Interplay between algebra and proof theory for substructural logics

Rostislav Horčík

Institute of Computer Science
Academy of Sciences of the Czech Republic

A Gathering of Prague-Based Logicians
15–16 February 2013

Our logical group

Outline

1. Substructural logics

Outline

1. Substructural logics

2. Algebraic semantics

Outline

1. Substructural logics
2. Algebraic semantics
3. Residuated frames

Sequent calculus LJ [Gentzen 1935]

Sequent calculus LJ [Gentzen 1935]

Let $A_1, \dots, A_n, B$ be formulas. Then a sequent is a formal expression:

$$A_1, \dots, A_n \Rightarrow B\,.$$

The interpretation of the above sequent is that one can infer B from the set of assumptions $\{A_1, \dots, A_n\}$.

Sequent calculus LJ [Gentzen 1935]

Let $A_1, \ldots, A_n, B$ be formulas. Then a sequent is a formal expression:

$$A_1, \ldots, A_n \Rightarrow B\,.$$

The interpretation of the above sequent is that one can infer B from the set of assumptions $\{A_1, \ldots, A_n\}$.

The sequent calculus LJ for intuitionistic logic consists of an axiom schema $A \Rightarrow A$ and two types of inference rules:

1. rules introducing logical connectives (operational inference figures),
2. structural rules (structural inferences figures).

Structural rules

1. $$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, A \wedge B, \Delta \Rightarrow C}\,(\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,(\Rightarrow \wedge)$$

Structural rules

1. $$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, A \wedge B, \Delta \Rightarrow C} (\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} (\Rightarrow \wedge)$$

2. $$\frac{\Gamma, A, A, \Delta \Rightarrow C}{\Gamma, A, \Delta \Rightarrow C} \text{ Contraction} \qquad \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C} \text{ Exchange}$$

$$\frac{\Gamma, \Delta \Rightarrow C}{\Gamma, A, \Delta \Rightarrow C} \text{ Weakening} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \text{ Weakening}$$

$$\frac{\Gamma \Rightarrow A \qquad \Sigma, A, \Delta \Rightarrow C}{\Sigma, \Gamma, \Delta \Rightarrow C} \text{ Cut}$$

Structural rules

1. $$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, A \wedge B, \Delta \Rightarrow C}\,(\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,(\Rightarrow \wedge)$$

2. $$\frac{\Gamma, A, A, \Delta \Rightarrow C}{\Gamma, A, \Delta \Rightarrow C}\,\text{Contraction} \qquad \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}\,\text{Exchange}$$

$$\frac{\Gamma, \Delta \Rightarrow C}{\Gamma, A, \Delta \Rightarrow C}\,\text{Weakening} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\,\text{Weakening}$$

$$\frac{\Gamma \Rightarrow A \qquad \Sigma, A, \Delta \Rightarrow C}{\Sigma, \Gamma, \Delta \Rightarrow C}\,\text{Cut}$$

Theorem (Gentzen 1935)

LJ admits cut elimination, i.e., proofs have a direct/normal form.

Substructural logics (Kosta Došen)

Several important nonclassical logics have in common that they reject or restrict some of the structural rules while keeping the rules introducing logical connectives the same as in intuitionistic/classical logic.

Substructural logics (Kosta Došen)

Several important nonclassical logics have in common that they reject or restrict some of the structural rules while keeping the rules introducing logical connectives the same as in intuitionistic/classical logic.

Definition (Kosta Došen 1990)

Substructural logics are logics which can be obtained from intuitionistic/classical logic by restricting structural rules.

Substructural logics (Kosta Došen)

Several important nonclassical logics have in common that they reject or restrict some of the structural rules while keeping the rules introducing logical connectives the same as in intuitionistic/classical logic.

Definition (Kosta Došen 1990)

Substructural logics are logics which can be obtained from intuitionistic/classical logic by restricting structural rules.

Example

Linear logic, relevant logics, Lambek calculus ...

Full Lambek calculus FL

- Additive connectives: $\wedge, \vee, (\bot, \top)$,

Full Lambek calculus FL

- Additive connectives: $\wedge, \vee, (\bot, \top)$,
- Multiplicative connectives: $\cdot, \backslash, /, 1, 0$.

Full Lambek calculus FL

- Additive connectives: $\wedge, \vee, (\bot, \top)$,
- Multiplicative connectives: $\cdot, \backslash, /, 1, 0$.
- **FL** is given by a single-conclusion sequent calculus:

$$A \Rightarrow A \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

$$\frac{\Gamma \Rightarrow A \qquad \Pi, A, \Sigma \Rightarrow C}{\Pi, \Gamma, \Sigma \Rightarrow C}\ (\text{cut})$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \cdot B, \Delta \Rightarrow C}\ (\cdot\Rightarrow) \qquad \frac{\Gamma \Rightarrow A \qquad \Sigma \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}\ (\Rightarrow\cdot)$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \varphi \qquad \Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Sigma \Rightarrow \varphi}\ (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow\vee)$$

$$\vdots$$

Substructural logics (Hiroakira Ono)

Definition (Hiroakira Ono 2003)

Substructural logics are (axiomatic) extensions of **FL**. Nowadays even a nonassociative version of FL is considered as a base logics.

Substructural logics (Hiroakira Ono)

Definition (Hiroakira Ono 2003)

Substructural logics are (axiomatic) extensions of **FL**. Nowadays even a nonassociative version of FL is considered as a base logics.

Example

Multiplicative additive fragment of linear logic **MALL**, relevant logic **R**, Full Lambek calculus **FL**, fuzzy logics, superintuitionistic logics, classical logic ...

My motivation

Substructural logics were studied at the beginning mainly from the proof-theoretical perspective.

My motivation

Substructural logics were studied at the beginning mainly from the proof-theoretical perspective.

Many important results are done by (triple) induction on the complexity of the proof tree.

My motivation

Substructural logics were studied at the beginning mainly from the proof-theoretical perspective.

Many important results are done by (triple) induction on the complexity of the proof tree.

The used methods depend closely on the corresponding sequent calculus and it is difficult to transfer them to another substructural logics.

My motivation

Substructural logics were studied at the beginning mainly from the proof-theoretical perspective.

Many important results are done by (triple) induction on the complexity of the proof tree.

The used methods depend closely on the corresponding sequent calculus and it is difficult to transfer them to another substructural logics.

An interesting project

Look for another methods which are more versatile.

Algebraic semantics

Definition

An FL-algebra is a pointed residuated lattice, i.e., an algebra

$$\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle \,,$$

Algebraic semantics

Definition

An FL-algebra is a pointed residuated lattice, i.e., an algebra

$$\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle ,$$

- $\langle A, \wedge, \vee \rangle$ is a lattice,

Algebraic semantics

Definition

An FL-algebra is a pointed residuated lattice, i.e., an algebra

$$\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle,$$

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,

Algebraic semantics

Definition

An FL-algebra is a pointed residuated lattice, i.e., an algebra

$$\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle,$$

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,
- 0 is an arbitrary element and

Algebraic semantics

Definition

An FL-algebra is a pointed residuated lattice, i.e., an algebra

$$\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle ,$$

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,
- 0 is an arbitrary element and
- the following condition holds:

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z \, .$$

Algebraic semantics

Definition

An FL-algebra is a pointed residuated lattice, i.e., an algebra

$$\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle ,$$

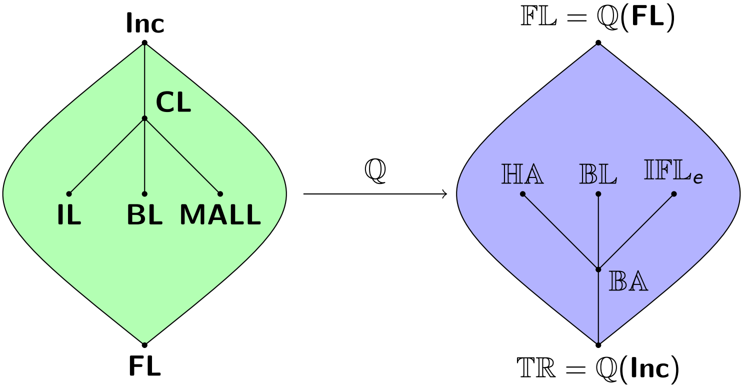
- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,
- 0 is an arbitrary element and
- the following condition holds:

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z \,.$$

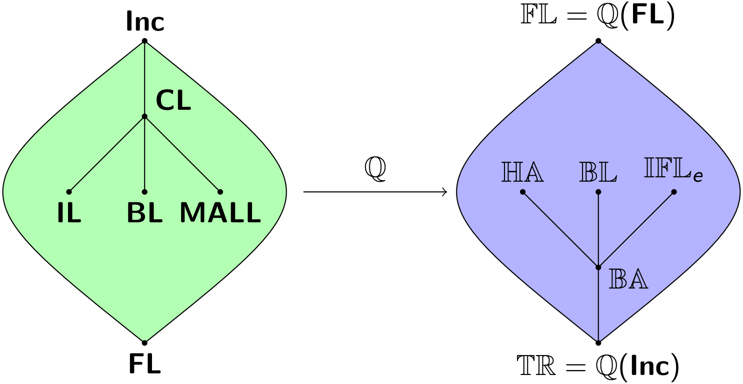
Fact

The class of FL-algebras forms a variety (i.e., an equational class).

Algebraizability

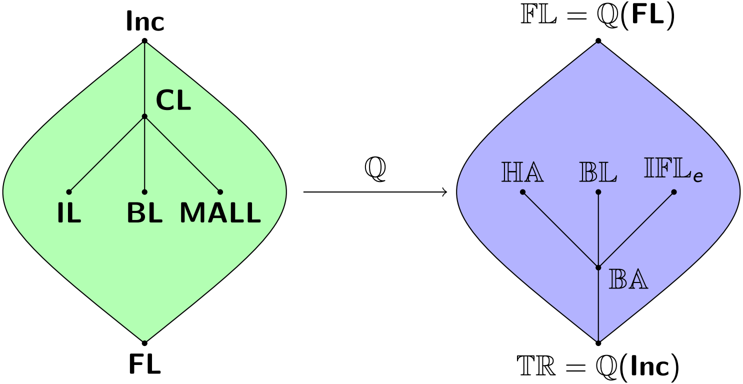


Algebraizability



$$\vdash_{\mathbf{L}} A_1, \ldots, A_n \Rightarrow B \quad \text{iff} \quad \models_{\mathbb{Q}(\mathbf{L})} A_1 \cdots A_n \leq B\,.$$

Algebraizability



$$\mathcal{S} \vdash_{\mathbf{L}} A_1, \ldots, A_n \Rightarrow B \quad \text{iff}$$
$$\{C_1 \cdots C_m \le D \mid C_1, \ldots, C_m \Rightarrow D \in \mathcal{S}\} \models_{\mathbb{Q}(\mathbf{L})} A_1 \cdots A_n \le B\,.$$

Disjunction Property (DP) and complexity

Theorem (Lincoln et al. 1992)

The set of theorems of **MALL** *is PSPACE-complete.*

Disjunction Property (DP) and complexity

Theorem (Lincoln et al. 1992)

The set of theorems of **MALL** *is PSPACE-complete.*

Definition

Let $\mathbf{L}$ be a substructural logic. Then $\mathbf{L}$ satisfies the disjunction property if

$$\vdash_{\mathbf{L}} \varphi \vee \psi \quad \text{implies} \quad \vdash_{\mathbf{L}} \varphi \;\; \text{or} \;\; \vdash_{\mathbf{L}} \psi.$$

Disjunction Property (DP) and complexity

Theorem (Lincoln et al. 1992)

The set of theorems of **MALL** *is PSPACE-complete.*

Definition

Let $\mathbf{L}$ be a substructural logic. Then $\mathbf{L}$ satisfies the disjunction property if

$$\vdash_{\mathbf{L}} \varphi \vee \psi \quad \text{implies} \quad \vdash_{\mathbf{L}} \varphi \ \text{ or } \ \vdash_{\mathbf{L}} \psi.$$

Theorem (H., Terui 2011)

Every consistent substructural logic having the DP is PSPACE-hard.

Disjunction Property (DP) and complexity

Theorem (Lincoln et al. 1992)

The set of theorems of **MALL** *is PSPACE-complete.*

Definition

Let $\mathbf{L}$ be a substructural logic. Then $\mathbf{L}$ satisfies the disjunction property if

$$\vdash_{\mathbf{L}} \varphi \vee \psi \quad \text{implies} \quad \vdash_{\mathbf{L}} \varphi \ \text{ or } \ \vdash_{\mathbf{L}} \psi.$$

Theorem (H., Terui 2011)

Every consistent substructural logic having the DP is PSPACE-hard.

Proof.

By reduction from TQBF using a modification of Švejdar's encoding for intuitionistic logic. □

Algebraic characterization of the DP

Definition

An FL-algebra $\boldsymbol{A}$ is called well-connected if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Algebraic characterization of the DP

Definition

An FL-algebra $\boldsymbol{A}$ is called well-connected if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Theorem (Maksimova 1986, Souma 2007)

Let $\mathbf{L}$ *be a substructural logic. Then* $\mathbf{L}$ *has the DP iff the following condition holds:*

> *for every* $\boldsymbol{A} \in \mathbb{Q}(\mathbf{L})$ *there is a well-connected FL-algebra* $\boldsymbol{C} \in \mathbb{Q}(\mathbf{L})$ *such that* $\boldsymbol{A}$ *is a homomorphic image of* $\boldsymbol{C}$.

DP for ℓ-monoidal extensions

Theorem (H., Terui 2011)

Every extension of **FL** *by any set of rules not containing implication has the DP.*

DP for ℓ-monoidal extensions

Theorem (H., Terui 2011)

Every extension of **FL** *by any set of rules not containing implication has the DP.*

Example

- Every extension of **FL** by any combination of structural rules enjoys the DP.

DP for ℓ-monoidal extensions

Theorem (H., Terui 2011)

*Every extension of **FL** by any set of rules not containing implication has the DP.*

Example

- Every extension of **FL** by any combination of structural rules enjoys the DP.
- The extension of **FL** by the rule

$$\frac{\Rightarrow A}{\Rightarrow A \cdot B}$$

has the DP. It defines a proper subquasivariety of $\mathbb{FL}$.

Examples of axioms

Axiom	Name
$AB \Rightarrow BA$	exchange (e)
$A \Rightarrow 1$	integrality, left weakening (i)
$0 \Rightarrow A$	right weakening (o)
$A \Rightarrow AA$	contraction (c)
$A^n \Rightarrow A^m$	knotted axioms ($n, m \geq 0$)
$A \wedge (A \backslash 0) \Rightarrow$	no-contradiction
$AB/B \Rightarrow A,\ A \backslash AB \Rightarrow B$	cancellativity
$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$	distributivity
$((A \wedge B) \vee C) \wedge B \Rightarrow (A \wedge B) \vee (C \wedge B)$	modularity
$AB \wedge AC \Rightarrow A(B \wedge C)$	$(\cdot, \wedge)$-distributivity
$A \wedge (BC) \Rightarrow (A \wedge B)(A \wedge C)$	$(\wedge, \cdot)$-distributivity

Preframes

Example

Preframes

Example

1. A sequent calculus for **L** naturally defines $\langle Fm^*, Fm, N\rangle$ by

$$A_1, \ldots, A_n \; N \; B \quad \text{iff} \quad \vdash_{\mathbf{L}} A_1, \ldots, A_n \Rightarrow B\,.$$

Preframes

Example

1. A sequent calculus for **L** naturally defines $\langle Fm^*, Fm, N\rangle$ by

$$A_1, \ldots, A_n \; N \; B \quad \text{iff} \quad \vdash_{\mathbf{L}} A_1, \ldots, A_n \Rightarrow B\,.$$

2. Every FL-algebra $\boldsymbol{A}$ naturally defines $\langle A, A, \leq\rangle$.

Preframes

Example

1. A sequent calculus for **L** naturally defines $\langle Fm^*, Fm, N \rangle$ by
$$A_1, \dots, A_n \;\; N \;\; B \quad \text{iff} \quad \vdash_{\mathbf{L}} A_1, \dots, A_n \Rightarrow B\,.$$
2. Every FL-algebra $\boldsymbol{A}$ naturally defines $\langle A, A, \leq \rangle$.

Definition

Let $\boldsymbol{W}$ be a monoid. Then a preframe is $\mathcal{W} = \langle \boldsymbol{W}, W', N \rangle$, where $N \subseteq W \times W'$.

Preframes

Example

1. A sequent calculus for **L** naturally defines $\langle Fm^*, Fm, N\rangle$ by

$$A_1, \dots, A_n \;\; N \;\; B \quad \text{iff} \quad \vdash_{\mathbf{L}} A_1, \dots, A_n \Rightarrow B\,.$$

2. Every FL-algebra $\boldsymbol{A}$ naturally defines $\langle A, A, \leq\rangle$.

Definition

Let $\boldsymbol{W}$ be a monoid. Then a preframe is $\mathcal{W} = \langle \boldsymbol{W}, W', N\rangle$, where $N \subseteq W \times W'$.

1. The powerset $\mathcal{P}(W)$ canonically bears the structure of a residuated lattice.

Preframes

Example

1. A sequent calculus for **L** naturally defines $\langle Fm^*, Fm, N \rangle$ by
$$A_1, \ldots, A_n \;\; N \;\; B \quad \text{iff} \quad \vdash_{\mathbf{L}} A_1, \ldots, A_n \Rightarrow B\,.$$
2. Every FL-algebra $\boldsymbol{A}$ naturally defines $\langle A, A, \leq \rangle$.

Definition

Let $\boldsymbol{W}$ be a monoid. Then a preframe is $\mathcal{W} = \langle \boldsymbol{W}, W', N \rangle$, where $N \subseteq W \times W'$.

1. The powerset $\mathcal{P}(W)$ canonically bears the structure of a residuated lattice.
2. The relation N induces a Galois connection between $\mathcal{P}(W)$ and $\mathcal{P}(W')$.

Residuated frames (Galatos, Jipsen)

Definition (Galatos, Jipsen)

A residuated frame is a preframe $\mathcal{W} = \langle \boldsymbol{W}, W', N\rangle$ such that there are $\backslash\!\!\backslash\colon W \times W' \to W'$ and $/\!\!/\colon W' \times W \to W'$ such that

$$x \cdot y \, N \, z \quad \text{iff} \quad y \, N \, x \backslash\!\!\backslash z \quad \text{iff} \quad x \, N \, z /\!\!/ y \,.$$

Residuated frames (Galatos, Jipsen)

Definition (Galatos, Jipsen)

A residuated frame is a preframe $\mathcal{W} = \langle \boldsymbol{W}, W', N \rangle$ such that there are $\backslash\!\!\backslash \colon W \times W' \to W'$ and $/\!\!/ \colon W' \times W \to W'$ such that

$$x \cdot y \; N \; z \quad \text{iff} \quad y \; N \; x \backslash\!\!\backslash z \quad \text{iff} \quad x \; N \; z /\!\!/ y \,.$$

Lemma

Every preframe $\langle \boldsymbol{W}, W', N \rangle$ can be extended to a residuated frame by enlarging W' and N.

Complex algebra of a residuated frame

$\mathcal{W} = \langle \boldsymbol{W}, W', N \rangle$: $\quad W \overset{N}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!/\;\;\;\;\;} W'$

Complex algebra of a residuated frame

$\mathcal{W} = \langle \boldsymbol{W}, W', N \rangle$:

$$W \xrightarrow{\quad N \quad} W'$$

$$\begin{tikzcd} \mathcal{P}(W) \arrow[r, bend left, "\rhd"] & \mathcal{P}(W') \arrow[l, bend left, "\lhd"] \end{tikzcd}$$

$$\begin{aligned} X^{\rhd} &= \{b \in B \mid (\forall a \in X)(a \mathrel{N} b)\}\,, \\ Y^{\lhd} &= \{a \in A \mid (\forall b \in Y)(a \mathrel{N} b)\}\,. \end{aligned}$$

Complex algebra of a residuated frame

$\mathcal{W} = \langle \boldsymbol{W}, W', N \rangle$:

$$W \overset{N}{\not\longrightarrow} W'$$

$$\begin{tikzcd} \mathcal{P}(W) \arrow[r, bend left, "\triangleright"] & \mathcal{P}(W') \arrow[l, bend left, "\triangleleft"] \end{tikzcd}$$

$$\begin{aligned} X^{\triangleright} &= \{b \in B \mid (\forall a \in X)(a\ N\ b)\}\,, \\ Y^{\triangleleft} &= \{a \in A \mid (\forall b \in Y)(a\ N\ b)\}\,. \end{aligned}$$

Galois connection $^{\triangleright}, ^{\triangleleft}$ induces a closure operator $\mathcal{P}(W)$.

Complex algebra of a residuated frame

$\mathcal{W} = \langle \boldsymbol{W}, W', N \rangle$:

$$\begin{tikzcd} W \arrow[r, "N", "/" marking] & W' \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{P}(W) \arrow[r, bend left, "\triangleright"] & \mathcal{P}(W') \arrow[l, bend left, "\triangleleft"] \end{tikzcd}$$

$$\begin{aligned} X^{\triangleright} &= \{b \in B \mid (\forall a \in X)(a\ N\ b)\}\,, \\ Y^{\triangleleft} &= \{a \in A \mid (\forall b \in Y)(a\ N\ b)\}\,. \end{aligned}$$

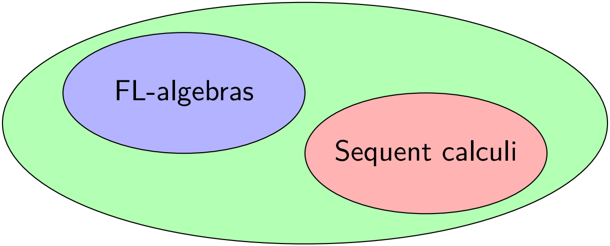
Galois connection $^{\triangleright}, ^{\triangleleft}$ induces a closure operator $\mathcal{P}(W)$.

The collection of closed sets forms a complete residuated lattice $\mathcal{W}^+$ called complex algebra of $\mathcal{W}$.

The map $x \mapsto \{x\}^{\triangleright\triangleleft}$ is a monoid homomorphism from $\boldsymbol{W}$ to $\mathcal{W}^+$.

Residuated frames

Residuated frames



$$\underset{\text{Residuated frame}}{\mathcal{W}} \xrightarrow{\text{quasi-DM-completion}} \underset{\text{Residuated lattice}}{\mathcal{W}^+}$$

Remark

[Belardinelli, Jipsen, Ono 2004] gave a purely algebraic proof of cut admissibility in $\mathbf{FL}_{\mathrm{ew}}$. For other logics [Galatos, Ono 2010].

An undecidability result

Theorem (Lincoln et al. 1992)

There is a finite set $\mathcal{S}$ of sequents such that the set of sequents provable from $\mathcal{S}$ in **MALL** *is undecidable.*

An undecidability result

Theorem (Lincoln et al. 1992)

*There is a finite set $\mathcal{S}$ of sequents such that the set of sequents provable from $\mathcal{S}$ in **MALL** is undecidable.*

Theorem (H.)

There is a finite set $\mathcal{S}$ of sequents such that the set of sequents

$$\{s \mid \mathcal{S} \vdash_{\mathbf{L}} s\}$$

*is undecidable for every substructural logic **L** weaker than the axiomatic extension of **FL** by axiom schemata $A \Rightarrow A^2$ and $A^3 \Rightarrow A^2$.*

Sketch of the proof

2CM consists of finitely many states $\{0, \ldots, m\}$ (0 is a final state), two counters holding $k, l \in \mathbb{N}$, program.

Sketch of the proof

2CM consists of finitely many states $\{0, \ldots, m\}$ (0 is a final state), two counters holding $k, l \in \mathbb{N}$, program.

Configurations of 2CM are $\langle i, k, l \rangle$ such that $i \in \{0, \ldots, m\}$ and $k, l \in \mathbb{N}$.

Sketch of the proof

2CM consists of finitely many states $\{0, \dots, m\}$ (0 is a final state), two counters holding $k, l \in \mathbb{N}$, program.

Configurations of 2CM are $\langle i, k, l \rangle$ such that $i \in \{0, \dots, m\}$ and $k, l \in \mathbb{N}$.

Theorem (Minsky 1961)

There is 2CM M such that given a configuration $\langle i, k, l \rangle$, it is undecidable whether M stops computation in $\langle 0, 0, 0 \rangle$ when computing from $\langle i, k, l \rangle$.

Sketch of the proof

2CM consists of finitely many states $\{0, \ldots, m\}$ (0 is a final state), two counters holding $k, l \in \mathbb{N}$, program.

Configurations of 2CM are $\langle i, k, l \rangle$ such that $i \in \{0, \ldots, m\}$ and $k, l \in \mathbb{N}$.

Theorem (Minsky 1961)

There is 2CM M such that given a configuration $\langle i, k, l \rangle$, it is undecidable whether M stops computation in $\langle 0, 0, 0 \rangle$ when computing from $\langle i, k, l \rangle$.

Configurations can be encoded into words over a finite alphabet Σ:

$$w(i, k, l) = A\overline{h^k(a)}Bq_iCh^l(a)A\,.$$

The program of M can be encoded into a finite set of sequents $\mathcal{S}$.

Sketch of the proof (cont.)

1. Easy part: If M stops with empty counters then

$$\mathcal{S} \vdash_{\mathbf{L}} w(i,k,l) \Rightarrow w(0,0,0)\,.$$

Sketch of the proof (cont.)

1. Easy part: If M stops with empty counters then

$$\mathcal{S} \vdash_{\mathbf{L}} w(i,k,l) \Rightarrow w(0,0,0)\,.$$

2. Converse implication is difficult one. Define a residuated frame $\mathcal{W} = \langle \Sigma^*, \Sigma^* \times \Sigma^*, N\rangle$ by

$x\,N\,\langle u, v\rangle$ iff uxv encodes a configuration $\langle i,k,l\rangle$ and M stops in $\langle 0,0,0\rangle$ when computing from $\langle i,k,l\rangle$.

Sketch of the proof (cont.)

1. Easy part: If M stops with empty counters then
$$\mathcal{S} \vdash_{\mathbf{L}} w(i,k,l) \Rightarrow w(0,0,0)\,.$$

2. Converse implication is difficult one. Define a residuated frame $\mathcal{W} = \langle \Sigma^*, \Sigma^* \times \Sigma^*, N\rangle$ by

$x\,N\,\langle u, v\rangle$ iff uxv encodes a configuration $\langle i,k,l\rangle$ and M stops in $\langle 0,0,0\rangle$ when computing from $\langle i,k,l\rangle$.

Then $\mathcal{W}^+$ is a residuated lattice satisfying $x \leq x^2$ and $x^3 \leq x^2$.

Sketch of the proof (cont.)

1. Easy part: If M stops with empty counters then

$$\mathcal{S} \vdash_{\mathbf{L}} w(i,k,l) \Rightarrow w(0,0,0)\,.$$

2. Converse implication is difficult one. Define a residuated frame $\mathcal{W} = \langle \Sigma^*, \Sigma^* \times \Sigma^*, N \rangle$ by

$x \, N \, \langle u, v \rangle$ iff uxv encodes a configuration $\langle i,k,l \rangle$ and M stops in $\langle 0,0,0 \rangle$ when computing from $\langle i,k,l \rangle$.

Then $\mathcal{W}^+$ is a residuated lattice satisfying $x \leq x^2$ and $x^3 \leq x^2$.

The $\mathcal{W}^+$-evaluation $p \in \Sigma \mapsto \{p\}^{\rhd\lhd} \in \mathcal{W}^+$ satisfies $\mathcal{S}$.

Sketch of the proof (cont.)

1. Easy part: If M stops with empty counters then

$$\mathcal{S} \vdash_{\mathbf{L}} w(i,k,l) \Rightarrow w(0,0,0)\,.$$

2. Converse implication is difficult one. Define a residuated frame $\mathcal{W} = \langle \Sigma^*, \Sigma^* \times \Sigma^*, N \rangle$ by

$x \, N \, \langle u, v \rangle$ iff uxv encodes a configuration $\langle i, k, l \rangle$ and M stops in $\langle 0, 0, 0 \rangle$ when computing from $\langle i, k, l \rangle$.

Then $\mathcal{W}^+$ is a residuated lattice satisfying $x \leq x^2$ and $x^3 \leq x^2$.

The $\mathcal{W}^+$-evaluation $p \in \Sigma \mapsto \{p\}^{\rhd\lhd} \in \mathcal{W}^+$ satisfies $\mathcal{S}$.

If $\mathcal{S} \vdash_{\mathbf{L}} w(i,k,l) \Rightarrow w(0,0,0)$ then we have in $\mathcal{W}^+$:

$$w(i,k,l) \in \{w(i,k,l)\}^{\rhd\lhd} \subseteq \{w(0,0,0)\}^{\rhd\lhd} \subseteq \{\langle \varepsilon, \varepsilon \rangle\}^{\lhd}\,.$$

Density rule

UL is the logic corresponding to the variety of FL-algebras generated by commutative linearly ordered FL-algebras (FL_e-chains).

Density rule

UL is the logic corresponding to the variety of FL-algebras generated by commutative linearly ordered FL-algebras ($\mathsf{FL_e}$-chains).

Theorem (Metcalfe, Montagna 2007; Ciabattoni, Metcalfe 2008)

UL *is strongly complete w.r.t. the class of densely ordered* $\mathsf{FL_e}$*-chains.*

Density rule

UL is the logic corresponding to the variety of FL-algebras generated by commutative linearly ordered FL-algebras ($\mathsf{FL_e}$-chains).

Theorem (Metcalfe, Montagna 2007; Ciabattoni, Metcalfe 2008)

UL *is strongly complete w.r.t. the class of densely ordered* $\mathsf{FL_e}$*-chains.*

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta}\ \text{Density}$$

Density rule

UL is the logic corresponding to the variety of FL-algebras generated by commutative linearly ordered FL-algebras ($\mathrm{FL_e}$-chains).

Theorem (Metcalfe, Montagna 2007; Ciabattoni, Metcalfe 2008)

UL *is strongly complete w.r.t. the class of densely ordered FL_e-chains.*

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta}\ \text{Density}$$

$$\frac{A \Rightarrow p \mid p \Rightarrow B}{A \Rightarrow B}$$

Corresponding residuated frame

Let $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ be an $\mathsf{FL_e}$-chain and $a, b \in A$ such that $a \prec b$.

Corresponding residuated frame

Let $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ be an $\mathsf{FL_e}$-chain and $a, b \in A$ such that $a \prec b$.

Define $\mathcal{W} = \langle Ap^*, A \cup \{p\}, N\rangle$ by

$$\begin{aligned} cp^n \; N \; d \quad &\text{iff} \quad cb^n \leq d \\ c \; N \; p \quad &\text{iff} \quad c \leq a \\ cp^m \; N \; p \quad &\text{iff} \quad cp^{m-1} \leq 1 \end{aligned}$$

where $c, d \in A$, $n \geq 1$ and $m > 1$.

Corresponding residuated frame

Let $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ be an $\mathrm{FL_e}$-chain and $a, b \in A$ such that $a \prec b$.

Define $\mathcal{W} = \langle Ap^*, A \cup \{p\}, N\rangle$ by

$$\begin{aligned} cp^n \ N \ d \quad &\text{iff} \quad cb^n \leq d \\ c \ N \ p \quad &\text{iff} \quad c \leq a \\ cp^m \ N \ p \quad &\text{iff} \quad cp^{m-1} \leq 1 \end{aligned}$$

where $c, d \in A$, $n \geq 1$ and $m > 1$.

Lemma

1. *$\mathcal{W}^+$ can be made into an $\mathrm{FL_e}$-chain.*
2. *$\boldsymbol{A}$ embeds into $\mathcal{W}^+$ via $x \mapsto \{x\}^{\rhd\lhd}$.*
3. $\{a\}^{\rhd\lhd} \subsetneq \{p\}^{\rhd\lhd} \subsetneq \{b\}^{\rhd\lhd}$.

Algebraic description of $\mathcal{W}^+$

Start with the given $\mathsf{FL_e}$-chain $\boldsymbol{A}$ with $a \prec b$ (we assume that it is complete). $\boldsymbol{A}$ has a semiring reduct.

Algebraic description of $\mathcal{W}^+$

Start with the given $\mathrm{FL_e}$-chain $\boldsymbol{A}$ with $a \prec b$ (we assume that it is complete). $\boldsymbol{A}$ has a semiring reduct.

Consider the semiring $\boldsymbol{A}\langle X\rangle$ of formal power series over $\boldsymbol{A}$:

$$\bigvee_{n\in\mathbb{N}} f_n X^n\,, \quad f_n \in A\,.$$

$\boldsymbol{A}\langle X\rangle$ forms in fact an FL-algebra.

Algebraic description of $\mathcal{W}^+$

Start with the given $\mathsf{FL_e}$-chain $\boldsymbol{A}$ with $a \prec b$ (we assume that it is complete). $\boldsymbol{A}$ has a semiring reduct.

Consider the semiring $\boldsymbol{A}\langle X\rangle$ of formal power series over $\boldsymbol{A}$:

$$\bigvee_{n\in\mathbb{N}} f_n X^n\,, \quad f_n \in A\,.$$

$\boldsymbol{A}\langle X\rangle$ forms in fact an FL-algebra.

Consider the subset consisting of the following formal series for $c, d \in A$:

$$\bigvee\nolimits_{n\in\mathbb{N}}(b^n \to c)X^n$$
$$(d \to a) \vee \bigvee\nolimits_{n\in\mathbb{N}}(b^n d \to 1)X^{n+1}$$

Algebraic description of $\mathcal{W}^+$

Start with the given $\mathsf{FL_e}$-chain $\boldsymbol{A}$ with $a \prec b$ (we assume that it is complete). $\boldsymbol{A}$ has a semiring reduct.

Consider the semiring $\boldsymbol{A}\langle X\rangle$ of formal power series over $\boldsymbol{A}$:

$$\bigvee_{n\in\mathbb{N}} f_n X^n\,, \quad f_n \in A\,.$$

$\boldsymbol{A}\langle X\rangle$ forms in fact an FL-algebra.

Consider the subset consisting of the following formal series for $c, d \in A$:

$$\bigvee_{n\in\mathbb{N}}(b^n \to c)X^n$$
$$(d \to a) \vee \bigvee_{n\in\mathbb{N}}(b^n d \to 1)X^{n+1}$$

This subset forms an $\mathsf{FL_e}$-chain isomorphic to $\mathcal{W}^+$.

Thank you!