Densification via Polynomial Extensions

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Introduction

Theorem (Cantor)

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Question

Let $\mathbf{A}$ be a totally ordered (countable) monoid (resp. residuated lattice). Is there a dense totally ordered (countable) monoid (resp. residuated lattice) $\bar{\mathbf{A}}$ into which $\mathbf{A}$ is embeddable?

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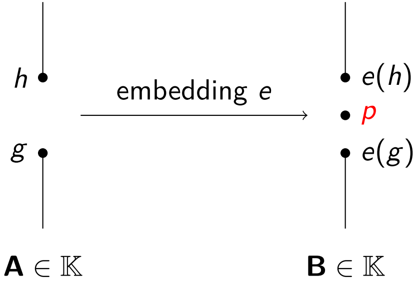
Theorem (Metcalfe, Montagna, Ciabattoni, Terui, Galatos, RH)

Let **A** *be a commutative totally ordered (countable) monoid (resp. residuated lattice). Then* **A** *is embeddable into a dense totally ordered (countable) monoid (resp. residuated lattice).*

One-step densification

Definition

A class $\mathbb{K}$ of ordered algebras is said to be densifiable if every gap (g, h) of a chain in $\mathbb{K}$ can be filled by another chain in $\mathbb{K}$.



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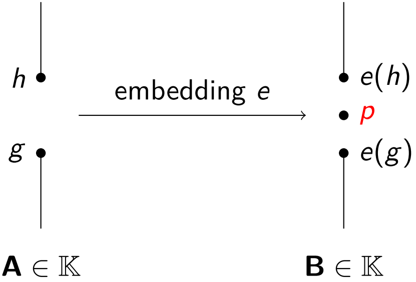

Theorem

Let $\mathbb{K}$ be a densifiable variety. Then every (nontrivial) at most countable chain in $\mathbb{K}$ is embeddable into a countable dense chain in $\mathbb{K}$.

Naive approach

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If $ab = 1$ for some $a, b \in A$, then

$$p = 1 \cdot p = (ab) \cdot p = a(b \cdot p) = a(bh) = (ab)h = h > p\,.$$

Idempotent semirings and residuated lattices

Definition

An idempotent commutative semiring (ic-semiring) is an algebra $\mathbf{A} = \langle A, \vee, \cdot, 1\rangle$ such that

- $\langle A, \vee\rangle$ is semilattice,
- $\langle A, \cdot, 1\rangle$ is a commutative monoid,
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A commutative residuated lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, 1 \rangle$ such that

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We assume that our algebras always have a bottom element $\perp$.

Nuclei

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1. A nucleus on **A** is a closure operator $\gamma\colon A \to A$ such that

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Fact

The semiring $\mathbf{A}/\theta$ forms a commutative residuated lattice $\mathbf{A}_\gamma$.

Fields extensions

1. Suppose we want to extend the field of rational numbers $\mathbb{Q}$ by $\sqrt{2}$.

   $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$,

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}\,.$$

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2. First, consider the polynomial ring $\mathbb{Q}[X]$ and the take a quotient by the principal congruence $\Theta(X^2, 2)$.

   Then the congruence class of X plays the role of $\sqrt{2}$.

   Every congruence class has a representative of the form $a + bX$ for $a, b \in \mathbb{Q}$.

   Note that $\mathbb{Q}[X]$ is not a field, the divisions are recoved in the quotient $\mathbb{Q}[X]/\Theta(X^2, 2)$.

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(p_0 \vee p_1 X) \to (q_0 \vee q_1 X) &= [(p_0 \to q_0) \wedge (p_1 \to q_1)] \vee [(p_0 \vee p_1 h) \to q_1] X
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Lemma

The algebra $\mathbf{A}(X) = \langle A(X), \wedge, \vee, \cdot, \to, 1 \rangle$ *is a commutative residuated lattice.*

Algebraic construction

Consider the subset $C \subseteq \mathbf{A}(X)$ consisting of the following linear polynomials for all $a \in A$:

$$\hat{a} = a \vee (h \to a)X\,,$$

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- *$\hat{g} < \tilde{1} < \hat{h}$.*

Thus $\mathbf{A}(X)_\gamma$ is a one-step densification of $\mathbf{A}$ at the gap $g < h$.

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Let $\mathbf{A} = \mathbf{L}_3$ be the 3-element MV-chain on the set $\{0, 1/2, 1\}$ and $g = 1/2$, $h = 1$.

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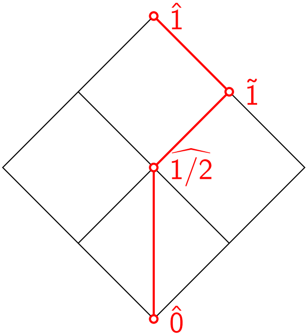

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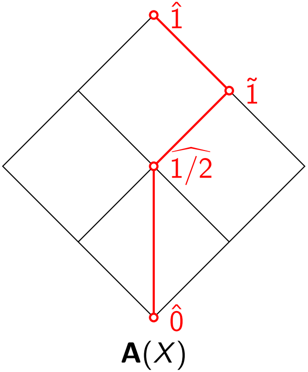

The resulting residuated chain is isomorphic to $\mathbf{L}_3 \oplus \mathbf{2}$.

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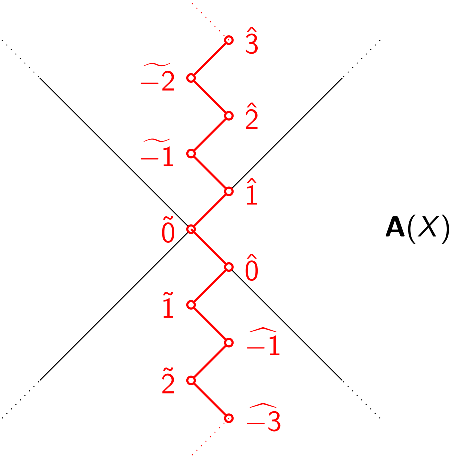

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Consider the congruence $\Theta(X^2, hX)$. Then $\mathbf{A}(X) \cong \mathbf{A}[X]/\Theta(X^2, hX)$.

Syntactic (Leibniz) congruence

Let $\mathbf{M}$ be a commutative monoid and $L \subseteq M$. Then the syntactic congruence $\sim_L$ of L is the following relation:

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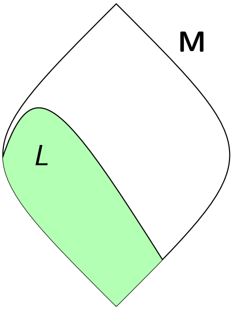
The congruence $\sim_L$ is the largest monoid congruence which saturates L.

In particular, $a \not\sim_L b$ for every $a \in L$ and $b \notin L$.



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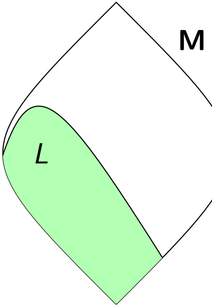

If **M** is an ic-semiring and L an ideal, then $\sim_L$ is a semiring congruence.

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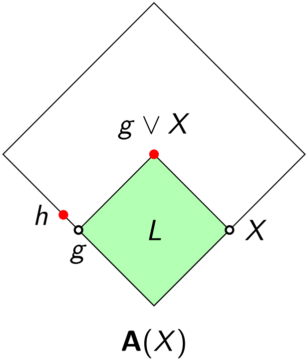

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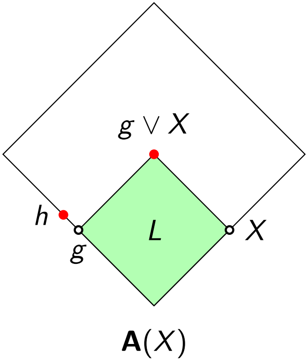

Lemma

$\mathbf{A}(X)_\gamma \cong \mathbf{A}(X)/\theta$ *where* $\theta = \Theta(X, h) \cap \sim_L$.

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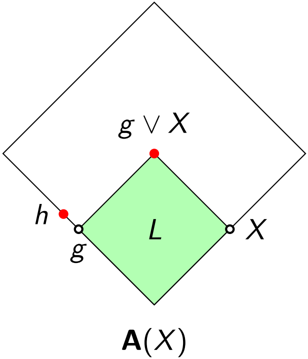
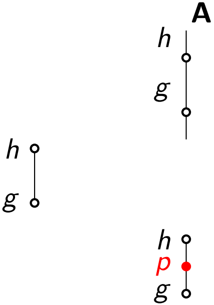


Lemma

$\mathbf{A}(X)_\gamma \cong \mathbf{A}(X)/\theta$ *where* $\theta = \Theta(X, h) \cap \sim_L$.

For other explanations see my webpage: www.cs.cas.cz/∼horcik

Mixed amalgamation



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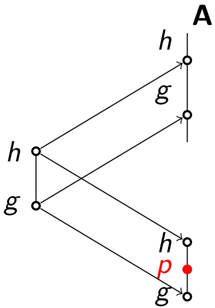

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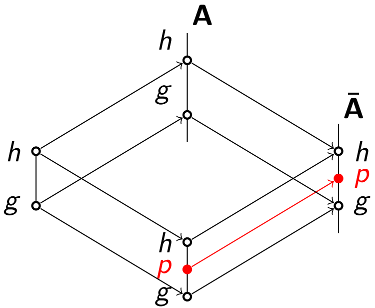

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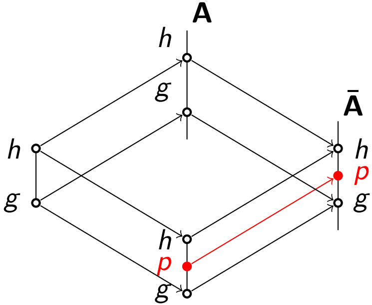
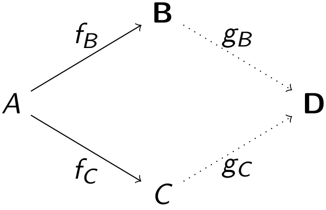
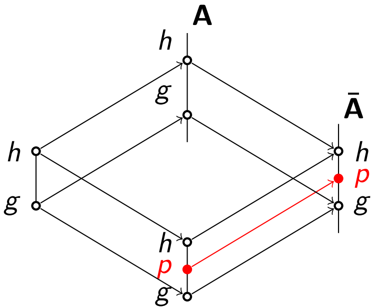
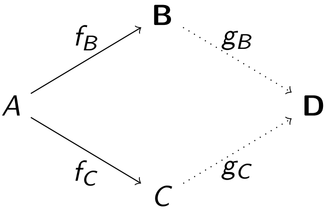




Mixed amalgamation





Theorem

Let A, C be chains, $\mathbf{B}$ a commutative residuated chain and let $f_B : A \to B$ and $f_C : A \to C$ be order embeddings. Then there is a commutative residuated chain $\mathbf{D}$, an order embedding $g_C : C \to D$ and a residuated lattice embedding $g_B : \mathbf{B} \to \mathbf{D}$ such that the right diagram commutes.

Thank you!