Characterization of FEP for (Distributive) Residuated Lattices via Regular (Tree) Languages

Rostislav Horčík

Introduction

Definition [Henkin 1951, Evans 1969]

A class of algebras $\mathbb{K}$ of the same type has the finite embeddability property (FEP) if every finite partial subalgebra $\mathbf{B}$ of any algebra $\mathbf{A} \in \mathbb{K}$ is embeddable into a finite algebra $\mathbf{D} \in \mathbb{K}$.

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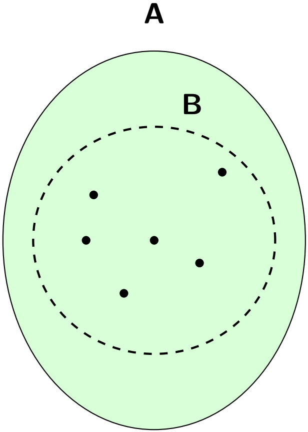
- We are going to rephrase the above contruction in terms of recognizable sets/languages.
- This simplifies the proofs by employing results from language theory.

FEP



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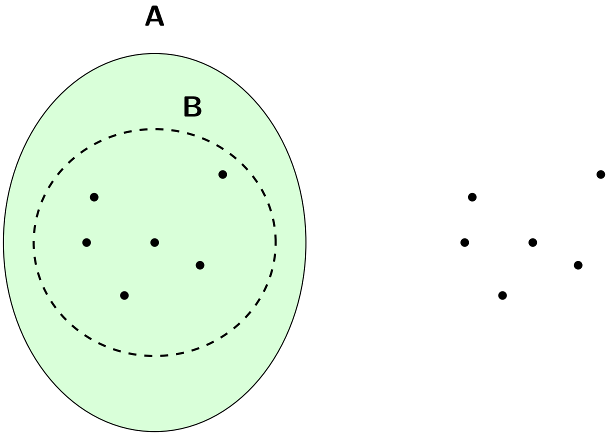

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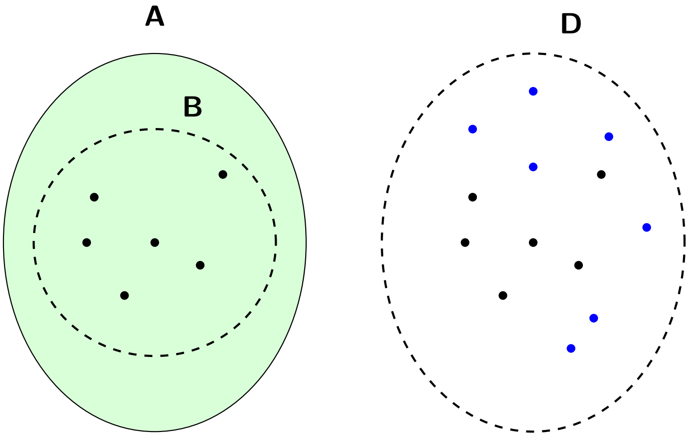

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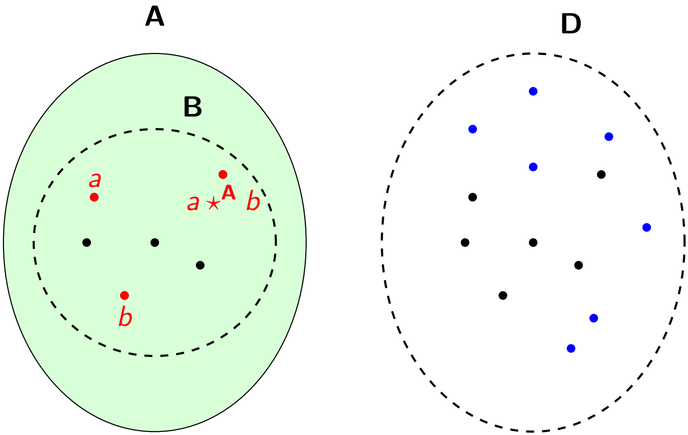

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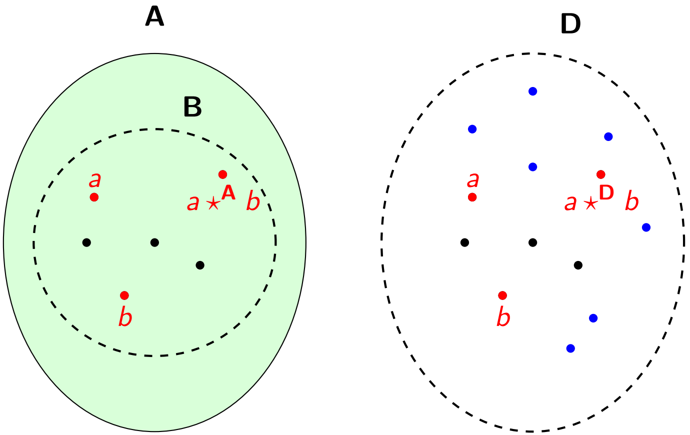

Recognizable sets

Definition

Let $\mathbf{A}$ be an algebra and $L \subseteq A$. The set L is recognizable if

- there is a finite algebra $\mathbf{D}$,
- homomorphism $h \colon \mathbf{A} \to \mathbf{D}$ and
- $\ker(h)$ saturates L, i.e., L is a union of congruence classes.

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Theorem [Kleene]

Recognizable sets $\mathrm{Rec}(\mathbf{B}^*)$ over finitely generated free monoids are precisely regular/rational languages.

Syntactic congruences

The set of unary, linear polynomials on algebra $\mathbf{A}$ is denoted $Tr(\mathbf{A})$ whose elements are called translations.

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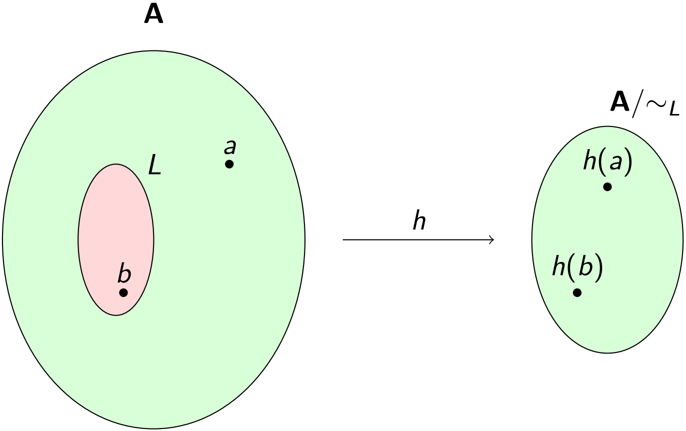
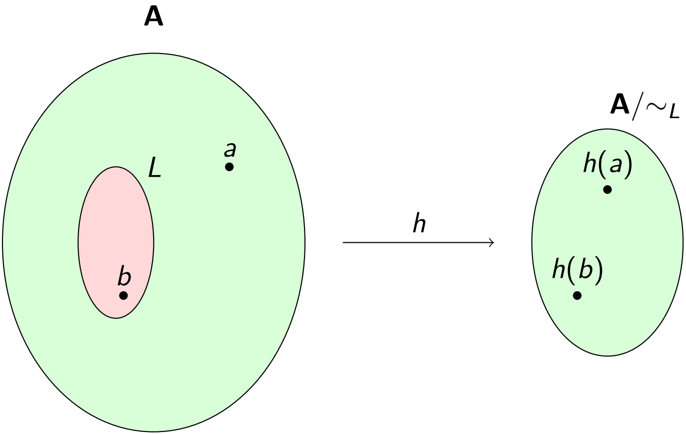
Lemma

The synt. congruence $\sim_L$ is the largest congruence saturating L. Thus $\mathbf{A}/{\sim_L}$ is finite iff L is recognizable.



A
L
a
b
h
A/∼L
h(a)
h(b)



Given a finite subset $B \subseteq A$, if we can find recognizable sets $L_1, \ldots, L_n \subseteq A$ separating elements of B, we obtain a finite algebra $\mathbf{D} \cong \mathbf{A}/\bigcap_{i=1}^{n} \sim_{L_i}$ and a homomorphism $h \colon \mathbf{A} \to \mathbf{D}$.

Residuated lattices

Definition

A residuated lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$, where

- $\langle A, \wedge, \vee \rangle$ is a lattice,
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Facts

- $\langle A, \vee, \cdot, 1 \rangle$ forms an idempotent semiring because $a(b \vee c)d = abd \vee acd$.
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When does a variety $\mathbb{K}$ of residuated lattices axiomatized over $\{\vee, \cdot, 1\}$ have the FEP?

Construction of the finite algebra D

$$\mathbf{A} \xleftarrow{id} \mathbf{B}$$

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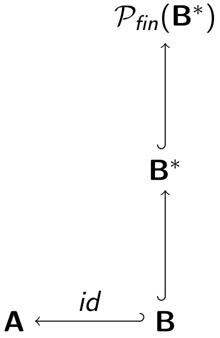

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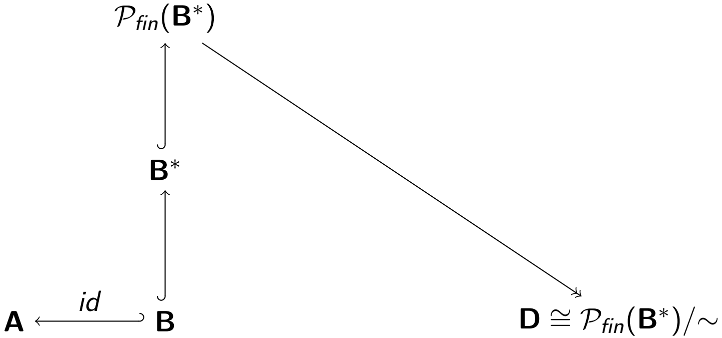

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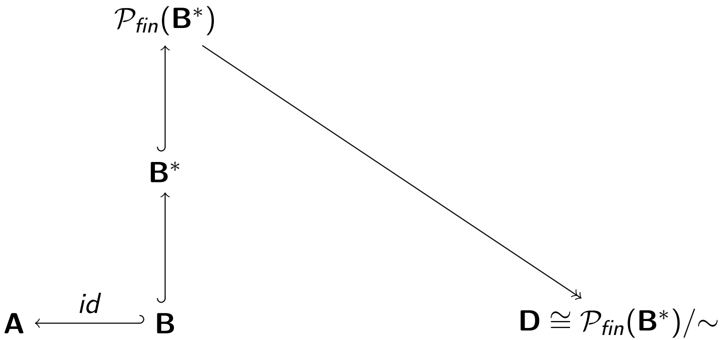

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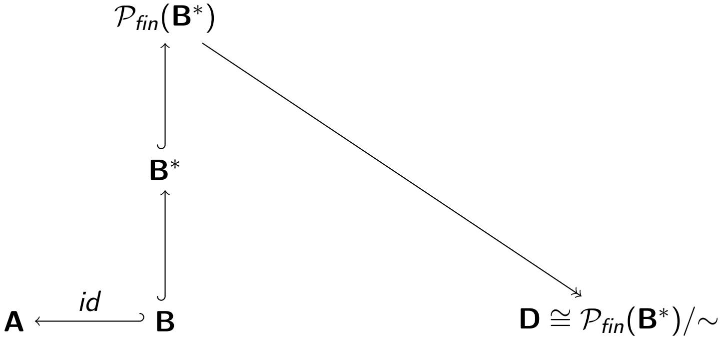

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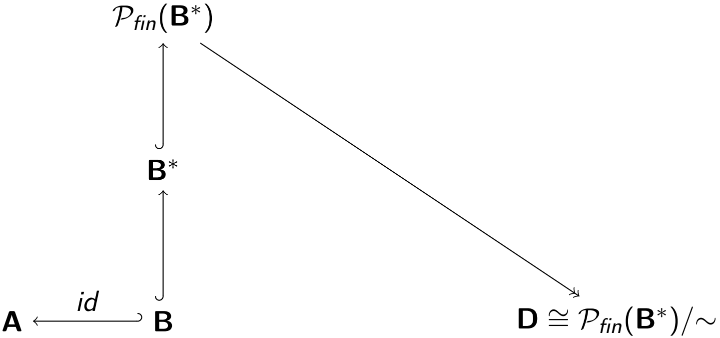

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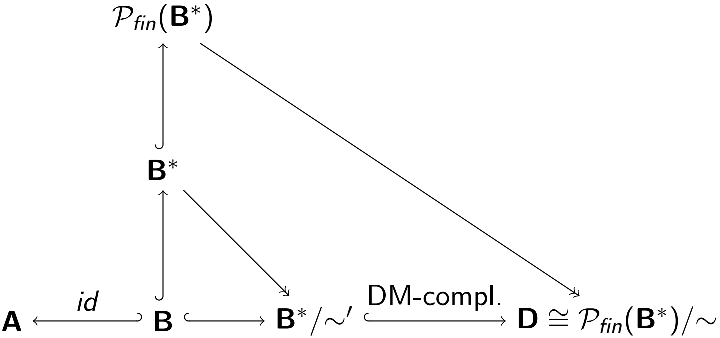

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for all $c \in B$, finite sets $X_1, \ldots, X_n \subseteq B^*$ and $p \in Tr(\mathbf{B}^*)$.

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Example

For instance $x^2 \leq x$ and $X = \{a, b\}$. Then $X^2 = \{a^2, ab, ba, b^2\}$.

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Gentzen rules ($a, b, c \in B$, $p \in Tr(\mathbf{B}^*)$)

$$\frac{p(a) \in L_b}{p(L_a) \subseteq L_b}\ (\mathrm{Cut}) \qquad \frac{}{b \in L_b}\ (\mathrm{Id})$$

$$\frac{p(b) \in L_c}{p(a(a\backslash b)) \in L_c}\ (\backslash\mathrm{L}) \qquad \frac{}{a\backslash L_b \subseteq L_{a\backslash b}}\ (\backslash\mathrm{R})$$

$$\frac{p(a) \in L_c}{p(a \wedge b) \in L_c}\ (\wedge\mathrm{L}) \qquad \frac{}{L_a \cap L_b \subseteq L_{a\wedge b}}\ (\wedge\mathrm{R})$$

$$\frac{p(ab) \in L_c}{p(a \cdot b) \in L_c}\ (\cdot\mathrm{L}) \qquad \frac{}{L_a L_b \subseteq L_{a\cdot b}}\ (\cdot\mathrm{R})$$

$$\frac{p(\{a, b\}) \in L_c}{p(a \vee b) \in L_c}\ (\vee\mathrm{L}) \qquad \frac{}{L_a \cup L_b \subseteq L_{a\vee b}}\ (\vee\mathrm{R})$$

$$\frac{p(\varepsilon) \in L_c}{p(1) \in L_c}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \in L_1}\ (1\mathrm{R})$$

Characterization

Theorem

Let $\mathbb{V}$ be a variety of residuated lattices axiomatized over $\{\vee, \cdot, 1\}$ by a set of inequalities $\mathcal{E}$. T.F.A.E.

1. $\mathbb{V}$ has the FEP.
2. For every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A} \in \mathbb{V}$ there is a collection
$$\{L_b \subseteq \mathrm{Rec}(\mathbf{B}^*) \mid b \in B\}$$

- separating elements of B,
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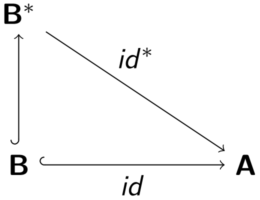
Existing results



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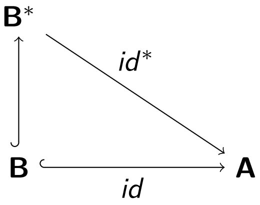

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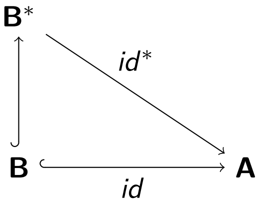

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To prove the FEP, it suffices to show that L_b's are recognizable.

Higman's lemma + Generalized Myhill Theorem imply

Theorem [Blok, van Alten, Galatos, Jipsen]

Every variety $\mathbb{V}$ of integral ($x \leq 1$) residuated lattices axiomatized over $\{\vee, \cdot, 1\}$ has the FEP.

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Let $\mathbb{V}$ be a variety of residuated lattices axiomatized over $\{\vee, \cdot, 1\}$ satisfying $xy = yx$ and $x^m \leq x^n$ for $m \neq n$. Then $\mathbb{V}$ has the FEP.

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Theorem [Cardona, Galatos]

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Existing result

Using Kruskal Tree Theorem and Generalized Myhill Theorem for tree languages, we immediately obtain:

Theorem [Galatos]

Every subvariety of distributive integral residuated lattices axiomatized over $\{\wedge, \vee, \cdot, 1\}$ has the FEP.

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- Is it possible to characterize varieties axiomatized over $\{\vee, \cdot, 1\}$ having the FEP via the characterization of recognizable/regular languages?
- Is it necessary to consider other sets than

$$L_b = \{x \in B^* \mid id^*(x) \leq b\}?$$

  Other sets are used in the proofs of FMP and undecidability proofs.

Thank you!