A Gentle Introduction to Mathematical Fuzzy Logic

6. Further lines of research and open problems

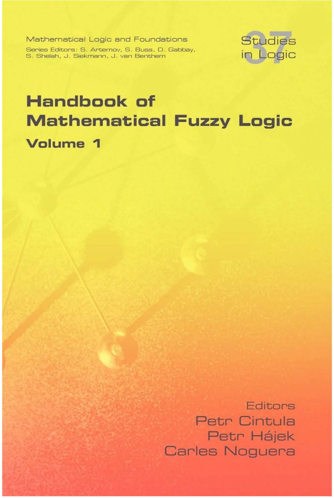
Petr Cintula[1] and Carles Noguera[2]

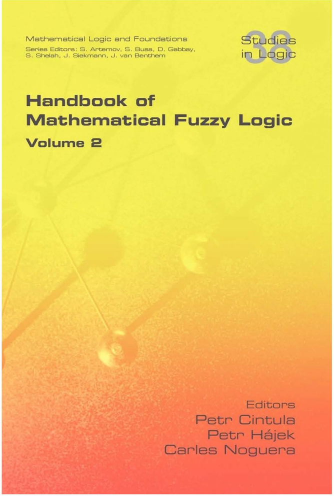
[1]Institute of Computer Science,
Czech Academy of Sciences, Prague, Czech Republic

[2]Institute of Information Theory and Automation,
Czech Academy of Sciences, Prague, Czech Republic

www.cs.cas.cz/cintula/MFL

PC, P. Hájek, CN. *Handbook of Mathematical Fuzzy Logic.*
Studies in Logic, Mathematical Logic and Foundations 37 and 38, 2011.





PC, P. Hájek, CN. *Handbook of Mathematical Fuzzy Logic.* Studies in Logic, Mathematical Logic and Foundations 37 and 38, 2011.

Volume 1

Volume 2

An even more general approach

Why should we stop at SL^ℓ?

fuzzy logics = logics of chains $\Rightarrow$ general theory of semilinear logics

Necessary ingredients:

- An order relation on all algebras (so, in particular, we have chains)
- An implication $\to$ s.t. for every $a, b \in A$, $a \leq b$ iif $a \to^{\boldsymbol{b}}$ is true in $\boldsymbol{A}$
- The implication gives a congruence w.r.t. all connectives (so, we can do the Lindenbaum–Tarski construction)

Using Abstract Algebraic Logic we can develop a theory of weakly implicative semilinear logics.

Basic syntactical notions – 1

Propositional language: a countable type $\mathcal{L}$, i.e. a function $ar\colon C_{\mathcal{L}} \to$ N, where $C_{\mathcal{L}}$ is a countable set of symbols called connectives, giving for each one its arity. Nullary connectives are also called truth-constants. We write $\langle c, n\rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$.

Formulae: Let Var be a fixed infinite countable set of symbols called variables. The set $Fm_{\mathcal{L}}$ of formulas in $\mathcal{L}$ is the least set containing Var and closed under connectives of $\mathcal{L}$, i.e. for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \dots, \varphi_n)$ is a formula.

Substitution: a mapping $\sigma\colon Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ holds for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$.

Basic syntactical notions – 2

Let L be relation between sets of formulas and formulas, we write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\langle \Gamma, \varphi \rangle \in \mathrm{L}$'.

Definition 6.1

A relation L between sets of formulas and formulas in $\mathcal{L}$ is called a (finitary) logic in $\mathcal{L}$ whenever

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \varphi$, then $\Gamma, \Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathrm{L}} \varphi$. (Finitarity)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

Observe that reflexivity and cut entail:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)

Basic syntactical notions – 3

Axiomatic system: a set $\mathcal{AS}$ of pairs $\langle \Gamma, \varphi \rangle$ closed under substitutions, where Γ is a finite set of formulas. If Γ is empty we speak about axioms otherwise we speak about deduction rules.

Proof: a proof of a formula φ from a set of formulas Γ in $\mathcal{AS}$ is a finite sequence of formulas whose each element is either

- an axiom of $\mathcal{AS}$, or
- an element of Γ, or
- the conclusion of a deduction rules whose premises are among its predecessors.

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$.

Basic syntactical notions – 4

Presentation: We say that $\mathcal{AS}$ is an axiomatic system for (or a presentation of) the logic L if $\mathrm{L} = \vdash_{\mathcal{AS}}$.

Theorem: a consequence of the empty set

Theory: a set of formulas T such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. By $\mathrm{Th}(\mathrm{L})$ we denote the set of all theories of L.

Basic semantical notions – 1

$\mathcal{L}$-algebra: $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in C_{\mathcal{L}} \rangle\rangle$, where $A \neq \emptyset$ (universe) and $c^{\boldsymbol{A}} \colon A^n \to A$ for each $\langle c, n\rangle \in \mathcal{L}$.

Algebra of formulas: the algebra $\boldsymbol{Fm}_{\mathcal{L}}$ with domain $Fm_{\mathcal{L}}$ and operations $c^{\boldsymbol{Fm}_{\mathcal{L}}}$ for each $\langle c, n\rangle \in \mathcal{L}$ defined as:

$$c^{\boldsymbol{Fm}_{\mathcal{L}}}(\varphi_1, \dots, \varphi_n) = c(\varphi_1, \dots, \varphi_n).$$

$\boldsymbol{Fm}_{\mathcal{L}}$ if the absolutely free algebra in language $\mathcal{L}$ with generators Var.

Homomorphism of algebras: a mapping $f \colon A \to B$ such that for every $\langle c, n\rangle \in \mathcal{L}$ and every $a_1, \dots, a_n \in A$,

$$f(c^{\boldsymbol{A}}(a_1, \dots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Note that substitutions are exactly endomorphisms of $\boldsymbol{Fm}_{\mathcal{L}}$.

Basic semantical notions – 2

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra called the algebraic reduct of $\mathbf{A}$, and F is a subset of A called the filter of $\mathbf{A}$. The elements of F are called designated elements of $\mathbf{A}$.

A matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is

- trivial if $F = A$.
- finite if A is finite.
- Lindenbaum if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$.

$\boldsymbol{A}$-evaluation: a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$, i.e. a mapping $e\colon Fm_{\mathcal{L}} \to A$, such that for each $\langle c, n \rangle \in \mathcal{L}$ and each n-tuple of formulas $\varphi_1, \ldots, \varphi_n$ we have:

$$e(c(\varphi_1, \ldots, \varphi_n)) = c^{\boldsymbol{A}}(e(\varphi_1), \ldots, e(\varphi_n)).$$

Basic semantical notions – 3

Semantical consequence: A formula φ is a semantical consequence of a set Γ of formulas w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models_{\mathbb{K}} \varphi$.

L-matrix: Let L be a logic in $\mathcal{L}$ and $\mathbf{A}$ an $\mathcal{L}$-matrix. We say that $\mathbf{A}$ is an L-matrix if $\mathrm{L} \subseteq \models_{\mathbf{A}}$. We denote the class of L-matrices by $\mathbf{MOD}(\mathrm{L})$.

Logical filter: Given a logic L in $\mathcal{L}$ and an $\mathcal{L}$-algebra $\boldsymbol{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$.

Example: Let $\boldsymbol{A}$ be a Boolean algebra. Then $\mathcal{F}i_{\mathrm{CPC}}(\boldsymbol{A})$ is the class of lattice filters on $\boldsymbol{A}$, in particular for the two-valued Boolean algebra $\mathbf{2}$:

$$\mathcal{F}i_{\mathrm{CPC}}(\mathbf{2}) = \{\{1\}, \{0, 1\}\}.$$

The first completeness theorem

Proposition 6.2

For any logic L *in a language* $\mathcal{L}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L})$.

Theorem 6.3

Let L *be a logic. Then for each set* Γ *of formulas and each formula* φ *the following holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$.

Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CPC})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CPC}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta\rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta\rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum–Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle} \varphi$.

  2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CPC})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\}\rangle} \varphi$. 3rd completeness theorem

Weakly implicative logics

Definition 6.4

A logic L in a language $\mathcal{L}$ is weakly implicative if there is a binary connective $\to$ (primitive or definable) such that:

(R)	$\vdash_{\mathrm{L}} \varphi \to \varphi$
(MP)	$\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$
(T)	$\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$
(sCng)	$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$

for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

Examples

The following logics are weakly implicative:

- CPC, BCI, and Inc
- global modal logics
- intuitionistic and superintuitionistic logic
- linear logic and its variants
- (the most of) fuzzy logics
- substructural logics

⋮

The following logics are not weakly implicative:

- local modal logics
- the conjunction-disjunction fragment of classical logic as it has no theorems
- logics of ortholattices

⋮

Congruence Property

Conventions

Unless said otherwise, L is a weakly implicative in a language $\mathcal{L}$ with an implication $\to$. We write:

- $\varphi \leftrightarrow \psi$ instead of $\{\varphi \to \psi, \psi, \to \varphi\}$
- $\Gamma \vdash \Delta$ whenever $\Gamma \vdash \chi$ for each $\chi \in \Delta$

Theorem 6.5

Let φ, ψ, χ be formulas. Then:

- $\vdash_{\mathrm{L}} \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \hat{\chi}$, *where $\hat{\chi}$ is obtained from χ by replacing some occurrences of φ in χ by ψ.*

Lindenbaum–Tarski matrix

Let L be a weakly implicative logic in $\mathcal{L}$ and $T \in Th(\mathrm{L})$. For every formula φ, we define the set

$$[\varphi]_T = \{\psi \in Fm_{\mathcal{L}} \mid \varphi \leftrightarrow \psi \subseteq T\}.$$

The Lindenbaum–Tarski matrix with respect to L and T, $\mathbf{LindT}_T$, has the filter $\{[\varphi]_T \mid \varphi \in T\}$ and algebraic reduct with the domain $\{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$ and operations:

$$c^{\mathbf{LindT}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [c(\varphi_1, \dots, \varphi_n)]_T$$

What are Lindenbaum–Tarski matrices in general?
Recall that Lindenbaum matrices have domain $Fm_{\mathcal{L}}$ and

$$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L}).$$

Leibniz congruence

A congruence θ of $\boldsymbol{A}$ is logical in a matrix $\langle \boldsymbol{A}, F\rangle$ if for each $a, b \in A$ if $a \in F$ and $\langle a, b\rangle \in \theta$, then $b \in F$.

Definition 6.6

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ be an L-matrix. We define the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ of $\mathbf{A}$ as

$$\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F) \qquad \text{iff} \qquad a \leftrightarrow^{\boldsymbol{A}} b \subseteq F$$

Theorem 6.7

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ be an L-matrix. Then $\Omega_{\boldsymbol{A}}(F)$ is the largest logical congruence of $\boldsymbol{A}$.

Algebraic counterpart

Definition 6.8

A L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is reduced, $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ in symbols, if $\Omega_{\boldsymbol{A}}(F)$ is the identity relation $\mathrm{Id}_{\boldsymbol{A}}$ (iff $\leq_{\mathbf{A}}$ is an order).

An algebra $\boldsymbol{A}$ is L-algebra, $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ in symbols, if there a set $F \subseteq A$ such that $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$.

Example: it is easy to see that

$$\Omega_{\mathbf{2}}(\{1\}) = \mathrm{Id}_{\mathbf{2}} \qquad \text{i.e., } \mathbf{2} \in \mathbf{ALG}^*(\mathrm{CPC}).$$

Actually for any Boolean algebra $\boldsymbol{A}$:

$$\Omega_{\boldsymbol{A}}(\{1\}) = \mathrm{Id}_{\boldsymbol{A}} \qquad \text{i.e., } \boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{CPC}).$$

But: $\Omega_{\mathbf{4}}(\{a, 1\}) = \mathrm{Id}_{\boldsymbol{A}} \cup \{\langle 1, a\rangle, \langle 0, \neg a\rangle\}$ i.e. $\langle \mathbf{4}, \{a, 1\}\rangle \notin \mathbf{MOD}^*(\mathrm{CPC})$.

Factorizing matrices

Let us take $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. We write:

- $\boldsymbol{A}^*$ for $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$
- $[\cdot]_F$ for the canonical epimorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ defined as:

$$[a]_F = \{b \in A \mid \langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F)\}$$

- $\mathbf{A}^*$ for $\langle \boldsymbol{A}^*, [F]_F \rangle$.

Theorem 6.9

Let T be a theory, $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$, and $a, b \in A$. Then:

1. $\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T \rangle^*$
2. $a \in F$ *iff* $[a]_F \in [F]_F$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ *iff* $a \to^{\boldsymbol{A}} b \in F$.
4. $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})$.

The second completeness theorem

Theorem 6.10

Let L *be a weakly implicative logic. Then for any set* Γ *of formulas and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Proof.

Using just the soundness part of the FCT it remains to prove:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ and take the theory $T = \mathrm{Th}_{\mathrm{L}}(\Gamma)$. Then

- $\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T \rangle^* \in \mathbf{MOD}^*(\mathrm{L})$ and for $\mathbf{LindT}_T$-evaluation $e(\psi) = [\psi]_T$ holds $e(\psi) \in [T]_T$ iff $\psi \in T$
- Thus $e[\Gamma] \subseteq e[T] = [T]_T$ and $e(\varphi) \notin [T]_T$ □

Order and Leibniz congruence

Definition 6.11

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix. We define the matrix preorder $\leq_{\mathbf{A}}$ of $\mathbf{A}$ as

$$a \leq_{\mathbf{A}} b \qquad \text{iff} \qquad a \to^{A} b \in F$$

Note that

$$\langle a, b \rangle \in \Omega_{A}(F) \qquad \text{iff} \qquad a \leq_{\mathbf{A}} b \ \text{ and } \ b \leq_{\mathbf{A}} a.$$

Thus the Leibniz congruence of $\mathbf{A}$ is the identity iff $\leq_{\mathbf{A}}$ is an order, and so all reduced matrices of L are ordered by $\leq_{\mathbf{A}}$.

Weakly implicative logics are the logics of ordered matrices.

Linear filters

Definition 6.12

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. Then

- F is *linear* if $\leq_{\mathbf{A}}$ is a total preorder, i.e. for every $a, b \in A$, $a \to^{\boldsymbol{A}} b \in F$ or $b \to^{\boldsymbol{A}} a \in F$
- $\mathbf{A}$ is a *linearly ordered model* (or just a *linear model*) if $\leq_{\mathbf{A}}$ is a linear order (equivalently: F is linear and $\mathbf{A}$ is reduced).

We denote the class of all linear models as $\mathbf{MOD}^{\ell}(\mathrm{L})$.

A theory T is linear in L if $T \vdash_{\mathrm{L}} \varphi \to \psi$ or $T \vdash_{\mathrm{L}} \psi \to \varphi$, for all φ, ψ

Lemma 6.13

Let $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$. Then F is linear iff $\mathbf{A}^* \in \mathbf{MOD}^{\ell}(\mathrm{L})$. In particular: a theory T is linear iff $\mathbf{LindT}_T \in \mathbf{MOD}^{\ell}(\mathrm{L})$

Semilinear implications and semilinear logics

Definition 6.14

We say that $\rightarrow$ is *semilinear* if

$$\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}.$$

We say that L is *semilinear* if it has a semilinear implication.

(Weakly implicative) *semilinear* logics are the logics of *linearly* ordered matrices.

Characterization of semilinear logics

Theorem 6.15

Let L *be a finitary weakly implicative logic. TFAE:*

1. L *is semilinear.*
2. L *has the* Semilinearity Property*, i.e., the following meta-rule is valid:*
$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$
3. L *has the* Linear Extension Property*, i.e., if for every theory* $T \in Th(\mathrm{L})$ *and every formula* $\varphi \in Fm_{\mathcal{L}} \setminus T$*, there is a linear theory* $T' \supseteq T$ *such that* $\varphi \notin T'$*.*
4. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^\ell(\mathrm{L})$.

Calculus for $\mathrm{FL_{ew}}$: structural rules

A sequent is a pair $\Gamma \Rightarrow \Delta$ where Γ is a multiset of formulas and Δ is a formula or the empty set.

The calculus has the following axiom and the structural rules:

(ID) $$\frac{}{\varphi \Rightarrow \varphi}$$

(Cut) $$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Delta \Rightarrow \chi}{\Gamma, \Delta \Rightarrow \chi}$$

(W-L) $$\frac{\Gamma \Rightarrow \chi}{\varphi, \Gamma \Rightarrow \chi}$$

(W-R) $$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}$$

Calculus for $\mathrm{FL_{ew}}$: operational rules

($\wedge$-L) $\dfrac{\varphi, \Gamma \Rightarrow \chi}{\varphi \wedge \psi, \Gamma \Rightarrow \chi}$, ditto ψ

($\wedge$-R) $\dfrac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$

($\&$-L) $\dfrac{\varphi, \psi, \Gamma \Rightarrow \chi}{\varphi \,\&\, \psi, \Gamma \Rightarrow \chi}$

($\&$-R) $\dfrac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \varphi \,\&\, \psi}$

($\vee$-L) $\dfrac{\varphi, \Gamma \Rightarrow \chi \qquad \psi, \Gamma \Rightarrow \chi}{\varphi \vee \psi, \Gamma \Rightarrow \chi}$

($\vee$-R) $\dfrac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}$, ditto ψ

($\to$-L) $\dfrac{\Gamma \Rightarrow \varphi \qquad \psi, \Delta \Rightarrow \chi}{\varphi \to \psi, \Gamma, \Delta \Rightarrow \chi}$

($\to$-R) $\dfrac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$

($\neg$-L) $\dfrac{\Gamma \Rightarrow \varphi}{\neg\varphi, \Gamma \Rightarrow}$

($\neg$-R) $\dfrac{\varphi, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$

From sequents to hypersequents

A hypersequent is a multiset of sequents. We add hypersequent context $\mathcal{G}$ to all rules:

(ID) $\dfrac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}$ (∨-R) $\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi}$, ditto ψ

What we need is Avron's communication rule

(COM) $\dfrac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \chi_1 \qquad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \chi_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \chi_1 \mid \Pi_1, \Pi_2 \Rightarrow \chi_2}$

Characterizations of completeness properties

Let L be core semilinear logic and $\mathbb{K}$ a class of L-chains.

Theorem 6.16 (Characterization of strong $\mathbb{K}$-completeness)

1. *For each $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$.*
2. $\mathbb{L} = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.
3. *Each countable L-chain is embeddable into some member of $\mathbb{K}$.*

Theorem 6.17 (Characterization of finite strong $\mathbb{K}$-completeness)

1. *For each finite $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$*
2. $\mathbb{L} = \mathbf{Q}(\mathbb{K})$, *i.e., $\mathbb{K}$ generates $\mathbb{L}$ as a quasivariety.*
3. *Each countable L-chain is embeddable into some ultrapower of $\mathbb{K}$.*
4. *Each finite subset of an L-chain is partially embeddable into an element of $\mathbb{K}$.*

Completeness properties

Let L be a core semilinear logic and $\mathbb{K}$ a class of L-chains.

Definition 6.18

- L has the $\mathrm{S}\mathbb{K}\mathrm{C}$ if:

$$\text{for every } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}},\ \Gamma \vdash_{\mathrm{L}} \varphi \text{ iff } \Gamma \models_{\mathbb{K}} \varphi$$

- L has the $\mathrm{FS}\mathbb{K}\mathrm{C}$ if:

$$\text{for every finite } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}},\ \Gamma \vdash_{\mathrm{L}} \varphi \text{ iff } \Gamma \models_{\mathbb{K}} \varphi$$

Distinguished semantics

Typical instances: $\mathbb{K} \in \{\mathcal{R}, \mathcal{Q}, \mathcal{F}\}$ (real, rational, finite-chain semantics).

Theorem 6.19 (Strong finite-chain completeness)

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$*,*
2. *all* L*-chains are finite,*
3. *there exists* $n \in \mathrm{N}$ *such each* L*-chain has at most* n *elements,*
4. *there exists* $n \in \mathrm{N}$ *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$*.*

Theorem 6.20 (Relation of Rational and Real completeness)

1. L *has the* $\mathrm{FS}\mathcal{Q}\mathrm{C}$ *iff it has the* $\mathrm{S}\mathcal{Q}\mathrm{C}$*.*
2. *If* L *has the* $\mathcal{R}\mathrm{C}$*, then it has the* $\mathcal{Q}\mathrm{C}$*.*
3. *If* L *has the* $\mathrm{FS}\mathcal{R}\mathrm{C}$*, then it has the* $\mathrm{S}\mathcal{Q}\mathrm{C}$*.*

Known results and open problems

Logic	$\mathcal{SRC}$	F$\mathcal{SRC}$	$\mathcal{SQC}$	F$\mathcal{SQC}$	F$\mathcal{SFC}$
FL^{ℓ}	No	No	No	No	No
$\mathrm{FL}^{\ell}_{\mathrm{c}}$	No	No	No	No	?
$\mathrm{FL}^{\ell}_{\mathrm{e}} = \mathrm{UL}$	Yes	Yes	Yes	Yes	No
$\mathrm{FL}^{\ell}_{\mathrm{w}} = \mathrm{psMTL}^{r}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{FL}^{\ell}_{\mathrm{ew}} = \mathrm{MTL}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{FL}^{\ell}_{\mathrm{ec}}$	?	?	?	?	?
$\mathrm{FL}^{\ell}_{\mathrm{wc}} = \mathrm{G}$	Yes	Yes	Yes	Yes	Yes

Problem 6.21

Solve the missing cases.

Known results and open problems

Logic	$\mathcal{SRC}$	F$\mathcal{SRC}$	S$\mathcal{QC}$	FS$\mathcal{QC}$	FS$\mathcal{FC}$
InFL^{ℓ}	No	No	No	No	No
InFL^{ℓ}_{c}	No	No	No	No	?
$\mathrm{InFL}^{\ell}_{e} = \mathrm{IUL}$	?	?	?	?	No
InFL^{ℓ}_{w}	Yes	Yes	Yes	Yes	?
$\mathrm{InFL}^{\ell}_{ew} = \mathrm{IMTL}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{InFL}^{\ell}_{ec}$	?	?	?	?	?
$\mathrm{InFL}^{\ell}_{wc}$ = CL	No	No	No	No	Yes

Problem 6.22

Solve the missing cases.

Known results in non-associative logics

Logic	$\mathcal{SRC}$	$\mathrm{F}\mathcal{SRC}$	$\mathcal{SQC}$	$\mathrm{F}\mathcal{SQC}$	$\mathrm{FS}\mathcal{FC}$
SL^{ℓ}	Yes	Yes	Yes	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{c}}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{e}}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{w}}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{ew}}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{ec}}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{wc}} = \mathrm{G}$	Yes	Yes	Yes	Yes	Yes

More interesting questions (no one addressed yet)

Logic	$\mathcal{SRC}$	$\mathrm{FS}\mathcal{RC}$	$\mathcal{SQC}$	$\mathrm{FS}\mathcal{QC}$	$\mathrm{FS}\mathcal{FC}$
InSL^{ℓ}	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{c}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{e}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{w}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{ew}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{ec}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{wc}}$ = CL	No	No	No	No	Yes

An extensive research field . . .

- based on the structural description of HL-chains
- classification and axiomatization of subvarieties
- amalgamation, interpolation, and Beth properties
- completions theory
- etc.

We have heard a lot about it already . . .

If you want to know more, read:

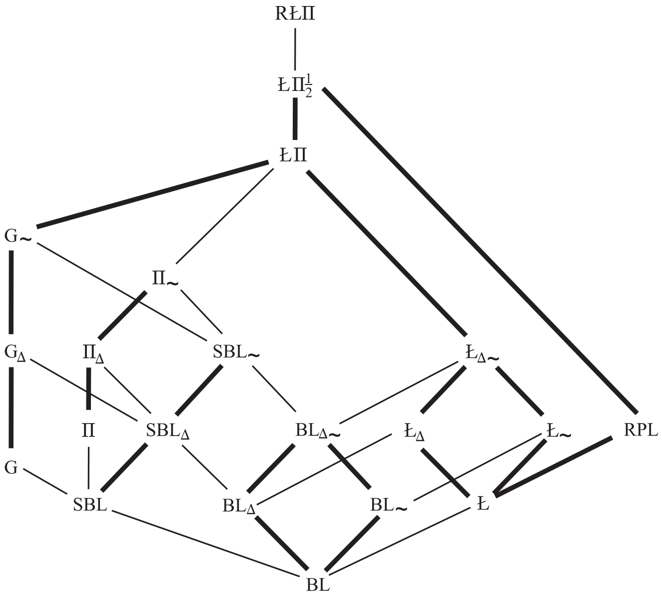
D. Mundici. Advanced Łukasiewicz calculus and MV-algebras. Trends in Logic, Vol. 35 Springer, New York, 2011.

We have heard a lot about it already . . .

If you want to know more, read:

anything from Vienna school: M. Baaz, N. Preining, C. Fermüller, R. Zach, etc.

A plethora of results not only about ...



Basic notions

We fix a logic L which is standard complete w.r.t. $[0,1]_{\mathrm{L}}$.

Definition 6.23

Function $f\colon [0,1]^n \to [0,1]$ is *represented* by formula φ of logic L if $e(\varphi) = f(e(v_1), e(v_2), \dots, e(v_m))$ for each $[0,1]_{\mathrm{L}}$-evaluation e.

Definition 6.24

Functional representation of logic L is a class of functions from any power of $[0,1]$ into $[0,1]$ s.t. each $C \in \mathcal{C}$ is represented by some formula φ and vice-versa (i.e., for each φ there is $C \in \mathcal{C}$ represented by φ).

An overview

Łukasiewicz logic:

Operations: truncated sum $\min\{1, x + y\}$ and involutive negation $1 - x$
Functions: continuous piece-wise linear functions with integer coeff.

$$f(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n + a_0\,, \quad a_i \in \mathbf{Z}$$

Ext.	**Added operations**	**Functions**
P	multiplication	polynomial
RP	rational constants	rational shift ($a_0 \in \mathbf{Q}$)
$\triangle$	$\triangle(x) = 1 \quad \text{if } x = 1$ $\triangle(x) = 0 \quad \text{if } x < 1$	non-continuity
δ	dividing by integers	rational coefficients ($a_i \in \mathbf{Q}$)
ŁΠ	fractions	fractions of functions

Some known results

Definition 6.25

A subset S of $[0,1]^n$ is Q-*semialgebraic* if it is a Boolean combination of sets of the form

$$\{\langle x_1, \ldots, x_n\rangle \in [0,1]^n \mid P(\langle x_1, \ldots, x_n\rangle) > 0\}$$

for polynomials P with integer coefficients. If all of the polynomials are linear, then S is *linear* Q-*semialgebraic.*

Logic	Contin.	Domains	Pieces
Ł	yes	linear	linear functions with integer coefficients
$Ł_\triangle$	no	linear	linear functions with integer coefficients
RPL	yes	linear	linear integer coefficients and a rational shift
$\delta Ł$	yes	linear	linear rational coefficients
$PŁ'$	yes	?	??? Pierce-Birkhoff conjecture ???
$PŁ'_\triangle$	no	all	polynomials with integer coefficients
$Ł\Pi$	no	all	fractions of polynomials with integer coeff.
$Ł\Pi\frac{1}{2}$	no	all	as above plus $f[\{0,1\}^n] \subseteq \{0,1\}$

Known results (see Handbook ch. X)

Logic L	THM(L)	CONS(L)	expansion by rational constants
HL	**coNP**-c.	**coNP**-c.	–
Ł	**coNP**-c.	**coNP**-c.	**coNP**-c.
L ⊃Ł	**coNP**-c.	**coNP**-c.	–
G	**coNP**-c.	**coNP**-c.	**coNP**-c.
Π	**coNP**-c.	**coNP**-c.	∈ **PSPACE**
L(∗) ⊃ HL	**coNP**-c.	**coNP**-c.	–
$Ł\Pi\frac{1}{2}$	∈ **PSPACE**	∈ **PSPACE**	–
MTL	decidable	decidable	–
IMTL	decidable	decidable	–
ΠMTL	decidable	decidable	–
NM	**coNP**-c.	**coNP**-c.	**coNP**-c.
WNM	**coNP**-c.	**coNP**-c.	–

Problem 6.26

Determine the precise complexity in all cases.

Known results (see Handbook ch. XI)

Logic	$stTAUT_1$	$stSAT_1$	$stTAUT_{pos}$	$stSAT_{pos}$
(I)MTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WCMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	Π_1-hard
ΠMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	Π_1-hard
(S)HL∀	Non-arithm.	Non-arithm.	Non-arithm.	Non-arithm.
Ł∀	Π_2-complete	Π_1-complete	Σ_1-complete	Σ_2-complete
Π∀	Non-arithm.	Non-arithm.	Non-arithm.	Non-arithm.
G∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
C$_n$MTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
C$_n$IMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WNM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
NM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete

Problem 6.27

Determine the precise complexity in all cases.

Volume III of the Handbook (in preparation)

- XII Algebraic Semantics: Structure of Chains (Vetterlein)
- XIII Dialogue Game-based Interpretations of Fuzzy Logics (Fermüller)
- XIV Ulam–Rényi games (Cicalese, Montagna)
- XV Fuzzy Logics with Evaluated Syntax (Novák)
- XVI Fuzzy Description Logics (Bobillo, Cerami, Esteva, García-Cerdaña, Peñaloza, Straccia)
- XVII States of MV-algebras (Flaminio, Kroupa)
- XVIII Fuzzy Logics in Theories of Vagueness (Smith)

(edited by Cintula, Fermüller, and Noguera)

Those that would deserve a Handbook chapter in some of the future volumes, but are not ready yet . . .

- Model Theory of Fuzzy Logics
- Model Theory in Fuzzy Logics
- Fuzzy Modal Logics
- Duality Theory
- Fragments of Fuzzy Logics
- Higher-Order Fuzzy logics
- Fuzzy Set Theories
- Fuzzy Arithmetics

Example I: model theory in fuzzy logics

Definition 6.28

Let $\langle \boldsymbol{B}_1, \mathbf{M}_1 \rangle$ and $\langle \boldsymbol{B}_2, \mathbf{M}_2 \rangle$ be two $\mathcal{P}$-models. $\langle \boldsymbol{B}_1, \mathbf{M}_1 \rangle$ is *elementarily equivalent* to $\langle \boldsymbol{B}_2, \mathbf{M}_2 \rangle$ if for each φ:

$$\langle \boldsymbol{B}_1, \mathbf{M}_1 \rangle \models \varphi \quad \text{iff} \quad \langle \boldsymbol{B}_2, \mathbf{M}_2 \rangle \models \varphi$$

Definition 6.29

An *elementary embedding* of a $\mathcal{P}_1$-model $\langle \boldsymbol{B}_1, \mathbf{M}_1 \rangle$ into a $\mathcal{P}_2$-model $\langle \boldsymbol{B}_2, \mathbf{M}_2 \rangle$ is a pair (f, g) such that:

1. f is an injection of the domain of $\mathbf{M}_1$ into the domain of $\mathbf{M}_2$.
2. g is an embedding of $\boldsymbol{B}_1$ into $\boldsymbol{B}_2$.
3. $g(\|\varphi(a_1, \dots, a_n)\|^{\langle \boldsymbol{B}_1, \mathbf{M}_1 \rangle}) = \|\varphi(f(a_1), \dots, f(a_n))\|^{\langle \boldsymbol{B}_2, \mathbf{M}_2 \rangle}$ holds for each $\mathcal{P}_1$-formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in \mathfrak{M}$.

The characterization of conservative expansions

Theorem 6.30

Let L *be a canonical fuzzy logic,* T_1 *and* T_2 *theories over* $\mathrm{L}\forall$*. Then the following claims are equivalent:*

1. T_2 *is a conservative extension of* T_1*.*
2. *Each model of* T_1 *is elementarily equivalent with restriction of some model of* T_2 *to the language of* T_1 *.*
3. *Each exhaustive model of* T_1 *is elementarily equivalent with restriction of some model of* T_2 *to the language of* T_1*.*
4. *Each exhaustive model of* T_1 *can be elementarily embedded into some model of* T_2*.*

But it is not equivalent to

6. Each model of T_1 can be elementarily embedded into some model of T_2.

Example II: evaluation games for Łukasiewicz logic

Let $\mathbf{M}$ be a witnessed $\mathcal{P}$-structure, then the labelled evaluation game for $\mathbf{M}$ is

- win-lose extensive game of two players (Eloise $\mathcal{E}$ and Abelard $\mathcal{A}$);
- its states are tuples $\langle \varphi, \mathrm{e}, \bowtie, r \rangle$, where
  - φ is a $\mathcal{P}$-formula
  - e is an $\mathbf{M}$-evaluation
  - $\bowtie \in \{\leq, \geq\}$
  - $r \in [0, 1]$
- it has terminal states $\langle \varphi, \mathrm{e}, \bowtie, r \rangle$ where either
  - φ is atomic formula,
  - $\bowtie = \leq$ and $r = 1$, or
  - $\bowtie = \geq$ and $r = 0$
- Eloise winning in first type of TS if $||\varphi||_{\mathbf{M},v} \bowtie r$ and is 'automatically' winning in the other two
- the game moves given by the following rules ...

Rules of the game — negation and disjunction(s)

($\neg$) $(\neg\psi, \mathrm{v}, \bowtie, r)$: the game continues as $(\psi, \mathrm{v}, \bowtie^{-1}, 1 - r)$

($\oplus$) $(\psi_1 \oplus \psi_2, \mathrm{v}, \bowtie, r)$:
$\mathcal{E}$ chooses $r' \leq r$,
$\mathcal{A}$ chooses whether to play $(\psi_1, \mathrm{v}, \bowtie r')$ or $(\psi_2, \mathrm{v}, \bowtie r - r')$.

($\vee^{\geq}$) $(\psi_1 \vee \psi_2, \mathrm{v}, \geq, r)$:
$\mathcal{E}$ chooses whether to play (ψ_1, v, r) or (ψ_2, v, r).

($\vee^{\leq}$) $(\psi_1 \vee \psi_2, \mathrm{v}, \leq, r)$:
$\mathcal{A}$ chooses whether to play (ψ_1, v, r) or (ψ_2, v, r).

Rules of the game — general quantifier

$((\forall x)\psi, \mathrm{v}, \geq, r)$: $\mathcal{E}$ claims that $\min\{||\psi||_{v[x]} \mid x \in M\} \geq r$

$\mathcal{A}$ has to provide a counterexample - an a such that $(||\psi||_{\mathrm{v}[x\to a]} < r)$

($\forall^{\geq}$) $((\forall x)\psi, \mathrm{v}, \geq, r)$:
$\mathcal{A}$ chooses $a \in M$,
game continues as $(\psi, \mathrm{v}[x\to a], \geq, r)$.

$((\forall x)\psi, \mathrm{v}, \leq, r)$: $\mathcal{E}$ claims that $\min\{||\psi||_{v[x]} \mid x \in M\} \leq r$

$\mathcal{E}$ has to provide a witness - an a such that $(||\psi||_{\mathrm{v}[x\to a]} \leq r)$

($\forall^{\leq}$) $((\forall x)\psi, \mathrm{v}, \leq, r)$:
$\mathcal{E}$ chooses $a \in M$,
game continues as $(\psi, \mathrm{v}[x\to a], \leq, r)$.

Correspondence theorem

Let us by $G_{\mathbf{M}}(\varphi, \mathrm{v}, \bowtie, r)$ denote that labelled evaluation game for $\mathbf{M}$ with initial state $(\varphi, \mathrm{v}, \bowtie, r)$. Then by Gale–Steward theorem:

Theorem 6.31 (Determinedness)

Either Eloise or Abelard has a winning strategy for every $G_{\mathbf{M}}(\varphi, \mathrm{v}, \bowtie, r)$.

Theorem 6.32 (Correspondence)

Let $\mathbf{M}$ be a structure, φ a formula, v an $\mathbf{M}$-valuation, $\bowtie \in \{\leq, \geq\}$, and $r \in [0, 1]$. Then

Eloise has a winning strategy in $G_{\mathbf{M}}(\varphi, \mathrm{v}, \bowtie, r)$ iff $||\varphi||_{\mathbf{M},v} \bowtie r$.

Corollary 6.33

Let $\mathbf{M}$ be a structure and φ a formula. Then $\mathbf{M} \models \varphi$ iff Eloise has a winning strategy for the game $G_{\mathbf{M}}(\varphi, \mathrm{v}, \geq, 1)$ for each $\mathbf{M}$-valuation v

This all is just a beginning . . . (see Handbook ch. XIII)

Indeed, having a (labelled evaluation) game semantics opens doors to many interesting opportunities, in particular

- it allows for including inperfect information, which lead to
- study of branching quantifiers and
- a new way of combining probability and vagueness.
- It provides a useful characterization of *safe* structures (in other than $[0, 1]$-based models) and
- gives some notion of 'truth' even in the non-safe ones.
- It gives a novel 'explanation/justification' of the semantics of Łukasiewicz logic
- . . .

Bibliography

- Libor Běhounek, Ondrej Majer. A semantics for counterfactuals based on fuzzy logic. In M. Peliš, V. Punčochař (eds.), The Logica Yearbook 2010, pp. 25–41, College Publications, 2011.
- Libor Běhounek. Fuzzy logics interpreted as logics of resources. In M. Peliš (ed.): The Logica Yearbook 2008, pp. 9–21, College Publications 2009.
- Libor Běhounek, Petr Cintula. From fuzzy logic to fuzzy mathematics: a methodological manifesto. *Fuzzy Sets and Systems* 157(5): 642–646, 2006.
- Roberto Cignoli, Itala M.L. D'Ottaviano and Daniele Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Trends in Logic, vol. 7, Kluwer, Dordrecht, 1999.

Bibliography

- Petr Cintula, Francesc Esteva, Joan Gispert, Lluís Godo, Franco Montagna and Carles Noguera. Distinguished Algebraic Semantics For T-Norm Based Fuzzy Logics: Methods and Algebraic Equivalencies, *Annals of Pure and Applied Logic* 160(1): 53–81, 2009.
- Petr Cintula, Petr Hájek, Carles Noguera (eds). Handbook of Mathematical Fuzzy Logic. Studies in Logic, Mathematical Logic and Foundations, vol. 37 and 38, London, College Publications, 2011.
- Petr Cintula, Rostislav Horčík and Carles Noguera. Non-Associative Substructural Logics and their Semilinear Extensions: Axiomatization and Completeness Properties, *The Review of Symbolic Logic* 6(3):394–423, 2013.
- Michael Dummett. A propositional calculus with denumerable matrix. Journal of Symbolic Logic, 24:97–106, 1959.

Bibliography

- Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Studies in Logic and the Foundations of Mathematics, vol. 151, Amsterdam, Elsevier, 2007.
- Joseph Amadee Goguen. The logic of inexact concepts. Synthese, 19:325–373, 1969.
- Siegfried Gottwald. A Treatise on Many-Valued Logics, volume 9 of Studies in Logic and Computation. Research Studies Press, Baldock, 2001.
- Petr Hájek. Metamathematics of Fuzzy Logic, volume 4 of Trends in Logic. Kluwer, Dordrecht, 1998.
- Jan Łukasiewicz and Alfred Tarski. Untersuchungen über den Aussagenkalkül. Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, 23:30–50, 1930.

Bibliography

- George Metcalfe, Nicola Olivetti and Dov M. Gabbay, *Proof Theory for Fuzzy Logics*, Applied Logic Series, vol. 36, Springer, 2008.
- Daniele Mundici, *Advanced Łukasiewicz Calculus and MV-Algebras*, Springer, New York, 2011.
- Vilém Novák, Irina Perfilieva, and Jiří Močkoř. Mathematical Principles of Fuzzy Logic. Kluwer, Dordrecht, 2000.
- Lotfi A. Zadeh. Fuzzy sets. Information and Control, 8:338–353, 1965.

Conclusions

- MFL is a well-developed field, with a genuine agenda of Mathematical Logic: axiomatization, completeness, proof theory, functional representation, computational complexity, model theory, etc.
- All these areas are active, mathematically deep and pose challenging open problems.
- The objects studied by MFL are semilinear logics, i.e. logics of chains.
- Graduality in the semantics (linearly ordered truth-values) is a flexible tool amenable for many interesting applications.
- MFL, its extensions and applications has still a long way to go, the best is yet to come, and so there are plenty of topics for potential Ph.D. theses.