A Gentle Introduction to Mathematical Fuzzy Logic

5. The growing family of fuzzy logics

Petr Cintula[1] and Carles Noguera[2]

[1]Institute of Computer Science,
Czech Academy of Sciences, Prague, Czech Republic

[2]Institute of Information Theory and Automation,
Czech Academy of Sciences, Prague, Czech Republic

www.cs.cas.cz/cintula/MFL

Adding Baaz delta

Let L be a logic of continuous t-norm, i.e., $\mathrm{L} = L(\mathbb{K})$ for some class $\mathbb{K}$ of continuous t-norms.

We add a unary connective $\triangle$ known as Baaz delta or 0–1 projector.

The logic $\mathrm{L}_\triangle$ is the extension of L by the axioms:

$$\begin{array}{l}
\triangle\varphi \vee \neg\triangle\varphi, \\
\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi), \\
\triangle\varphi \to \varphi, \\
\triangle\varphi \to \triangle\triangle\varphi, \\
\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi).
\end{array}$$

and the rule of $\triangle$-necessitation: from φ infer $\triangle\varphi$.

Adding Baaz delta: syntactic properties

Lemma 5.1

$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}_\triangle} \triangle\varphi \leftrightarrow \triangle\psi \qquad \varphi \vee \chi \vdash_{\mathrm{L}_\triangle} \triangle\varphi \vee \chi$

Theorem 5.2

- $T, \varphi \vdash_{\mathrm{L}_\triangle} \psi$ *iff* $T \vdash_{\mathrm{L}_\triangle} \triangle\varphi \to \psi$ *(Delta Deduction Theorem)*
- *If* $\Gamma, \varphi \vdash_{\mathrm{L}_\triangle} \chi$ *and* $\Gamma, \psi \vdash_{\mathrm{L}_\triangle} \chi$, *then* $\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}_\triangle} \chi$. *(Proof by Cases Property)*
- *If* $\Gamma, \varphi \to \psi \vdash_{\mathrm{L}_\triangle} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\mathrm{L}_\triangle} \chi$, *then* $\Gamma \vdash_{\mathrm{L}_\triangle} \chi$. *(Semilinearity Property)*
- *If* $\Gamma \nvdash_{\mathrm{L}_\triangle} \varphi$, *then there is a linear* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{L}_\triangle} \varphi$. *(Linear Extension Property)*

Exercise 26

Prove this lemma and theorem.

Adding Baaz delta: semantics and completeness

An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}, \triangle\rangle$ is an $\mathrm{L}_\triangle$-algebra if:

(0) $\langle A, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}\rangle$ is an L-algebra,

(1) $\triangle x \vee (\triangle x \rightarrow 0) = \overline{1}$,

(2) $\triangle(x \vee y) \leq (\triangle x \vee \triangle y)$

(3) $\triangle x \leq x$

(4) $\triangle x \leq \triangle\triangle x$

(5) $\triangle(x \rightarrow y) \leq \triangle x \rightarrow \triangle y$

(6) $\triangle\overline{1} = \overline{1}$.

Let $\boldsymbol{A}$ be an $\mathrm{L}_\triangle$-chain. Then for every $x \in A$, $\triangle x = \begin{cases} \overline{1} & \text{if } x = \overline{1} \\ \overline{0} & \text{otherwise.} \end{cases}$

Theorem 5.3

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$:

1. $\Gamma \vdash_{\mathrm{L}_\triangle} \varphi$
2. $\Gamma \models_{(\mathbb{L}_\triangle)_{lin}} \varphi$

If Γ is finite we can add:

4. $\Gamma \models_{[0,1]_{*,\triangle}} \varphi$ *for any* $* \in \mathbb{K}$

Adding an involutive negation

Let $\text{L}_\sim$ be $\text{L}_\triangle$ plus a new unary connective $\sim$ and the following axioms:

$$(\sim 1)\ \sim\sim\varphi \leftrightarrow \varphi,$$
$$(\sim 2)\ \triangle(\varphi \to \psi) \to (\sim\psi \to \sim\varphi).$$

An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \overline{0}, \overline{1}, \triangle, \sim\rangle$ is a $\text{L}_\sim$-algebra if:

(0) $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \overline{0}, \overline{1}, \triangle\rangle$ is an $\text{L}_\triangle$-algebra,

(1) $x = \sim\sim x$,

(2) $\triangle(x \to y) \leq \sim y \to \sim x$,

Theorem 5.4

$\text{L}_\sim$ is complete w.r.t. $\text{L}_\sim$-chains and w.r.t. standard L chains expanded with $\triangle$ and some involutive negation.
Furthermore $\text{G}_\sim$ is complete w.r.t. $\text{G}_\sim$-chains and w.r.t. $[0, 1]_{\text{G}_\triangle}$ expanded with the involutive negation $1 - x$.

Adding multiplication

We add a binary connective $\odot$ and define the Product Lukasiewicz logic $\mathrm{PŁ}$ by adding the following axioms to $\mathrm{Ł}$:

(P1)	$(\chi \odot \varphi) \ominus (\chi \odot \psi) \leftrightarrow \chi \odot (\varphi \ominus \psi)$	(distributivity)
(P2)	$\varphi \odot (\psi \odot \chi) \leftrightarrow (\varphi \odot \psi) \odot \chi$	(associativity)
(P3)	$\varphi \to \varphi \odot \overline{1}$	(neutral element)
(P4)	$\varphi \odot \psi \to \varphi$	(monotonicity)
(P5)	$\varphi \odot \psi \to \psi \odot \varphi$	(commutativity)

$\mathrm{PŁ}'$ is the extension of $\mathrm{PŁ}$ with a new rule: (ZD) from $\neg(\varphi \odot \varphi)$ infer $\neg\varphi$.

Lemma 5.5

$\varphi \leftrightarrow \psi \vdash_{\mathrm{PŁ}} \varphi \odot \chi \leftrightarrow \psi \odot \chi$ $\qquad$ $\neg(\varphi \odot \varphi) \vee \chi \vdash_{\mathrm{PŁ}} \neg\varphi \vee \chi$

$\varphi \leftrightarrow \psi \vdash_{\mathrm{PŁ}'} \varphi \odot \chi \leftrightarrow \psi \odot \chi$

Theorem 5.6 (Deduction theorem)

$\Gamma, \varphi \vdash_{\mathrm{PŁ}} \psi$ *iff there is* n *such that* $\Gamma \vdash_{\mathrm{PŁ}} \varphi^n \to \psi$. *does not hold for* $\mathrm{PŁ}'$.

$\mathrm{P}Ł$-algebras and $\mathrm{P}Ł'$-algebras:

A $\mathrm{P}Ł$-algebra is a structure $\boldsymbol{A} = \langle A, \oplus, \neg, \odot, \overline{0}, \overline{1}\rangle$ such that $\langle A, \oplus, \neg, \overline{0}\rangle$ is an MV-algebra and the following equations hold:

(1)	$(x \odot y) \ominus (x \odot z) \approx x \odot (y \ominus z)$	(distributivity)
(2)	$x \odot (y \odot z) \approx (x \odot y) \odot z$	(associativity)
(3)	$x \odot \overline{1} \approx x$	(neutral element)
(4)	$x \odot y \approx y \odot x$	(commutativity)

A $\mathrm{P}Ł'$-algebra is a $\mathrm{P}Ł$-algebra where the following quasiequation holds:

(5) $x \odot x \approx \overline{0} \Rightarrow x \approx \overline{0}$ (domain of integrity)

$[0, 1]_{\mathrm{P}Ł} = \langle [0, 1], \oplus, \neg, \odot, 0, 1\rangle$ (where $\odot$ is the usual algebraic product) is both the standard $\mathrm{P}Ł$ and $\mathrm{P}Ł'$-algebra

Both logics enjoy the completeness w.r.t. their chains but only $\mathrm{P}Ł'$ enjoys the standard completeness.

Adding truth constants: Rational Pavelka Logic

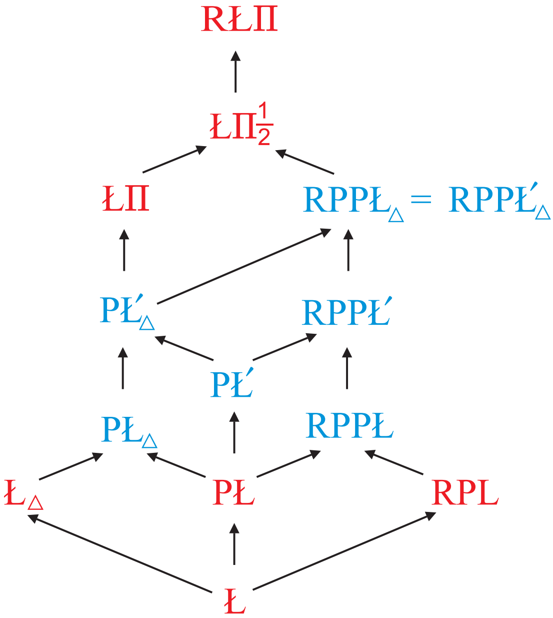
RPL is the expansion of Ł with a constant $\bar{r}$ for each $r \in [0,1] \cap \mathbb{Q}$ and axioms: $\bar{r} \oplus \bar{s} \leftrightarrow \overline{\min\{1, r+s\}}$ and $\neg\bar{r} \leftrightarrow \overline{1-r}$.

We define:

- The truth degree of φ over T is $||\varphi||_T = \inf\{e(\varphi) \mid e[T] \subseteq \{1\}\}$
- The provability degree of φ over T is $|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}} \bar{r} \to \varphi\}$.

Theorem 5.7 (Pavelka style completeness)

$||\varphi||_T = |\varphi|_T$, *for each set of formulas* $T \cup \{\varphi\}$.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ŁΠ½
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RPPŁ′
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Ł△
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RPL
Ł

$Ł\Pi$ and $Ł\Pi\frac{1}{2}$ logics: connectives

Logic $Ł\Pi$ has the following basic connectives:

$\overline{0}$	0	truth constant falsum
$\varphi \to_Ł \psi$	$x \to_Ł y = \min(1, 1 - x + y)$	Łukasiewicz implication
$\varphi \to_\Pi \psi$	$x \to_\Pi y = \min(1, \frac{x}{y})$	product implication
$\varphi \odot \psi$	$x \odot y = x \cdot y$	product conjunction

Logic $Ł\Pi\frac{1}{2}$ has an additional truth constant $\overline{\frac{1}{2}}$ with std. semantics $\frac{1}{2}$.

We define the following derived connectives:

$\neg_Ł \varphi$	is	$\varphi \to_Ł \overline{0}$	$\neg_Ł x = 1 - x$
$\neg_\Pi \varphi$	is	$\varphi \to_\Pi \overline{0}$	$\neg_Ł x = \frac{0}{x}$
$\triangle \varphi$	is	$\neg_\Pi \neg_Ł \varphi$	$\triangle 1 = 1$; $\triangle x = 0$ otherwise
$\varphi \,\&\, \psi$	is	$\neg_Ł(\varphi \to_Ł \neg_Ł \psi)$	$x \,\&\, y = \max(0, x + y - 1)$
$\varphi \oplus \psi$	is	$\neg_Ł \varphi \to_Ł \psi$	$x \oplus y = \min(1, x + y)$
$\varphi \ominus \psi$	is	$\varphi \,\&\, \neg_Ł \psi$	$x \ominus y = \max(0, x - y)$
$\varphi \wedge \psi$	is	$\varphi \,\&\, (\varphi \to_Ł \psi)$	$x \wedge y = \min(x, y)$
$\varphi \vee \psi$	is	$(\varphi \to_Ł \psi) \to_Ł \psi$	$x \vee y = \max(x, y)$
$\varphi \to_G \psi$	is	$\triangle(\varphi \to_Ł \psi) \vee \psi$	$x \to_G y = 1$ if $x \leq y$, otherwise y

$\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$ logics: axiomatic system

Logic $\text{Ł}\Pi$ is given by the following axioms:

(Ł)	Axioms of Łukasiewicz logic,
(Π)	Axioms of product logic,
(Ł$\triangle$)	$\triangle(\varphi \to_{\text{Ł}} \psi) \to_{\text{Ł}} (\varphi \to_{\Pi} \psi)$,
($\Pi\triangle$)	$\triangle(\varphi \to_{\Pi} \psi) \to_{\text{Ł}} (\varphi \to_{\text{Ł}} \psi)$,
(Dist)	$\varphi \odot (\chi \ominus \psi) \leftrightarrow_{\text{Ł}} (\varphi \odot \chi) \ominus (\varphi \odot \psi)$.

The deduction rules are modus ponens and $\triangle$-necessitation (from φ infer $\triangle\varphi$).

The logic $\text{Ł}\Pi\frac{1}{2}$ results from the logic $\text{Ł}\Pi$ by adding axiom $\overline{\frac{1}{2}} \leftrightarrow \neg_{\text{Ł}}\overline{\frac{1}{2}}$.

Alternative axiomatization (in the language of $\mathrm{L}_\sim$)

(Π) axioms and deduction rules of $\Pi_\sim$,

(A) $(\varphi \to_{\text{Ł}} \psi) \to_{\text{Ł}} ((\psi \to_{\text{Ł}} \chi) \to_{\text{Ł}} (\varphi \to_{\text{Ł}} \chi))$,

where $\varphi \to_{\text{Ł}} \psi$ is defined as $\sim(\varphi \,\&\, {\sim}(\varphi \to \psi))$.

$Ł\Pi$ and $Ł\Pi\frac{1}{2}$ logics: algebras

An $Ł\Pi$-algebra is a structure: $\boldsymbol{A} = (A, \oplus, \sim, \rightarrow_\Pi, \odot, \overline{0}, \overline{1})$

(1) $(A, \oplus, \neg, \odot, 0)$ is a PŁ-algebra

(2) $z \leq (x \rightarrow_\Pi y)$ iff $x \odot z \leq y$

OR

$(1'')$ $(A, \oplus, \sim, 0)$ is an MV-algebra

$(2'')$ $(A, \rightarrow_\Pi, \odot, \wedge, \vee, 0, 1)$ is a Π-algebra

$(3'')$ $x \odot (y \ominus z) = (x \odot y) \ominus (x \odot z)$

$(4'')$ $\triangle(x \rightarrow_Ł y) \rightarrow_Ł (x \rightarrow_\Pi y) = 1$

OR

$(1')$ $(A, \odot, \rightarrow_\Pi, \wedge, \vee, \sim, 0, 1)$ is $\Pi_\sim$-algebra

$(2')$ $(x \rightarrow_Ł y) \leq ((y \rightarrow_Ł z) \rightarrow_Ł (x \rightarrow_Ł z))$

$(3')$ $x \rightarrow_Ł y = \sim(x \odot \sim(x \rightarrow_\Pi y))$

Some theorems about $\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$ logics

- Both logics $\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$ have
  - $\triangle$-deduction theorem
  - Proof by Cases Property
  - Semilinearity Property
  - Linear Extension Property
  - general/linaer completeness
  - finite standard completeness
- In $\text{Ł}\Pi\frac{1}{2}$ we can define truth constants for each rational from [0,1]
- Let $*$ be a continuous t-norm s.t. $*$ is finite ordinal sum (it the sense of Mostert–Shields Theorem). Then the logic $L(*)$ is interpretable in $\text{Ł}\Pi\frac{1}{2}$

Monoidal t-norm logic MTL

The most prominent example of post-1998 fuzzy logics ...

We know that left-continuity of $*$ is sufficient
for the residuum (ie, $\Rightarrow$ such that $z * x \leq y$ iff $z \leq x \Rightarrow y$ holds)
to be defined as $(x \Rightarrow y) = \sup\{z \mid z * x = y\}$

$\Rightarrow$ We can weaken the condition of the continuity of $*$
... MTL = the logic of left-continuous t-norms
(turns out to be even more important than HL)

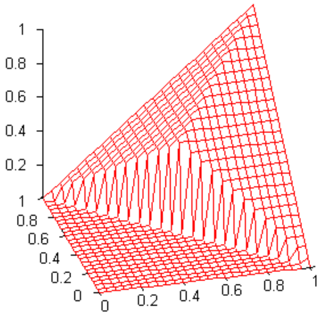
Differences from HL:

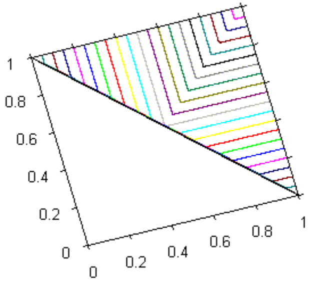
- The minimum is no longer definable from $*, \Rightarrow, 0$
  ($\wedge$ has to be added as a primitive connective)
- The HL axiom $(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$ fails in MTL
  (it has to be replaced by three weaker axioms
  ensuring the lattice behavior of $\wedge$)

Example of left-, not right-continuous t-norm

$*_{\mathrm{NM}}$ nilpotent minimum: $x *_{\mathrm{NM}} y = \begin{cases} \min\{x, y\} & x + y > 1, \\ 0 & \text{otherwise} \end{cases}$ (Fodor 1995)

Its logic NM = MTL+ $\neg\neg\varphi \to \varphi + \neg(\varphi \,\&\, \psi) \vee ((\varphi \wedge \psi) \to (\varphi \,\&\, \psi))$
(Wang 1997; Esteva&Godo 2001)





Changing the language

We consider a new set of primitive connectives $\mathcal{L}_{\mathrm{MTL}} = \{\overline{0}, \&, \wedge, \to\}$ and defined now are connectives $\neg$, $\vee$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \,\&\, (\psi \to \varphi)$$

$$\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

We keep the symbol $Fm_{\mathcal{L}}$ for the set of formulas.

Recall our axioms

The shared part

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)′	$\varphi \,\&\, \psi \to \varphi$	weakening
(Ex)′	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$	exchange
($\mathrm{Res_a}$)	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$	residuation
($\mathrm{Res_b}$)	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$	residuation
(Prl)′	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*

In HL we had

(Div) $\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\psi \to \varphi)$ divisibility

Recall that in the original systems we also had:

(∧a)	$\varphi \wedge \psi \to \varphi$
(∧b)	$\varphi \wedge \psi \to \psi$
(∧c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(∨a)	$\varphi \to \varphi \vee \psi$
(∨b)	$\psi \to \varphi \vee \psi$
(∨c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$

The logic MTL

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(MTL1)
(We)$'$	$\varphi \,\&\, \psi \to \varphi$	(MTL2)
(Ex)$'$	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$	(MTL3)
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	(MTL4a)
($\wedge$b)	$\varphi \wedge \psi \to \psi$	(MTL4b)
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	(MTL4c)
($\mathrm{Res_a}$)	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$	(MTL5a)
($\mathrm{Res_b}$)	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$	(MTL5b)
(Prl)$'$	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$	(MTL6)
(EFQ)	$\overline{0} \to \varphi$	(MTL7)

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{\mathrm{MTL}} \varphi$ if there is a proof of φ from Γ.

Note: axioms (MTL2) and (MTL3) are redundant, the others are independent.

Syntactical properties

Theorem 5.8

- $T, \varphi \vdash_{\mathrm{MTL}} \psi$ *iff there is* n *such that* $T \vdash_{\mathrm{MTL}} \varphi^n \to \psi$
  (Local Deduction Theorem)
- *If* $\Gamma, \varphi \vdash_{\mathrm{MTL}} \chi$ *and* $\Gamma, \psi \vdash_{\mathrm{MTL}} \chi$, *then* $\Gamma, \varphi \vee \psi \vdash_{\mathrm{MTL}} \chi$.
  (Proof by Cases Property)
- *If* $\Gamma, \varphi \to \psi \vdash_{\mathrm{MTL}} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\mathrm{MTL}} \chi$, *then* $\Gamma \vdash_{\mathrm{MTL}} \chi$.
  (Semilinearity Property)
- *If* $\Gamma \nvdash_{\mathrm{MTL}} \varphi$, *then there is a linear* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{MTL}} \varphi$.
  (Linear Extension Property)

Recall the HL-algebras

An HL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1}\rangle$ such that:

(1)	$\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,	
(2)	$\langle B, \&, \overline{1}\rangle$ is a commutative monoid,	
(3)	$z \leq x \to y$ iff $x \,\&\, z \leq y$,	(residuation)
(4)	$x \,\&\, (x \to y) = x \wedge y$	(divisibility)
(5)	$(x \to y) \vee (y \to x) = \overline{1}$	(prelinearity)

We say that $\boldsymbol{B}$ is

- linearly ordered (or HL-chain) if $\leq$ is a total order. $\mathbb{HL}_{\mathrm{lin}}$
- standard $B = [0, 1]$ and $\leq$ is the usual order on reals. $\mathbb{HL}_{\mathrm{std}}$
- G-algebra if $x \,\&\, x = x$
- MV-algebra if $\neg\neg x = x$

Introducing: MTL-algebras

An MTL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid,

(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$, (residuation)

(5) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

We say that $\boldsymbol{B}$ is

- linearly ordered (or MTL-chain) if $\leq$ is a total order. $\mathbb{MTL}_{\mathrm{lin}}$
- standard $B = [0, 1]$ and $\leq$ is the usual order on reals. $\mathbb{MTL}_{\mathrm{std}}$
- IMTL-algebra if $\neg\neg x = x$.

An exercise

Exercise 27

(a) Prove that HL-algebras are exactly MTL-algebras satisfying
$$x \mathbin{\&} (x \to y) \approx x \wedge y.$$

(b) Prove that G-algebras are exactly MTL-algebras satisfying $x \mathbin{\&} x \approx x$.

(c) Prove that all MV-algebras are IMTL-algebras but not vice versa.

(d) Prove that a structure $\boldsymbol{B} = \langle [0,1], \min, \max, \&, \to, 0, 1\rangle$ is an MTL-algebra IFF $\&$ is a left-continuous t-norm and $\to$ its residuum.

General/linear/standard completeness theorem

Theorem 5.9

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{MTL}} \varphi$
2. $\Gamma \models_{\mathbb{MTL}} \varphi$
3. $\Gamma \models_{\mathbb{MTL}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{\mathbb{MTL}_{\mathrm{std}}} \varphi$

Exercise 28

Prove the equivalence of the first three claims.

Three stages of development of an area of logic

Chagrov (*K voprosu ob obratnoi matematike modal'noi logiki,*
Online Journal Logical Studies, 2001)

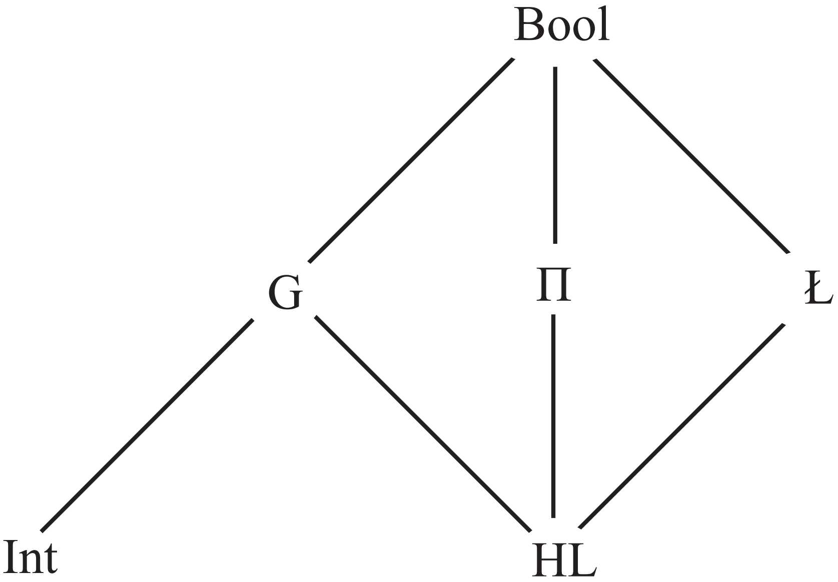
distinguishes three stages in the development of a field in logic.

Three stages of development of MFL

First stage: Emerging of the area (since 1965)

- 1965: Zadeh's fuzzy sets, 1968: 'fuzzy logic' (Goguen)
- 1970s: systems of fuzzy 'logic' lacking a good metatheory
- 1970s–1980s: first 'real' logics (Pavelka, Takeuti–Titani, ...),
  discussion of many-valued logics in the fuzzy context

'Culminated' in Hájek's monograph (1998): G, Ł, HL, Π


Bool
G
Π
Ł
Int
HL

Three stages of development of MFL

Second stage: development of particular logics and introduction of many new ones (since the 1990s)

- New logics: MTL, SHL, UL, $\Pi_{\sim}$, ŁΠ, ...
- Algebraic semantics, proof theory, complexity
  Kripke-style and game-theoretic semantics, ...
- First-order, higher-order, and modal fuzzy logics

Systematic treatment of particular fuzzy logics

Basic fuzzy logic?

Hájek called the logic HL the Basic fuzzy Logic BL

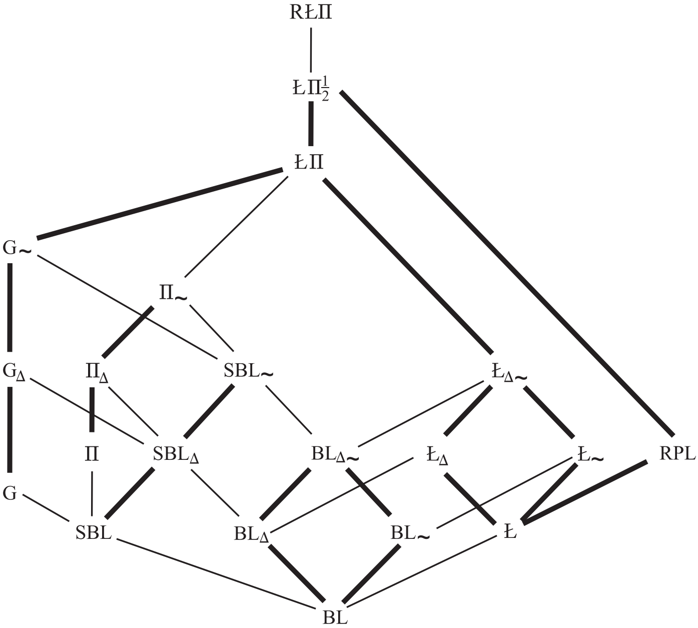
HL was *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties*
2. *it provided a base for the study of all fuzzy logics.*

Because:

- HL is complete w.r.t. the semantics given by *all* continuous t-norms
- All then known fuzzy logics were expansions of HL. The methods to introduce, algebraize, and study HL could be modified for all expansions of HL.

fuzzy logics = expansions of HL


RŁΠ
ŁΠ½
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G∆
Π∆
SBL∼
Ł∆∼
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BL∆∼
Ł∆
Ł∼
RPL
G
SBL
BL∆
BL∼
Ł
BL

“Removing legs from the flea”

In the 3rd EUSFLAT (Zittau, Germany, September 2003) Petr Hájek started his lecture *Fleas and fuzzy logic: a survey* with a joke.

“Upon removing the last leg the flea loses sense of hearing.”

7 Gödel logic

A G-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}\rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1}\rangle$ is a commutative monoid

(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

(5) $x \,\&\, (x \rightarrow y) = x \wedge y$ (divisibility)

(6) $x \,\&\, y = x \wedge y$

6 Hájek's logic

An HL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1}\rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1}\rangle$ is a commutative monoid

(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \to y) \vee (y \to x) = \overline{1}$ (prelinearity)

(5) $x \,\&\, (x \to y) = x \wedge y$ (divisibility)

Hájek logic HL is the logic of continuous t-norms

(well designed to jump)

5 Monoidal t-norm logic MTL

An MTL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1}\rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1}\rangle$ is a commutative monoid

(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \to y) \vee (y \to x) = \overline{1}$ (prelinearity)

MTL is the logic of left-continuous of t-norms

(designed to jump even further)

4 Uninorm logic: the non-integral case

A UL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1}, \bot, \top \rangle$ such that:

(1) $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid

(3) $z \leq x \to y$ iff $x \& z \leq y$, (residuation)

(4) $((x \to y) \wedge \overline{1}) \vee ((y \to x) \wedge \overline{1}) = \overline{1}$ (prelinearity)

UL is the logic of residuated uninorms

(designed to jump even further in one direction)

3 psMTL^r: the non commutative case

A psMTL^r*-algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a monoid,

(3) $z \leq x \to y$ iff $x \& z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

(4) something ugly (prelinearity)

psMTL^r is the logic of residuated pseudo t-norms

(designed to jump even further in other direction)

2 psUL: the non commutative and non integral case

A psUL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ s.t.:

(1) $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a monoid,

(3) $z \leq x \to y$ iff $x \& z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

(4) something even uglier (prelinearity)

psUL is NOT the logic of residuated pseudo uninorms

(lost all sense of hearing?)

1 SL^ℓ: the non associative case

An SL^ℓ-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top\rangle$ s.t.:

(1) $\langle B, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice,

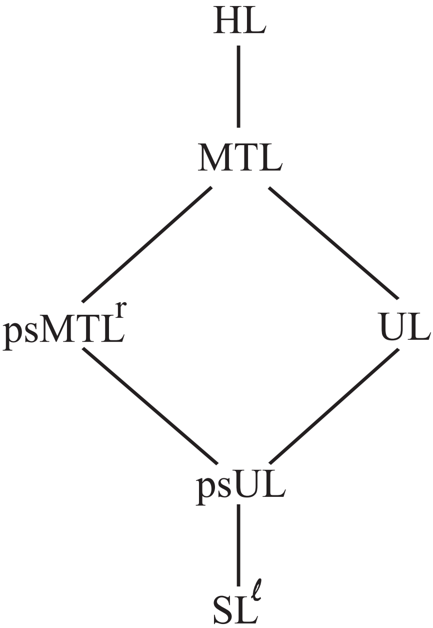
(2) $\langle B, \&, \overline{1}\rangle$ is a unital groupoid,

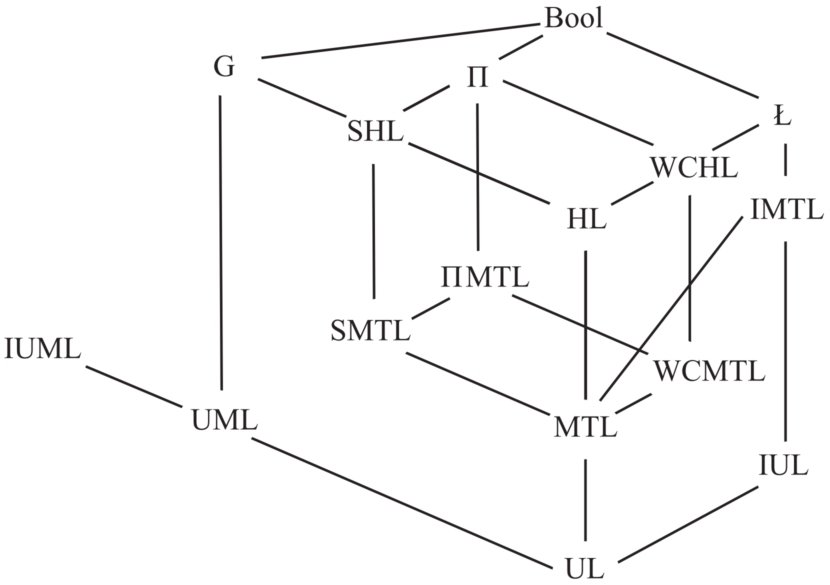
(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

(4) the ugliest thing possible (prelinearity)

SL^ℓ is the logic of residuated unital grupoids on [0,1]

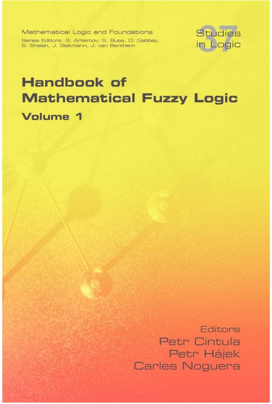
it jumps again!

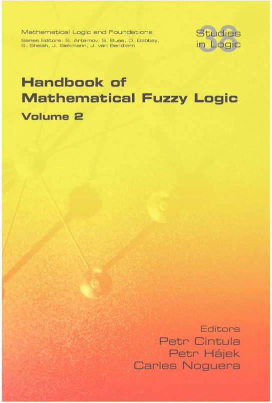

HL
MTL
psMTL^r
UL
psUL
SL$^\ell$


Bool
G
Π
Ł
SHL
WCHL
HL
IMTL
ΠMTL
SMTL
IUML
WCMTL
UML
MTL
IUL
UL

Three stages of development of MFL

The second stage is still ongoing; the state of the art is summarized in:





P. Cintula, P. Hájek, C. Noguera (editors). Vol. 37 and 38 of *Studies in Logic: Math. Logic and Foundations*. College Publications, 2011.

Three stages of development of MFL

Third stage: universal methods (since ∼2006)

- General methods to prove metamathematical properties
- Classification of existing fuzzy logics
- Systematic treatment of classes of fuzzy logics
- Determining the position of fuzzy logics in the logical landscape

Changing the language

We consider a new set of primitive connectives $\mathcal{L}_{\mathrm{SL}} = \{\overline{0}, \overline{1}, \bot, \top, \&, \to, \rightsquigarrow, \vee, \wedge\}$, and a defined connective $\leftrightarrow$:

$$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

We keep the symbol $Fm_{\mathcal{L}}$ for the set of formulas.

The 'minimal' algebraic semantics

Definition 5.10

An SL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ such that:

(1) $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a unital groupoid,

(3) $z \leq x \to y$ iff $x \& z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

Hilbert-system for $\mathcal{SL}$ – axioms

$$
\begin{array}{rl}
(\mathrm{Adj}_{\&}) & \varphi \to (\psi \to \psi \mathbin{\&} \varphi) \\
(\mathrm{Adj}_{\&\rightsquigarrow}) & \varphi \to (\psi \rightsquigarrow \varphi \mathbin{\&} \psi) \\
(\&\wedge) & (\varphi \wedge \overline{1}) \mathbin{\&} (\psi \wedge \overline{1}) \to \varphi \wedge \psi \\
(\wedge 1) & \varphi \wedge \psi \to \varphi \\
(\wedge 2) & \varphi \wedge \psi \to \psi \\
(\wedge 3) & (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi) \\
(\vee 1) & \varphi \to \varphi \vee \psi \\
(\vee 2) & \psi \to \varphi \vee \psi \\
(\vee 3) & (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi) \\
(\mathrm{Push}) & \varphi \to (\overline{1} \to \varphi) \\
(\mathrm{Pop}) & (\overline{1} \to \varphi) \to \varphi \\
(\mathrm{Res}') & \psi \mathbin{\&} (\varphi \mathbin{\&} (\varphi \to (\psi \to \chi))) \to \chi \\
(\mathrm{Res}'_{\rightsquigarrow}) & (\varphi \mathbin{\&} (\varphi \to (\psi \rightsquigarrow \chi))) \mathbin{\&} \psi \to \chi \\
(\mathrm{T}') & (\varphi \to (\varphi \mathbin{\&} (\varphi \to \psi)) \mathbin{\&} (\psi \to \chi)) \to (\varphi \to \chi) \\
(\mathrm{T}'_{\rightsquigarrow}) & (\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi) \mathbin{\&} \varphi) \mathbin{\&} (\psi \to \chi)) \to (\varphi \rightsquigarrow \chi)
\end{array}
$$

Hilbert-system for SL – rules

$$
\begin{array}{rl}
(\mathbf{MP}) & \varphi, \varphi \to \psi \vdash \psi \\
(\mathbf{Adj_u}) & \varphi \vdash \varphi \wedge \overline{1} \\
(\alpha) & \varphi \vdash \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \varphi) \\
(\alpha') & \varphi \vdash \delta \,\&\, \varepsilon \to (\delta \,\&\, \varphi) \,\&\, \varepsilon \\
(\beta) & \varphi \vdash \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \varphi) \\
(\beta') & \varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \varphi)
\end{array}
$$

Convention

Convention

A logic is a provability relation on formulas in a language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$ s.t.

- it is axiomatized by adding axioms Ax and finitary rules (R) to the logic SL
- for each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$, $\mathcal{L}$-formulas $\varphi, \psi, \chi_1, \ldots, \chi_n$, and each $i \leq n$ the following holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$$

Let us fix a logic L in language $\mathcal{L}$ which is the expansion of SL by axioms Ax and rules R.

Algebraic semantics for arbitrary logic L

Definition 5.11

Let $\boldsymbol{B}$ be an $\mathcal{L}$-algebra. A $\boldsymbol{B}$-evaluation is a mapping $e\colon Fm_{\mathcal{L}} \to B$ s.t.

- $e(*) = *^{\boldsymbol{B}}$ for truth constant $*$
- $e(\circ(\varphi_1, \dots, \varphi_n)) = \circ^{\boldsymbol{B}}(e(\varphi_1), \dots, e(\varphi_n))$ for each $n-$ary $\circ \in \mathcal{L}$

Definition 5.12

An $\mathcal{L}$-algebra $\boldsymbol{A}$ is an L-algebra, $\boldsymbol{A} \in \mathbb{L}$, if

- its reduct $\boldsymbol{A}_{\mathrm{SL}} = \langle A, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ is an SL-algebra,
- for each $\varphi \in Ax$, $\boldsymbol{A}$ satisfies the identity $\varphi \wedge \overline{1} = \overline{1}$,
- for each $\langle \{\psi_1, \dots, \psi_n\}, \varphi \rangle \in R$, $\boldsymbol{A}$ satisfies the quasi-identity

$$\texttt{If } \psi_1 \wedge \overline{1} = \overline{1} \texttt{ and } \cdots \texttt{ and } \psi_n \wedge \overline{1} = \overline{1} \texttt{ then } \varphi \wedge \overline{1} = \overline{1}$$

$\boldsymbol{A}$ is a linearly ordered (or L-*chain*), $\boldsymbol{A} \in \mathbb{L}_{\mathrm{lin}}$, if its lattice order is total.

Logical consequence w.r.t. a class of algebras

Definition 5.13

A formula φ is a logical consequence of set of formulas Γ w.r.t. a class $\mathbb{K}$ of L-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

$$\text{if } e(\gamma) \geq \overline{1} \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) \geq \overline{1}.$$

Observation

1. An $\mathcal{L}$-algebra $\boldsymbol{A}$ is an L-algebra iff
   - its reduct $\boldsymbol{A}_{\mathrm{SL}} = \langle A, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ is an SL-algebra,
   - if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma \models_{\boldsymbol{A}} \varphi$.
2. $\mathbb{L}$ is the largest class $\mathbb{K}$ of $\mathcal{L}$-algebras such that $\vdash_{\mathrm{L}} \subseteq \models_{\mathbb{K}}$

General completeness theorem

Theorem 5.14 (Completeness theorem)

For every set of formulas Γ and every formula φ we have:

$$\Gamma \vdash_{\mathrm{L}} \varphi \textit{ if, and only if, } \Gamma \models_{\mathbb{L}} \varphi.$$

Each L is an algebraizable logic and $\mathbb{L}$ is its equivalent algebraic semantics with translations:

$$E(p,q) = \{p \leftrightarrow q\} \text{ and } \mathcal{T}(p) = \{p \wedge \overline{1} \approx \overline{1}\}.$$

Indeed, all we have to do is to prove:

$$p \vdash p \wedge \overline{1} \leftrightarrow \overline{1} \qquad \text{and} \qquad p \wedge \overline{1} \leftrightarrow \overline{1} \vdash p$$

Core semilinear logics

Definition 5.15

A logic L is core semilinear logic whenever it is complete w.r.t. linearly ordered L-algebras, i.e.,

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad T \models_{\mathbb{L}_{\mathrm{lin}}} \varphi$$

Core semilinear logics — syntactic characterization

Theorem 5.16 (Syntactic characterization)

Let L *be axiomatized by axioms* Ax *and rules* R*. TFAE:*

1. L *is a core semilinear logic*
2. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and if* $\langle \Gamma, \varphi \rangle \in R$*, then* $\Gamma \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi$ *for every* χ
3. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and if* $\Gamma \vdash_{\mathrm{L}} \varphi$*, then* $\Gamma \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi$ *for every* χ
4. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and for every set of formulas* $\Gamma \cup \{\varphi, \psi, \chi\}$*:*
   $$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad \Gamma, \psi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad \Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi.$$
5. *For every set of formulas* $\Gamma \cup \{\varphi, \psi, \chi\}$*:*
   $$\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad \Gamma \vdash_{\mathrm{L}} \chi.$$
6. *If* $\Gamma \nvdash_{\mathrm{L}} \varphi$ *then there is a linear theory* $\Gamma' \supseteq \Gamma$ *s.t.* $\Gamma \nvdash_{\mathrm{L}} \varphi$

Core semilinear logics — semantic characterization

Theorem 5.17 (Semantic characterization)

Let L *be a logic. TFAE:*

1. L *is a core semilinear logic*
2. *finitely relatively subdirectly irreducible* L*-algebras are exactly the* L*-chains*
3. *relatively subdirectly irreducible* L*-algebras are linearly ordered*

Weakest semilinear extension

Definition 5.18

By L^{ℓ} we denote the least core semilinear logic extending L.

Exercise 29

(a) Prove that the previous definition is sound (show that the class of core semilinear logics is closed under arbitrary intersections).

(b) Prove that $\mathbb{L}_{\mathrm{lin}}^{\ell} = \mathbb{L}_{\mathrm{lin}}$.

Theorem 5.19

If L *is axiomatized by rules* R*, then* L^{ℓ} *is axiomatized by adding axiom* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and rules:* $\langle \Gamma \vee \chi, \varphi \vee \chi \rangle$ *for each* $\langle \Gamma, \varphi \rangle \in R$*.*

In many cases we can prove that L^{ℓ} is an axiomatic extension of L.

Hilbert-system for SL^{ℓ} – axioms

To the axioms of SL we add

$$(\mathrm{PRL}\alpha) \quad [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, [(\psi \to \varphi) \wedge \overline{1}])$$

$$(\mathrm{PRL}\alpha') \quad [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \,\&\, \varepsilon \to (\delta \,\&\, [(\psi \to \varphi) \wedge \overline{1}]) \,\&\, \varepsilon)$$

$$(\mathrm{PRL}\beta) \quad [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, [(\psi \to \varphi) \wedge \overline{1}]))$$

$$(\mathrm{PRL}\beta') \quad [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, [(\psi \to \varphi) \wedge \overline{1}]))$$

A linear/standard completeness theorem of SL^{ℓ}

Let us by $\mathbb{SL}^{\ell}_{\mathrm{std}}$ denote the class of SL-algebras with the domain $[0,1]$ and the usual order.

Theorem 5.20 (Standard completeness theorem of SL^{ℓ})

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{SL}^{\ell}} \varphi$
2. $\Gamma \models_{\mathbb{SL}^{\ell}} \varphi$
3. $\Gamma \models_{\mathbb{SL}^{\ell}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{\mathbb{SL}^{\ell}_{\mathrm{std}}} \varphi$

Is SL^{ℓ} the new basic fuzzy logic?

We need to show that it is *basic* in the following two senses:

1. *it cannot be made weaker without losing essential properties and*
2. *it provides a base for the study of all fuzzy logics.*

And indeed we have seen that

1. SL^{ℓ} is complete w.r.t. a hardly-to-be-made-weaker semantics over real numbers.
2. Almost all reasonable fuzzy logics expands SL^{ℓ}. The methods to introduce, algebraize, and study SL^{ℓ} could be utilized for any such logic. We can develope a uniform mathematical theory for MFL based on SL^{ℓ}.

fuzzy logics = core semilinear logics