Minimal Varieties of Representable Commutative Residuated Lattices

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Logic, Algebra and Truth Degrees
Prague, 2010

Introduction

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- The above-mentioned examples correspond respectively to the atoms $V(\mathbf{2})$, $V(\mathbf{Z})$, $V(\mathbf{Z}^-)$.
- It is known that there are continuum many atoms $\Lambda(\mathsf{FL})$.
- What about atoms satisfying some additional properties like representability, commutativity, integrality?

Introduction (cont.)

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- In this talk we are going to show that this was a false conjecture.

- Furthermore, we solve related open problems on cardinality of atoms in $\Lambda(\mathsf{FL_{ei}})$ and $\Lambda(\mathsf{FL_{eo}})$.

FL-algebras

Definition

An algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle$ is called FL-algebra if

1. $\langle A, \wedge, \vee \rangle$ is a lattice,
2. $\langle A, \cdot, 1 \rangle$ is a monoid,
3. $x \cdot y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z / y$.

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3. $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$.

Other properties

- $\mathbf{A}$ is a residuated lattice if $1 = 0$,
- $\mathbf{A}$ is an $\mathrm{FL_e}$-algebra (commutative) if $x \cdot y = y \cdot x$,
- $\mathbf{A}$ is an $\mathrm{FL_i}$-algebra (integral) if $x \leq 1$,
- $\mathbf{A}$ is an $\mathrm{FL_o}$-algebra if $0 \leq x$,
- $\mathbf{A}$ is n-potent if $x^{n+1} = x^n$,
- $\mathbf{A}$ is representable (semilinear) if it is a subdirect product of chains.

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- Let $\gamma : A \to A$ be an operator on A. The image of γ is denoted A_γ.

Nuclear retraction and conuclear contraction

Lemma

- *An operator σ on **A** is conucleus iff A_σ is a submonoid of **A** and*

$$\max\{a \in A_\sigma \mid a \leq x\} \text{ exists for all } x \in A.$$

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- *An operator γ on **A** is nucleus iff A_γ satisfies*
$$\min\{a \in A_\gamma \mid x \leq a\} \text{ exists for all } x \in A.$$
and
$$x \to y \in A_\gamma \text{ for all } x \in A \text{ and } y \in A_\gamma.$$
A_γ is called nuclear retraction.

Resulting residuated algebras

Lemma

- *Let* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1\rangle$ *be an FL-algebra,* γ *a nucleus on* $\mathbf{A}$ *and* σ *a conucleus on* $\mathbf{A}$*.*

Resulting residuated algebras

Lemma

- *Let* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1\rangle$ *be an FL-algebra,* γ *a nucleus on* $\mathbf{A}$ *and* σ *a conucleus on* $\mathbf{A}$.

- *Then the algebra* $\mathbf{A}_\gamma = \langle A_\gamma, \wedge, \vee_\gamma, \circ_\gamma, /, \backslash, , \gamma(0), \gamma(1)\rangle$ *is an FL-algebra, where*
  - $x \vee_\gamma y = \gamma(x \vee y)$,
  - $x \circ_\gamma y = \gamma(x \cdot y)$.

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- *Then the algebra* $\mathbf{A}_\gamma = \langle A_\gamma, \wedge, \vee_\gamma, \circ_\gamma, /, \backslash, , \gamma(0), \gamma(1)\rangle$ *is an FL-algebra, where*
  - $x \vee_\gamma y = \gamma(x \vee y)$,
  - $x \circ_\gamma y = \gamma(x \cdot y)$.
- *Further, the algebra* $\mathbf{A}_\sigma = \langle A_\sigma, \wedge_\sigma, \vee, \cdot, /_\sigma, \backslash_\sigma, \sigma(0), 1\rangle$ *is an FL-algebra, where*
  - $x \wedge_\sigma y = \sigma(x \wedge y)$,
  - $x /_\sigma y = \sigma(x/y)$,
  - $x \backslash_\sigma y = \sigma(x \backslash y)$.

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Our results can be summarized as follows:

- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(RL)$.
- There are $2^{\aleph_0}$ representable atoms in $\Lambda(FL_{ei})$.
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- There are $2^{\aleph_0}$ representable atoms in $\Lambda(\mathsf{FL_{ei}})$.
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On the other hand, we also prove the following result:

- There are 2^1 representable commutative integral atoms in $\Lambda(\mathsf{RL})$.

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- The conucleus σ_S is defined by its conuclear contraction:

$$G_{\sigma_S} = \{\langle 0,0\rangle, \langle -1,0\rangle, \langle -1,-1\rangle\} \cup \{\langle -1,z\rangle \in A \mid z \in S\} \cup \{\langle x,y\rangle \in A \mid x \leq -2\}\,.$$

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- Since S is infinite and dually well ordered, we get the following lemma.

Lemma

The set G_{σ_S} forms a conuclear contraction.

Construction of $\mathbf{A}_S$ (cont.)

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1. *The algebra $\mathbf{A}_S$ is simple 4-potent integral commutative residuated lattice.*
2. *Let $n \in \mathbb{N}$. Then $\langle -2, n\rangle, \langle -3, n\rangle \in A_S$.*

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3. *Let $z \in S$. Then $\langle -1, z\rangle \in A_S$.*
4. $\mathbf{A}_R \cong \mathbf{A}_S$ *iff* $R = S$.

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Theorem (Galatos)

1. *Let $\mathbf{A}$ be a strictly simple FL-algebra with bottom element $\bot$ nearly term definable by an n-ary term t. Then, $\mathsf{V}(\mathbf{A})$ is an atom.*

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Theorem (Galatos)

1. *Let $\mathbf{A}$ be a strictly simple FL-algebra with bottom element $\perp$ nearly term definable by an n-ary term t. Then, $\mathsf{V}(\mathbf{A})$ is an atom.*
2. *Moreover, if $\mathbf{A}'$ is a strictly simple FL-algebra with bottom element nearly term definable by the same term t, then $\mathsf{V}(\mathbf{A}) \subseteq \mathsf{V}(\mathbf{A}')$ iff $\mathbf{A}$ and $\mathbf{A}'$ are isomorphic.*

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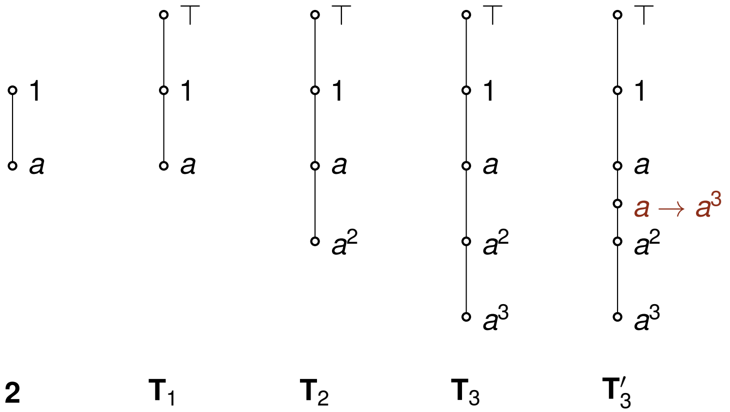
Theorem

There are only finitely many 3-potent representable commutative atoms in $\Lambda(\mathsf{RL})$. Namely, varieties generated by $\mathbf{2}, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}'_3$.

3-potent atoms



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Proof.

1. We use the FL-algebras living on $\mathbf{A}_S$ where 0 is interpreted by any element different from $\langle -3, -1\rangle, \langle 0, 0\rangle$.

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2. We use the FL-algebras living on $\mathbf{A}_S^\top$ where 0 is interpreted by $\langle -3, -1\rangle$.

□

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- Let θ be the congruence on $\mathbf{B}$ corresponding to the convex subalgebra generated by the congruence classes containing the constant mappings and $\mathbf{a} = \langle a^k \rangle_{k\in\mathbb{N}^+}/U$.

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- Let θ be the congruence on $\mathbf{B}$ corresponding to the convex subalgebra generated by the congruence classes containing the constant mappings and $\mathbf{a} = \langle a^k \rangle_{k\in\mathbb{N}^+}/U$.
- Then the subalgebra of $\mathbf{B}/\theta$ generated by $\mathbf{a}$ is isomorphic to $\mathbf{Z}^-$.

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Theorem

Each finitely generated ICRC can be embedded into a 1-generated ICRC.

Corollary

The variety of representable integral commutative residuated lattices is generated by 1-generated finite totally ordered members.

Lexicographic product

Lemma

Let $\mathbf{A}, \mathbf{B}$ be ICRCs such that $\mathbf{A}$ is cancellative. Then the lexicographic product $\mathbf{A} \overset{\rightarrow}{\times} \mathbf{B}$ is an ICRC.

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$$\langle a, x\rangle \to \langle b, y\rangle = \begin{cases} \langle a \to_A b, 1_B\rangle & \text{if } a \cdot_A (a \to_A b) <_A b, \\ \langle a \to_A b, x \to_B y\rangle & \text{otherwise.} \end{cases}$$

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Lemma

Let $\mathbf{A}, \mathbf{B}$ *be ICRCs such that* $\mathbf{A}$ *is cancellative. Then the lexicographic product* $\mathbf{A} \overrightarrow{\times} \mathbf{B}$ *is an ICRC.*

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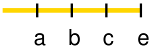
In particular, if $\mathbf{A} = \mathbf{Z}^-$, then for $\langle a, x\rangle > \langle b, y\rangle$ we have

$$\langle a, x\rangle \to \langle b, y\rangle = \langle b - a, x \to_B y\rangle\,.$$

Sketch of the proof



Let **A** be an ICRC generated by $\{a, b, c\}$.
We will construct a 1-generated ICRC in which **A** can be embedded.

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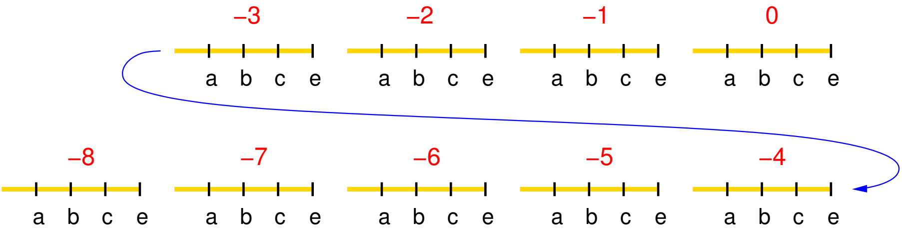

Consider the lexicographic product $\mathbf{Z}^- \overrightarrow{\times} \mathbf{A}$.
The elements are tuples $\langle x, y\rangle$ where $x \in \mathbf{Z}^-$ and $y \in \mathbf{A}$.

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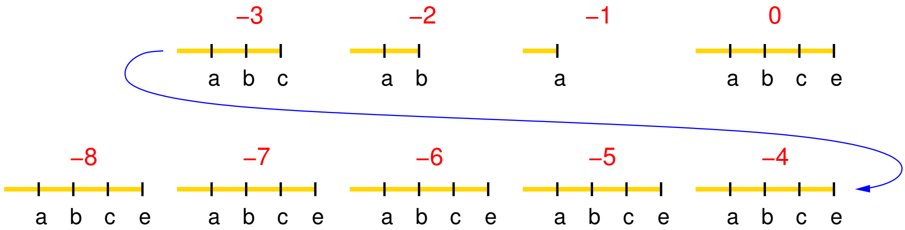
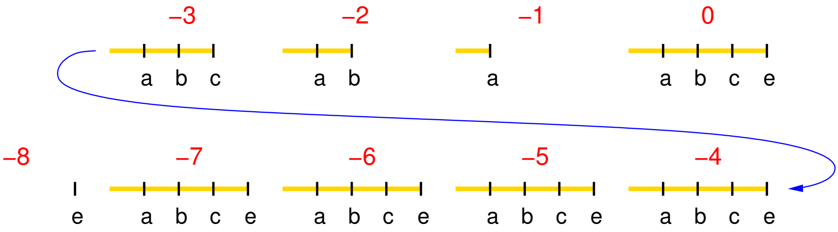


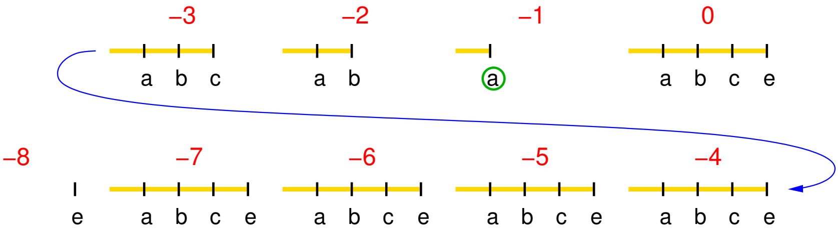
Take the conuclear contraction of $\mathbf{Z}^{-} \overset{\rightarrow}{\times} \mathbf{A}$ by deleting $\{\langle -1, y\rangle \mid y > a\} \cup \{\langle -2, y\rangle \mid y > b\} \cup \{\langle -3, y\rangle \mid y > c\}$. Denote the corresponding conucleus σ.

Sketch of the proof



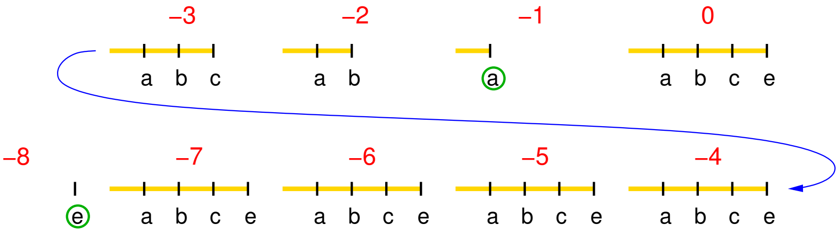
Consider the nucleus $\gamma(x) = x \vee \langle -8, e\rangle$ and its corresponding nuclear retraction.

Sketch of the proof



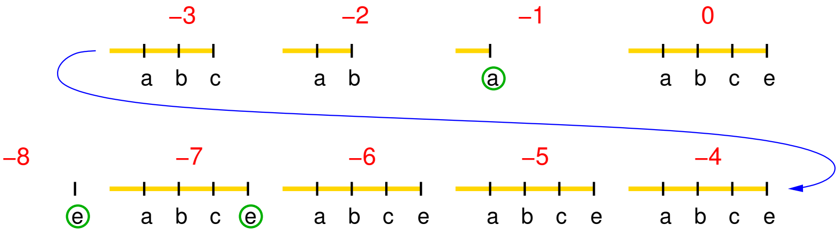
Finally, let **C** be the subalgebra generated by the element $g = \langle -1, a \rangle$. We will prove that **A** can be embedded into **C**.

Sketch of the proof



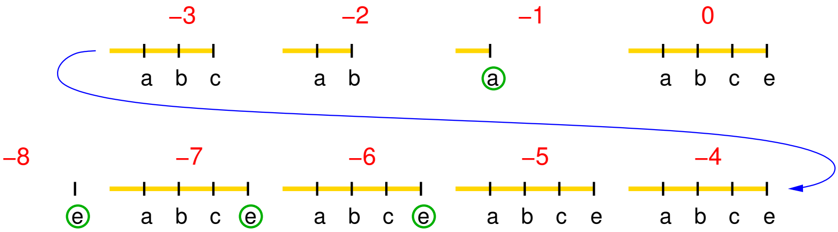
First, we have $g^8 = \gamma(\langle -1, a\rangle^8) = \gamma(\langle -8, a^8\rangle) = \langle -8, e\rangle$.

Sketch of the proof



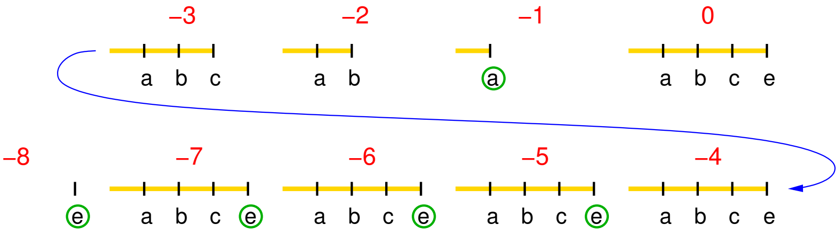
Then $g \to_\sigma g^8 = \sigma(\langle -1, a\rangle \to \langle -8, e\rangle) = \sigma(\langle -7, e\rangle) = \langle -7, e\rangle$.

Sketch of the proof



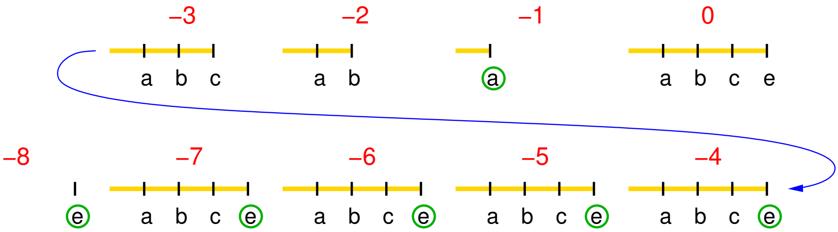
Then $g^2 \to_\sigma g^8 = \sigma(\langle -2, a^2 \rangle \to \langle -8, e \rangle) = \sigma(\langle -6, e \rangle) = \langle -6, e \rangle$.

Sketch of the proof



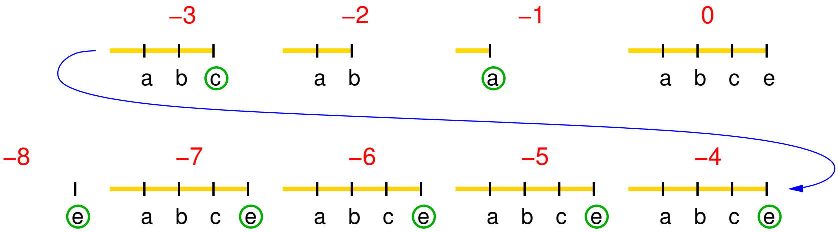
Then $g^3 \to_\sigma g^8 = \sigma(\langle -3, a^3\rangle \to \langle -8, e\rangle) = \sigma(\langle -5, e\rangle) = \langle -5, e\rangle$.

Sketch of the proof



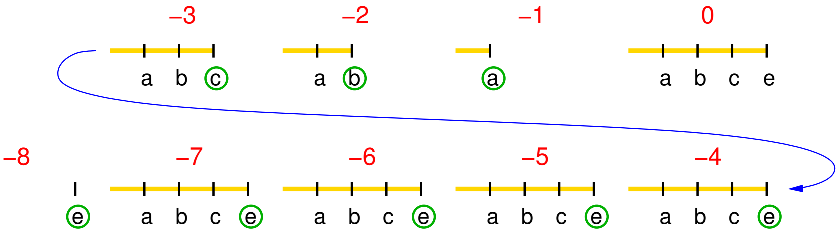
Then $g^4 \to_\sigma g^8 = \sigma(\langle -4, a^4\rangle \to \langle -8, e\rangle) = \sigma(\langle -4, e\rangle) = \langle -4, e\rangle$.

Sketch of the proof



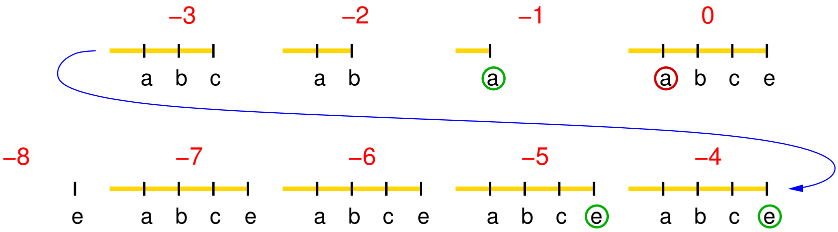
Then $g^5 \to_\sigma g^8 = \sigma(\langle -5, a^5\rangle \to \langle -8, e\rangle) = \sigma(\langle -3, e\rangle) = \langle -3, c\rangle$.

Sketch of the proof



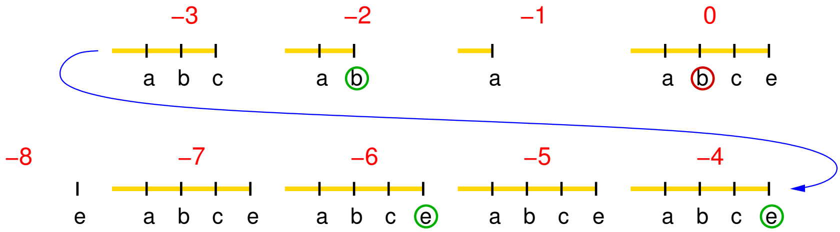
Then $g^6 \to_\sigma g^8 = \sigma(\langle -6, a^6\rangle \to \langle -8, e\rangle) = \sigma(\langle -2, e\rangle) = \langle -2, b\rangle$.

Sketch of the proof



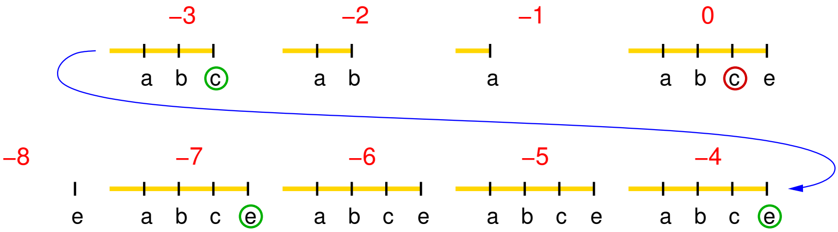
We have
$\langle -5, e\rangle \rightarrow_\sigma \langle -1, a\rangle\langle -4, e\rangle = \sigma(\langle -5, e\rangle \rightarrow \langle -5, a\rangle) = \sigma(\langle 0, a\rangle) = \langle 0, a\rangle$.

Sketch of the proof



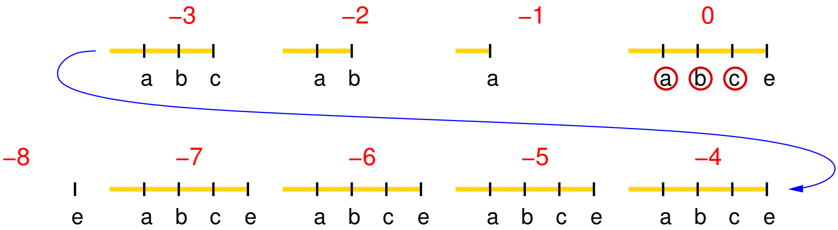
We have
$\langle -6, e\rangle \rightarrow_\sigma \langle -2, b\rangle\langle -4, e\rangle = \sigma(\langle -6, e\rangle \rightarrow \langle -6, b\rangle) = \sigma(\langle 0, b\rangle) = \langle 0, b\rangle$.

Sketch of the proof



We have
$\langle -7, e\rangle \rightarrow_\sigma \langle -3, c\rangle\langle -4, e\rangle = \sigma(\langle -7, e\rangle \rightarrow \langle -7, c\rangle) = \sigma(\langle 0, c\rangle) = \langle 0, c\rangle$.

Sketch of the proof



Thus $\langle 0, a\rangle, \langle 0, b\rangle, \langle 0, c\rangle \in \mathbf{C}$, i.e. **C** contains an isomorphic copy of **A**.

Thank you for your attention!