Residuated Lattices, Regular Languages, and Burnside Problem

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Topology, Algebra, and Categories in Logic
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Outline

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1. Residuated lattices
2. Analogy between languages and logics

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Residuated lattices

Definition

Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a monoid. A quasi-order $\leq$ on M is called compatible if for all $x, y, u, v \in M$:

$$x \leq y \implies uxv \leq uyv\,.$$

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A residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a monoid such that $\langle A, \wedge, \vee\rangle$ is a lattice and for all $a, b, c \in A$:

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b\,.$$

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Logic = a substructural logic, i.e., an axiomatic extension of FL.

Powerset monoid

Example

Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a monoid. Then

$$\mathcal{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle$$

is a residuated lattice, where

$$\begin{aligned} X \cdot Y &= \{xy \in M \mid x \in X, y \in Y\}, \\ X \backslash Z &= \{y \in M \mid X \cdot \{y\} \subseteq Z\}, \\ Z / Y &= \{x \in M \mid \{x\} \cdot Y \subseteq Z\}. \end{aligned}$$

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Other examples can be obtained by introducing a suitable closure operator on $\mathcal{P}(M)$.

Nuclei

Definition

Let **M** be a monoid and γ a closure operator on $\mathcal{P}(M)$. The collection of γ-closed sets is denoted $\mathcal{P}(M)_\gamma$. Then γ is called a nucleus if for every $u, v \in M$ we have

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Let **M** be a monoid and γ a nucleus on $\mathcal{P}(M)$. Then $\mathcal{P}(\mathbf{M})_\gamma = \langle \mathcal{P}(M)_\gamma, \cap, \cup_\gamma, \cdot_\gamma, \backslash, /, \gamma\{1\}\rangle$ is a residuated lattice, where

$$\begin{aligned} X \cup_\gamma Y &= \gamma(X \cup Y)\,, \\ X \cdot_\gamma Y &= \gamma(X \cdot Y)\,. \end{aligned}$$

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Theorem

1. *The syntactic congruence $\sim_L$ is the largest congruence saturating L, i.e., $L = \bigcup_{w \in L} w/\!\sim_L$.*
2. $\mathbf{M}(L)$ *is finite iff L is regular (Myhill-Nerode Theorem).*

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Eilenberg variety theorem

The assignment $L \mapsto \mathbf{M}(L)$ induces a correspondence between varieties of regular languages and pseudovarieties of finite monoids.

$$\mathcal{L} \mapsto \text{the pseudovariety generated by } \{\mathbf{M}(L) \mid L \in \mathcal{L}\}.$$
$$\mathcal{V} \mapsto \text{the variety } \mathcal{L} \text{ of regular languages } L \text{ s.t. } \mathbf{M}(L) \in \mathcal{V}.$$

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Theorem

Let L be a logic. The map $L \mapsto \mathbf{Fm}/{\sim_L}$ induces a dual-isomorphism between the lattice of axiomatic extensions of L and the subvariety lattice of the variety generated by $\mathbf{Fm}/{\sim_L}$.

Analogy table

Language theory	Logic
language	theorems
syntactic congruence	Leibniz congruence
syntactic monoid	Lindenbaum-Tarski algebra
Eilenberg variety theorem	axiomatic extensions $\leftrightsquigarrow$ subvarieties

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- Lindenbaum-Tarski algebra is used to prove the completeness theorem for a logic L.
- Nevertheless, there is also another construction used in order to prove it.
- Does it have its analogy on the language side?

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$$S_\varphi = \{\Gamma \in Fm^* \mid \vdash_L \Gamma \Rightarrow \varphi\}.$$

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- Then $\mathcal{P}(Fm^*)_\gamma$ is the algebra used to prove the completeness theorem.
- If δ is a nucleus on $\mathcal{P}(Fm^*)$ making all S_φ's δ-closed then $\delta(X) \subseteq \gamma(X)$ for all $X \subseteq Fm^*$.

Syntactic residuated lattice

Definition

Let $L \subseteq \Sigma^*$ be a language. The pointwise largest nucleus γ_L making L a closed set is called syntactic nucleus. Then $\mathbf{R}(L) = \mathcal{P}(\Sigma^*)_{\gamma_L}$ is called a syntactic residuated lattice.

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Theorem

1. $\{\gamma\{x\} \mid x \in \Sigma^*\}$ *forms a submonoid isomorphic to the syntactic monoid* $\mathbf{M}(L)$.
2. $\mathbf{R}(L)$ *is finite iff* L *is regular.*

Is it good for something?

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Example (Sakarovitch)

Consider the following languages over $\Sigma = \{0, 1\}$:

$$\begin{aligned} L_1 &= \{ww^R \mid w \in \Sigma^*\}, \\ L_2 &= \{w \in \Sigma^* \mid w \text{ is prime}\}. \end{aligned}$$

Then $\mathbf{M}(L_1) = \mathbf{M}(L_2) = \Sigma^*$.

Beyond regular languages

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Theorem

A language L is closed under (r) *iff* $\mathbf{R}(L)$ *satisfies*

$$1 \leq x \vee x^2 \vee x \backslash y\,.$$

Thus the languages L_1, L_2 can be separated by a variety of residuated lattices.

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- Use residuated frames (Galatos, Jipsen).

Frames

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$$\mathcal{P}(M) \underset{\lhd}{\overset{\rhd}{\rightleftarrows}} \mathcal{P}(B)$$

$$\begin{aligned} X^{\rhd} &= \{b \in B \mid (\forall a \in X)(a \; N \; b)\}, \\ Y^{\lhd} &= \{a \in M \mid (\forall b \in Y)(a \; N \; b)\}. \end{aligned}$$

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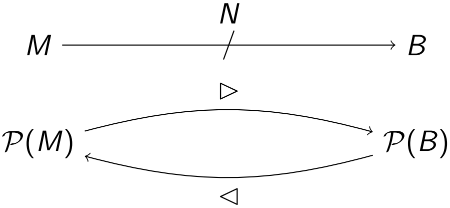

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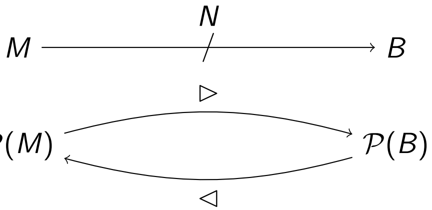

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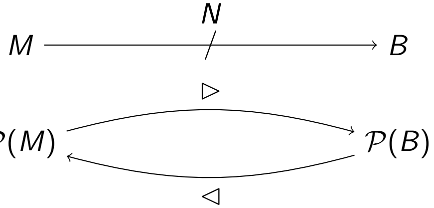

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- $\gamma(X) = X^{\rhd\lhd}$ is a closure operator on $\mathcal{P}(M)$.
- It is the pointwise largest closure operator making all sets in its basis $\{\{b\}^{\lhd} \mid b \in B\}$ γ-closed.
- The collection of closed sets forms a complete lattice $\mathbf{W}^+ = \langle \mathcal{P}(M)_\gamma, \cap, \cup_\gamma \rangle$, where

$$X \cup_\gamma Y = \gamma(X \cup Y).$$

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- The closure operator γ induced by $\widehat{N}$ is a nucleus.
- Then $\widehat{\mathbf{W}}^+ = \mathcal{P}(\mathbf{M})_\gamma$ forms a complete residuated lattice.
- Moreover, γ is the pointwise largest nucleus making all $\{1, 1, b\}^{\lhd}$'s γ-closed.

Construction of $\mathbf{R}(L)$

- Let L be a logic and consider the frame $\mathbf{W} = \langle Fm^*, Fm, N \rangle$ where

$$\Gamma\ N\ \varphi \quad \text{iff} \quad \Gamma \in S_\varphi \quad \text{iff} \quad \vdash_L \Gamma \Rightarrow \varphi\,.$$

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- Let $L \subseteq \Sigma^*$ be a language. Define frame $\mathbf{W} = \langle \Sigma^*, \{L\}, N\rangle$, where $N \subseteq \Sigma^* \times \{L\}$ is defined by

$$x\; N\; L \quad \text{iff} \quad x \in L\,.$$

  Then $\mathbf{R}(L) = \widehat{\mathbf{W}}^+$ is the syntactic residuated lattice of L.

FEP

Definition (Evans)

A class of algebras $\mathcal{K}$ of the same type has the finite embeddability property (FEP) if every finite partial subalgebra $\mathbf{B}$ of any algebra $\mathbf{A} \in \mathcal{K}$ is embeddable into a finite algebra $\mathbf{D} \in \mathcal{K}$.

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- Is $\widehat{\mathbf{W}}^{+}$ finite? Does $\widehat{\mathbf{W}}^{+}$ belong to $\mathcal{K}$?

Generalized Myhill Theorem

Theorem

Let $\mathbf{M}$ *be a monoid and* $\mathbf{W} = \langle M, B, N \rangle$ *a frame where* B *is finite. Then* $\widehat{\mathbf{W}}^+$ *is finite iff there is a compatible dual well quasi-order* $\sqsubseteq$ *on* $\mathbf{M}$ *such that*

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Corollary (Generalized Myhill Theorem – Ehrenfeucht, Rozenberg)

A language $L \subseteq \Sigma^*$ *is regular iff* L *is downward closed w.r.t. a compatible dual well quasi-order on* Σ^*.

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A language $L \subseteq \Sigma^*$ *is regular iff* L *is downward closed w.r.t. a compatible dual well quasi-order on* Σ^*.

$\mathcal{V}$ = finitely gen. subpomonoids of members from $\mathcal{K}$

Find a compatible dual well quasi-order $\sqsubseteq$ on Σ^* s.t. all pomonoids from $\mathcal{V}$ are homomorphic images of $\Sigma^*/\sqsubseteq$.

Weakening rule

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Every language L closed under the following rule is regular:

$$\frac{uv \in L}{uxv \in L}.$$

Exchange and knotted rules

Let $m \geq 1$, $n \geq 0$ and $m \neq n$.

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Inspired by language theory

Theorem (de Luca, Varricchio)

Language L is regular iff L is permutable and quasi-periodic or co-quasi-periodic.

Let $\sigma \in S_k \setminus \{id\}$ for $k \geq 2$ and $m, n \in \mathbb{N}$ such that $m > n \geq 1$.

In particular, they prove that the least compatible quasi-order $\sqsubseteq$ on Σ^+ such that $\Sigma^+/\sqsubseteq$ satisfies $x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$ and $x^m \leq x^n$ is dually well.

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Theorem

Let $\sigma \in S_k \setminus \{id\}$ for $k \geq 2$ and $m, n \in \mathbb{N}$ such that $m > n \geq 1$. Then the variety of residuated lattice-ordered semigroups axiomatized by $x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$ and $x^m \leq x^n$ has the FEP.

Knotted axioms

Let $m, n \geq 1$ and $m \neq n$. The variety of res. lattices defined by $x^m \leq x^n$ is denoted $\mathcal{RL}_m^n$.

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The only remaining cases are $x^m \leq x$ for $m \geq 2$.

Partial order

Let $m \geq 2$. The variety $\mathcal{RL}^1_m$ can be axiomatized by

$$ux_1v \leq z \;\&\; \ldots \;\&\; ux_mv \leq z \implies ux_1 \cdots x_m v \leq z\,. \qquad (q_m)$$

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Lemma

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Let $\mathcal{G}_m$ be the variety of groups satisfying $x^m = 1$.

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The answer is negative for odd $m \geq 665$.

Theorem (Ivanov)

The answer is negative for $m \geq 2^{48}$.

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3. *Every language closed under the following rule is regular:*

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Corollary

The partial order $\leq_m$ is not dual well for even $m \geq 666$ and $m \geq 2^{48}$.

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The variety $\mathcal{RL}_2^1$ *has the FEP.*

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Every language $L \subseteq \Sigma^*$ closed under the following rule is regular:

$$\frac{uxv \in L \quad uyv \in L}{uxyv \in L}\,. \tag{r_2}$$

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The variety $\mathcal{RL}^1_2$ has the FEP.

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Every language $L \subseteq \Sigma^*$ closed under the following rule is regular:

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Example

The language $a^+(b(a+b+c)^*b+b)c^+$ is closed under (r_2).

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Lemma

Let $w \in \Sigma^*$ and $\mathrm{Alph}(w) = \Gamma$. Then $wuw \leq_2 w$ for every $u \in \Gamma^*$.

Higman's lemma

Definition

Let $\langle Q, \leq \rangle$ be a quasi-ordered set. Define a binary relation $\leq^*$ on Q^* by

$a_1 \ldots a_n \leq^* b_1 \ldots b_m$ iff there is a strictly increasing map $f\colon [1,n] \to [1,m]$ s.t. $a_i \leq b_{f(i)}$ for all $i \in [1,n]$.

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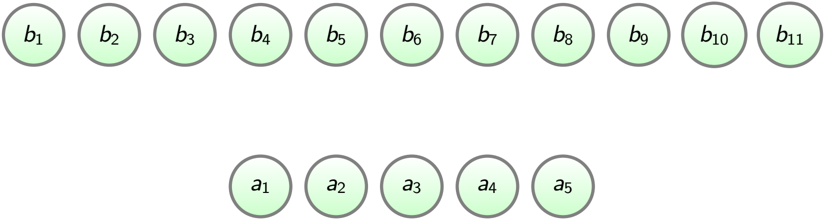

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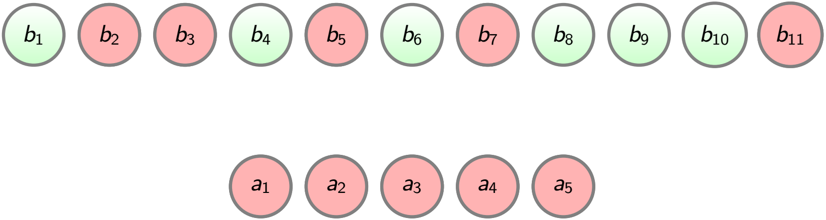

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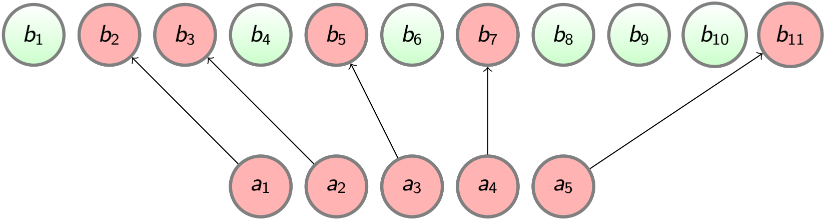

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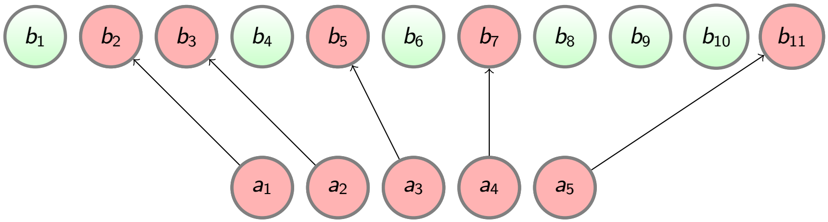

Lemma (Higman's lemma)

If $\langle Q, \leq \rangle$ *is a well quasi-ordered set then so is* $\langle Q^*, \leq^* \rangle$.

Modified Higman's lemma

Definition

Let $\langle Q, \leq \rangle$ be a quasi-ordered set. Define a binary relation $\leq^+$ on Q^+ by

> $a_1 \dots a_n \leq^+ b_1 \dots b_m$ *iff there is a strictly increasing map* $f : [1, n+1] \to [1, m+1]$ *such that*
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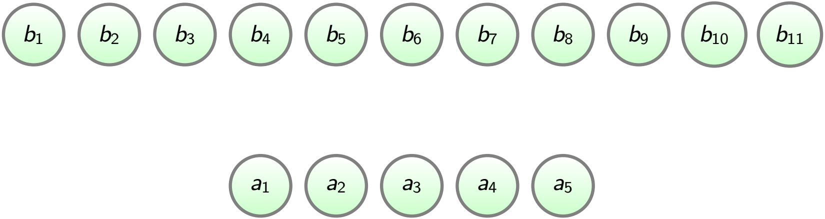

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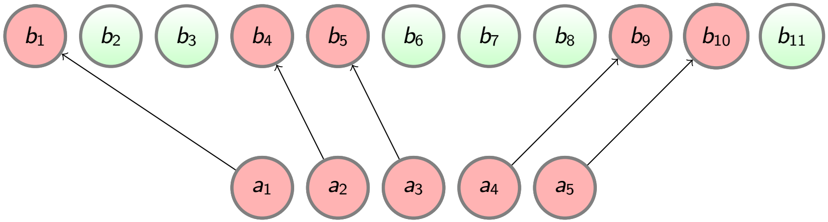

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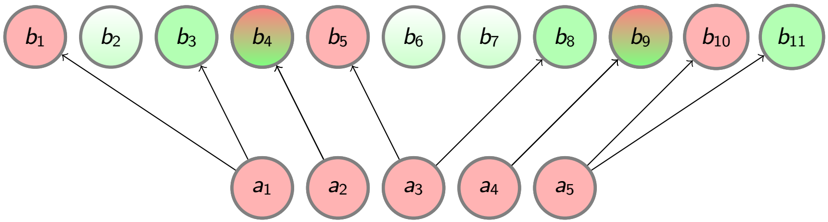

Modified Higman's lemma (cont.)

Lemma

If $\langle Q, \leq \rangle$ is a well quasi-ordered set then $\langle Q^+, \leq^+ \rangle$ forms a well quasi-ordered set as well.

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- Could it be interesting for people working in language theory?
- Is the compatible quasi-order $\leq_m$ on Σ^* dually well for $m = 3, 4, 5, \ldots, 665, 667, 669, \ldots, 2^{48} - 1$?

Thank you!