tfrac{1}{2}$$

Thus we define *two* different conjunctions:

$$\varphi \mathbin{\&} \psi = \neg(\varphi \to \neg\psi) \qquad e(\varphi \mathbin{\&} \psi) = \max\{0, e(\varphi) + e(\psi) - 1\}$$

$$\varphi \wedge \psi = \neg((\psi \to \varphi) \to \neg\psi) \qquad e(\varphi \wedge \psi) = \min\{e(\varphi), e(\psi)\}$$

The two conjunctions play two different algebraic roles:

1. $a \mathbin{\&} b \leq c$ iff $b \leq a \to c$ (residuation)
2. $a \to b = 1$ iff $a \wedge b = a$ iff $a \leq b$ ($\wedge = \min$)

Splitting of conjunction properties

They also have different 'linguistic' interpretation, Girard's example:

A) If I have one dollar, I can buy a pack of Marlboros $D \to M$

B) If I have one dollar, I can buy a pack of Camels $D \to C$

Therefore: $D \to M \wedge C$ i.e.,

C) If I have one dollar, I can buy a pack of Ms
and I can buy a pack of Cs

BETTER: $D \mathbin{\&} D \to M \mathbin{\&} C$ i.e.,

C′) If I have one dollar and I have one dollar,
I can buy a pack of Ms and I can buy a pack of Cs

1913 L.E.J. Brouwer proposes intuitionism as a new (genuine) form of mathematics.

1920 Jan Łukasiewicz publishes the first work ever on many-valued logic (a three-valued logic to deal with future contingents).

1922 He generalizes it to an n-valued logic for each $n \geq 3$.

1928 Heyting considers the logic behind intuitionism and endowes it with a Hilbert-style calculus.

1930 Together with Alfred Tarski, Łukasiewicz generalizes his logics to a $[0, 1]$-valued logic. They also provide a Hilbert-style calculus with 5 axioms and *modus ponens* and conjecture that it is complete w.r.t. the infinitely-valued logic.

1932 Kurt Gödel studies an infinite family of finite linearly ordered matrices for intuitionistic logic. They are not a complete semantics.

1934 Gentzen introduces natural deduction and sequent calculus for intuitionistic logic.

1935 Mordchaj Wajsberg claims to have proved Łukasiewicz's conjecture, but he never shows the proof.

1937 Tarski and Stone develop topological interpretations of intuitionistic logic.

1958 Rose and Rosser publish a proof of completeness of Łukasiewicz logic based on syntactical methods.

1959 Meredith shows that the fifth axiom of Łukasiewicz logic is redundant.

1959 Chang publishes a proof of completeness of Łukasiewicz logic based on algebraic methods.

- 1959 Michael Dummett resumes Gödel's work from 1932 and proposes a denumerable linearly ordered matrix for intuitionism. He gives a sound and complete Hilbert-style calculus for this matrix which turns out to be an axiomatic extension of intuitionism: Gödel-Dummett logic.
- 1963 Hay shows the finite strong completeness of Łukasiewicz logic.
- 1965 Saul Kripke introduces his relational semantics for intuitionistic logic.
- 1965 Lotfi Zadeh proposes Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.
- 1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve some vagueness logical paradoxes.