Density Elimination and the Corresponding Algebraic Construction

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Workshop on algebraic proof theory
Tbilisi Symposium on Language, Logic and Computation
September 23–27, 2013

Outline

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3. Corresponding algebraic construction

Uninorm logic (UL)

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  UL is the logic given by the hypersequent calculus for Full Lambek extended by exchange and Avron's communication rule.

- Algebraically:

  UL is the logic of commutative totally ordered FL-algebras (FL_e-chains).

Algebraic semantics

Definition

An $\mathrm{FL_e}$-algebra is a pointed commutative residuated lattice, i.e., an algebra

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- A totally ordered $\mathrm{FL_e}$-algebra is called $\mathrm{FL_e}$-chain.
- An $\mathrm{FL_e}$-chain $\boldsymbol{A}$ is dense if $a < b$ implies $a < c < b$ for some $c \in A$.

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Theorem (Metcalfe, Montagna 2007; Ciabattoni, Metcalfe 2008)

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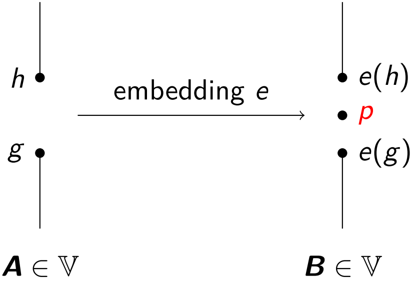
- The right occurrences of p in d are replaced as follows:

$$\Lambda \Rightarrow p \ \rightsquigarrow \ \Lambda, \Sigma \Rightarrow \Pi\,.$$

Densifiable varieties

Definition

A variety $\mathbb{V}$ ordered algebras is said to be densifiable if every gap (g, h) of a chain in $\mathbb{V}$ can be filled by another chain in $\mathbb{V}$.



Theorem

Let $\mathbb{V}$ be a densifiable variety. Then every (nontrivial) finite or countable chain in $\mathbb{V}$ is embeddable into a countable dense chain in $\mathbb{V}$.

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$$\begin{aligned} ap^n \ N \ b \quad &\text{iff} \quad ah^n \leq b \\ a \ N \ p \quad &\text{iff} \quad a \leq g \\ ap^m \ N \ p \quad &\text{iff} \quad ah^{m-1} \leq 1 \end{aligned}$$

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Lemma

- $\boldsymbol{W}$ *can be extended so that it is residuated.*
- $\boldsymbol{W}^+$ *forms an* $\mathsf{FL_e}$*-chain.*
- $\boldsymbol{A}$ *embeds into* $\boldsymbol{W}^+$ *via* $x \mapsto \{x\}^{\rhd\lhd}$.
- $\{g\}^{\rhd\lhd} \subsetneq \{p\}^{\rhd\lhd} \subsetneq \{h\}^{\rhd\lhd}$.

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Fact

Every complete $\mathrm{FL_e}$-algebra forms an ic-semiring.

Formal power series

Let $\boldsymbol{A} = \langle A, \vee, \cdot, 1, \bot \rangle$ be an ic-semiring. Then the ic-semiring of formal power series $\boldsymbol{A}[\![X]\!]$ consists of:

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$\boldsymbol{A}$ embeds into $\boldsymbol{A}[\![X]\!]$ via $a \mapsto a \vee \bigvee_{n \geq 1} \bot X^n$.

Formal power series (cont.)

Let $\boldsymbol{A} = \langle A, \vee, \wedge, \cdot, \to, 1, 0 \rangle$ be a complete $\mathrm{FL_e}$-algebra. Then $\boldsymbol{A}[\![X]\!]$ is a complete $\mathrm{FL_e}$-algebra as well where

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Nuclei

Let $\boldsymbol{A}$ be an $\mathrm{FL_e}$-algebra. Then the following concepts are equivalent:

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Fact

Then $\boldsymbol{A}/\theta$ is (not only an ic-semiring but also) an $\mathsf{FL_e}$-algebra.

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- $\hat{g} < \tilde{1} < \hat{h}$.

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   Generate the nuclear retraction by $\{\hat{a} \mid a \in A\} \cup \{\tilde{1}\}$. Thus we introduce elements $\tilde{a} = a \to \tilde{1}$.

Thank you!