# Holland's Theorem for Idempotent Semirings and Applications to Residuated Lattices

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# Outline

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1. Motivation
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4. Holland's theorem for idempotent semirings
5. FEP for integral idempotent semirings

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# Motivation

## Theorem (Cayley 1854)

*Every group* $\mathbf{G}$ *is embeddable into* $\mathbf{Sym}(G)$.

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## Theorem (Anderson-Edwards 1984)

*Every distributive* $\ell$*-monoid can be embedded in the* $\ell$*-monoid* **End**(**C**) *of the order-preserving maps on a chain* **C**.

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**Theorem (Paoli-Tsinakis 2010)**

*Every distributive residuated lattice in which multiplication distributes over meets can be embedded as* $\ell$*-monoid into* $\mathbf{Res}(\mathbf{C})$ *for a complete chain* $\mathbf{C}$.

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# Idempotent semirings

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A structure $\mathbf{R} = \langle R, +, \cdot, 1\rangle$ is called a (unital) semiring if

- $\langle R, +\rangle$ is a commutative semigroup,
- $\langle R, \cdot, 1\rangle$ is a monoid,
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- $\mathbf{R}^{+} = \langle R, +\rangle$,
- $\mathbf{R}$ is called idempotent if $a + a = a$. In that case $\mathbf{R}^{+}$ forms a (join)-semilattice.

# Examples

## Definition

Let **P** be a poset. A map $f\colon P \to P$ is said to be residuated iff it has a (left) residual $f^\dagger\colon P \to P$, i.e.

$$f(x) \leq y \qquad \text{iff} \qquad x \leq f^\dagger(y)\,.$$

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Let $\mathbf{L} = \langle L, \vee\rangle$ be a join-semilattice.

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- $\mathrm{Res}(\mathbf{L})$ forms a subsemiring $\mathbf{Res}(\mathbf{L})$ of $\mathbf{End}(\mathbf{L})$ since residuated maps are closed under composition and pointwise join.

# Relational endomorphisms

Let **L** be a join-semilattice. Recall that ideals on **L** forms an algebraic lattice $\mathcal{I}(\mathbf{L}) = \langle \mathcal{I}(\mathbf{L}), \cap, \vee \rangle$. We identify binary relations on **L** with functions from $L$ to $\mathcal{P}(L)$.

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The set of all compatible relations on **L** forms an idempotent semiring $\mathbf{REnd}(\mathbf{L}) = \langle \mathrm{REnd}(\mathbf{L}), \vee, \circ, Id \rangle$, where $Id(x) = {\downarrow}x$.

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## Lemma

$\mathbf{REnd}(\mathbf{L}) \cong \mathbf{Res}(\mathcal{I}(\mathbf{L}))$.

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# Idempotent semimodules

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Let $\mathbf{R} = \langle R, +, \cdot, 1 \rangle$ be a semiring. A left **R**-semimodule **M** is a commutative semigroup $\langle M, + \rangle$ together with a map $\star \colon R \times M \to M$ such that:

- $r \star (m + n) = r \star m + r \star n$,
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- A right $\mathbf{R}$-semimodule is defined analogously.

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Every commutative semigroup (join-semilattice) $\mathbf{L} = \langle L, +\rangle$ can be turned into an (idempotent) **End**($\mathbf{L}$)-semimodule where the left action $\star\colon \mathrm{End}(\mathbf{L}) \times L \to L$ is defined by $f \star m = f(m)$.

# Separation set

Let **M** be a left **R**-semimodule over a semiring **R**.

## Definition

A subset $E \subseteq M$ is called a separating set in **M** if for all $r, s \in R$ we have the following implication:

$$r \neq s \quad \Longrightarrow \quad (\exists e \in E)(r \star e \neq s \star e)\,.$$

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- If **M** has a separating set $E$, then $\phi$ is an embedding.
- The above holds also for a right **R**-semimodule if we replace $\mathbf{End}(\mathbf{M}^+)$ by $\mathbf{End}(\mathbf{M}^+)^{op}$.

# Cayley's theorem for idempotent semirings

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- Every idempotent semiring $\mathbf{R}$ can be viewed as an idempotent $\mathbf{R}$-semimodule whose left action is just the multiplication in $\mathbf{R}$.

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### Proof.

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- Moreover, $\{1\}$ is a separating set in the semimodule $\mathbf{R}$.

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# Residuated semimodules

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*Let* **M** *be a residuated* **R***-semimodule such that* $\mathbf{M}^+$ *forms a lattice. Then*

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*Thus* $(\mathbf{M}^+)^\partial$ *together with* $\backslash$ *forms a right idempotent* **R***-semimodule.*

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## Theorem

*Any idempotent semiring* **R** *is embeddable into* $\mathbf{REnd}(\mathbf{R}^+)$ *which is isomorphic to* $\mathbf{Res}(\mathcal{I}(\mathbf{R}^+))$*.*

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## Lemma

*An ideal $I \in \mathcal{I}(\mathbf{M}^+)$ is linear iff $\mathbf{M}/{\sim_I}$ is linearly ordered.*

# (EC) condition

- Consider the following quasi-identity in the language of semimodules:

$$u \leq h \vee c \star a \quad \& \quad u \leq h \vee d \star b \quad \Longrightarrow \quad u \leq h \vee c \star b \vee d \star a \,. \quad \text{(EC)}$$

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- The same quasi-identity can be considered also in the language of semirings:

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- If an idempotent semiring **R** satisfies (EC) as a semiring then it satisfies (EC) when viewed as an **R**-semimodule and also vice versa.

# Quasivariety generated by chains

Let $\mathbf{R}$ be an idempotent semiring.

## Lemma

1. *Every linearly ordered left idempotent* $\mathbf{R}$*-semimodule* $\mathbf{M}$ *satisfies* (EC)*.*
2. *Conversely, if* $\mathbf{M}$ *satisfies* (EC) *then every ideal* $I \in \mathcal{I}(\mathbf{M}^{+})$ *maximal with respect to not containing an element* $u$ *is linear.*

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## Theorem

*Let* **M** *be a left idempotent* **R***-semimodule. T.F.A.E:*

1. **M** *satisfies* (EC)*.*
2. **M** *is embeddable into* $\prod_{i\in K}\mathbf{N}_i$ *for some family* $\{\mathbf{N}_i \mid i \in K\}$ *of linearly ordered left idempotent* **R***-semimodules.*

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- $\{\mathbf{M}_i \mid i \in K\}$ a family of left idempotent **R**-semimodules whose left actions are denoted $\star_i$.
- Then the ordinal sum $\bigoplus_{i \in K} \mathbf{M}_i$ is a left idempotent **R**-semimodule, whose underlying join-semilattice is the ordinal sum of $\{\mathbf{M}_i^+ \mid i \in K\}$ and its left action is given by

$$r \star m = r \star_i m \quad \text{if } m \in M_i \,.$$

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*The following are equivalent:*

1. **R** *satisfies* (EC)*.*
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3. **R** *is embeddable into* $\mathbf{REnd}(\mathbf{C}) \cong \mathbf{Res}(\mathcal{I}(\mathbf{C}))$ *for some chain* **C***.*

# Outline

# Finite embeddability property

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- An idempotent semiring **R** is said to be integral if 1 is a top element with respect to the join-semilattice order on **R**.
- We denote the variety of all integral idempotent semirings by ISR and Q its sub-quasivariety axiomatized by (EC).

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## Corollary

*Let* $\mathsf{K}$ *be a subvariety of* $\mathsf{ISR}$ *and* $\mathsf{Q}$ *the quasivariety of idempotent semirings axiomatized by* (EC). *Then* $\mathsf{K}$ *and* $\mathsf{K} \cap \mathsf{Q}$ *have the finite embeddability property.*

# Outline

# Residuated lattices

## Definition

A residuated lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 1\rangle$, where

- $\langle A, \wedge, \vee\rangle$ is a lattice,
- $\langle A, \cdot, 1\rangle$ is a monoid and
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- If $\mathbf{L}$ is a complete lattice then $\mathbf{Res}(\mathbf{L})$ is a complete residuated lattice.
- If $\mathbf{L}$ is a lattice then $\mathbf{REnd}(\mathbf{L}) \cong \mathbf{Res}(\mathcal{I}(\mathbf{L}))$ is a complete residuated lattice.

# Interior operators

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Let **P** be a poset. A map $\sigma\colon P \to P$ is called an interior operator if

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## Observation

Let **L** be a complete lattice and $S \subseteq L$. Then $S$ induces an interior operator on **L**:

$$\sigma_S(x) = \bigvee \{s \in S \mid s \leq x\} .$$

# Conuclei

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## Theorem

*Let $\sigma$ be a conucleus on a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$. Then $\mathbf{L}_\sigma = \langle \sigma[L], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1\rangle$ is a residuated lattice called conuclear contraction, where $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x/y = \sigma(x/y)$.*

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## Lemma

*Let $\mathbf{A}$ be a complete residuated lattice and $S$ a submonoid of $\mathbf{A}$. Then the interior operator $\sigma_S$ on $\mathbf{A}$ is a conucleus.*

# Key lemma

## Lemma

*Assumptions:*

- *Let* $\mathbf{A}, \mathbf{B}$ *be residuated lattices such that* $\mathbf{B}$ *is complete and* $\mathbf{C}$ *a partial subalgebra of* $\mathbf{A}$.

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*Conclusions:*

- *Then* $\sigma_{\phi[D]}$ *is a conucleus and*
- $\phi\colon \mathbf{C} \to \mathbf{B}_{\sigma_{\phi[D]}}$ *is an embedding of residuated lattices.*

# Cayley's theorem for residuated lattices

## Theorem

*Let $\mathbf{A}, \mathbf{B}$ be residuated lattices such that $\mathbf{B}$ is complete. If $\mathbf{A}$ embeds into $\mathbf{B}$ via $\phi$ as an idempotent semiring, then $\mathbf{A}$ embeds into $\mathbf{B}_{\sigma_{\phi[A]}}$ as a residuated lattice.*

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## Corollary (Cayley's theorem for residuated lattices)

*Let* $\mathbf{A}$ *be a residuated lattice and* $\mathbf{A}^+$ *its join-semilattice reduct. Then* $\mathbf{A}$ *embeds into a conuclear contraction of* $\mathbf{REnd}(\mathbf{A}^+) \cong \mathbf{Res}(\mathcal{I}(\mathbf{A}^+))$*.*
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## Theorem (Blount-Tsinakis)

*Every residuated lattice embeds into a nuclear retraction of a powerset monoid.*

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3. $\mathbf{A}$ *embeds into a conuclear contraction of* $\mathbf{Res}(\mathbf{C}')$ *for a complete chain* $\mathbf{C}'$*.*

# Classes where Holland's theorem (does not) applies

- Prelinear residuated lattices, i.e., those where $1 = (x \setminus y \wedge 1) \vee (y \setminus x \wedge 1)$ holds. This class includes all semilinear varieties and $\ell$-groups.

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- There are also non-distributive integral residuated lattices satisfying (EC).
- Let $\mathbf{Z}_2 = \langle \{0, 1\}, +, 0 \rangle$ be the two-element group (ordered discretely). Consider its extension by a top and bottom element $\top, \bot$. Then its lattice reduct is distributive and (EC) does not hold in this extension.

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## Sketch of the proof for $\mathsf{V}_2$

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## Sketch of the proof for $\mathsf{V}_2$

- Let K be a subvariety of ISR axiomatized by $\mathcal{E}$ and

# FEP for integral residuated lattices

Let IRL be the variety of integral residuated lattices (i.e., $x \leq 1$).

## Theorem

- *Let* $\mathsf{V}_1$ *be a subvariety of* IRL *axiomatized by the set* $\mathcal{E}$ *of identities using only* $\vee, \cdot, 1$.
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## Sketch of the proof for $\mathsf{V}_2$

- Let K be a subvariety of ISR axiomatized by $\mathcal{E}$ and
- Q the quasivariety of idempotent semirings axiomatized by (EC).
- Suppose $\mathbf{A} \in \mathsf{V}_2$. Then its semiring reduct belongs to $\mathsf{K} \cap \mathsf{Q}$.

# Sketch of the proof (cont.)

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# Sketch of the proof (cont.)

- Let $\mathbf{C}$ be a finite partial subalgebra of $\mathbf{A}$.
- Consider the subsemiring $\mathbf{R}$ of $\mathbf{A}$ generated by $C$.
- Then $\mathbf{R} \in \mathsf{K} \cap \mathsf{Q}$.

## Sketch of the proof (cont.)

- Let **C** be a finite partial subalgebra of **A**.
- Consider the subsemiring **R** of **A** generated by $C$.
- Then **R** $\in \mathsf{K} \cap \mathsf{Q}$.
- By FEP for $\mathsf{K} \cap \mathsf{Q}$ there is a finite semiring **S** $\in \mathsf{K} \cap \mathsf{Q}$ and

# Sketch of the proof (cont.)

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- there is a surjective semiring homomorphism $\phi\colon \mathbf{R} \to \mathbf{S}$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for all $r \in R$ and $c \in C$.

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- Since **A** is integral, it is possible to show that **S** is in fact a finite residuated lattice.
- Thus **C** embeds as a residuated lattice into $\mathbf{S}_\sigma$ for some conucleus $\sigma$
- Since $\mathbf{S}_\sigma$ is a subsemiring of **S**, $\mathbf{S}_\sigma$ is finite and belongs to $\mathsf{K} \cap \mathsf{Q}$.

Thank you!