Disjunction Property and Complexity of Substructural Logics

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Basic substructural logics

- Substructural logics are logics lacking some of the structural rules (contraction, exchange, left and right weakening).

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}\ \text{(c)} \qquad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}\ \text{(e)}$$

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}\ \text{(i)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ \text{(o)}$$

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$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi} \text{(i)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \text{(o)}$$

- The weakest substructural logic called FL is a logic obtained from the sequent calculus for Intuitionistic logic by omitting (c), (e), (i) and (o). The language of FL: $\wedge, \vee, \cdot, /, \backslash, 0, 1$.
- Let $S \subseteq \{c, e, i, o\}$. Then FL_S denotes the extension of FL by rules from S.

Cut elimination

Theorem (Komori, Ono)

Let $\{c\} \neq S \subseteq \{e, c, i, o\}$*. Then* FL_S *enjoys cut elimination.*

The logic FL_c has not cut elimination. However, it can be equally presented by a different sequent calculus FL_{gc} arising from FL_c by replacing the contraction rule (c) by the global contraction rule (gc):

$$\frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \varphi}{\Gamma, \Sigma, \Delta \Rightarrow \varphi}\ \text{(gc)}$$

where Σ is any finite sequence of formulas. This means that a sequent is provable in FL_c iff it is provable in FL_{gc}.

Theorem

Cut elimination holds for FL_{gc}*.*

Axiomatic extensions

- By a substructural logic we mean an axiomatic extension of FL.

- Let Γ be a set of axiomatic schemata. The axiomatic extension of FL by Γ is the calculus obtained from FL by adding new initial sequents $\Rightarrow \varphi$ for all formulas $\varphi \in \Gamma$.

- Let L be a substructural logic. The provability relation of L is denoted by $\vdash_{\mathrm{L}}$.

- Given $S \subseteq \{c, e, i, o\}$, the logic FL_S can be viewed as an axiomatic extension of FL. The following schemata correspond respectively to (c),(e), (i) and (o):

$$\alpha \backslash (\alpha \cdot \alpha)\,, \qquad (\alpha \cdot \beta) \backslash (\beta \cdot \alpha)\,, \qquad \alpha \backslash 1\,, \qquad 0 \backslash \alpha\,.$$

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- Thus there is a dual isomorphism V between the lattice of axiomatic extensions of FL and the subvariety lattice of FL-algebras.
- Let L be a substructural logic. Then we have the following equivalences:

$$\vdash_L \varphi \quad \text{iff} \quad \models_{\mathsf{V}(L)} 1 = 1 \wedge \varphi \quad [1 \leq \varphi]\,.$$

$$\models_{\mathsf{V}(L)} \varphi = \psi \quad \text{iff} \quad \vdash_L (\varphi \backslash \psi) \wedge (\psi \backslash \varphi)\,.$$

Algebraizability

- FL is algebraizable and its equivalent algebraic semantics is the variety of FL-algebras.
- Thus there is a dual isomorphism V between the lattice of axiomatic extensions of FL and the subvariety lattice of FL-algebras.
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- By complexity of a logic L we mean the complexity of its set of theorems. Due to algebraizability it is the same as the complexity of the equational theory for V(L).

Algebraic semantics

Definition

An FL-algebra is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1\rangle$, where $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid and

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z\,.$$

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$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z\,.$$

Lemma

Let $\mathbf{A}$ be a nontrivial FL-algebra. Then there is $a \in A$ such that $a < 1$.

Proof.

Since $\mathbf{A}$ is nontrivial, there is $b \in A$ such that $b \neq 1$. If $1 \not\leq b$ then $a = b \wedge 1 < 1$. If $b > 1$ then we take $a = b \backslash 1 \leq 1 \backslash 1 = 1$. Moreover, $a < 1$ otherwise $b = b \cdot a = b \cdot (b \backslash 1) \leq 1$. □

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  - The fact the MALL is in PSPACE follows from the existence of a cut-free sequent calculus.
  - PSPACE-hardness is proved by reduction to QBF.
  - The proof is proof-theoretical, long, and very technical (does not work in the presence of weakening).
- FL is PSPACE-complete (Kanovich 94, Kanazawa 99). The proof is a modification of the proof for MALL. The coding is simplified, uses QBF where the propositional part is in DNF.

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3. We will show by algebraic means that any substructural logic having a stronger version of disjunction property is PSPACE-hard.

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2. We would like to find method which is more general and can be applied possibly to a wider class of substructural logics. Our proof of PSPACE-hardness should be more algebraic therefore less dependent on the sequent calculus.

3. We will show by algebraic means that any substructural logic having a stronger version of disjunction property is PSPACE-hard.

4. In fact, this result is analogous to that we have for superintuitionistic logics saying that each superintuitionistic logic with the disjunction property is PSPACE-hard.

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Definition (Strong Disjunction Property)

Let L be a substructural logic. Then L satisfies *Strong Disjunction Property* (SDP) if for all formulas φ, ψ and atoms $v_1, \ldots, v_n$ we have $\vdash_{\mathrm{L}} (v_1 \cdots v_n) \backslash (\varphi \vee \psi)$ implies $\vdash_{\mathrm{L}} (v_1 \cdots v_n) \backslash \varphi$ or $\vdash_{\mathrm{L}} (v_1 \cdots v_n) \backslash \psi$.

Theorem

Let $S \subseteq \{e, c, i, o\}$*. Then* FL_S *has SDP.*

Proof of SDP

- Let $\alpha = a_1 \cdots a_5$ where a_i's are atoms.
- Provability of $\alpha \backslash (\varphi \vee \psi)$ means that the sequent $a_1, a_2, a_3, a_4, a_5 \Rightarrow \varphi \vee \psi$ has a cut-free proof.
- If $S = \{c\}$, then consider a cut-free proof in FL_{gc}.

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Our results

Theorem

Let L be a consistent substructural logic having SDP. Then L is PSPACE-hard.

Corollary

Let $S \subseteq \{e, c, i, o\}$. Then FL_S is PSPACE-hard.

For basic substructural logics without contraction we can even obtain PSPACE-completeness.

Theorem

Let $S \subseteq \{e, i, o\}$. Then FL_S is PSPACE-complete.

Quantified Boolean formulas (QBF)

- Let us start with a quantified Boolean formula

$$Q_n x_n \cdots Q_1 x_1 B(x_1, \ldots, x_n)\,,$$

where $Q_i \in \{\forall, \exists\}$ and $B(x_1, \ldots, x_n)$ is in DNF.

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- Thus $B(x_1, \ldots, x_n) = D_1 \vee \cdots \vee D_k$, where D_i's are conjunctions of literals (i.e., x_j or $\neg x_j$) and every x_j appears in D_i at most once.

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- $\forall x \exists y\, (x \wedge \neg y) \vee (\neg x \wedge y)$ is true.

- $\exists y \forall x\, (x \wedge \neg y) \vee (\neg x \wedge y)$ if false.

Coding – propositional part

- Given $A = Q_n x_n \cdots Q_1 x_1 B(x_1, \ldots, x_n)$, we define FL-formula A^* in variables $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$.

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- Let e be a classical evaluation of $x_1, \ldots, x_n$. Define

$$x_j^e = \begin{cases} x_j & \text{if } e(x_j) = 1, \\ \bar{x}_j & \text{if } e(x_j) = 0. \end{cases}$$

Then $e^* = x_1^e \cdots x_n^e$.

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- For each D_i we define $D_i^* = y_1 \cdots y_n$, where

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- Then $B^* = D_1^* \vee \cdots \vee D_k^*$.

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Lemma

Let V *be a nontrivial subvariety of* FL *and* e *a classical evaluation of* $x_1, \dots, x_n$*. Then* $e(B) = 1$ *iff* $\models_{\mathsf{V}} 1 \leq e^* \setminus B^*$*.*

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- Case 2 – $\neg x_j \in D_i$: then (i) $e(x_j) = 0$, i.e., $x_j^e = \bar{x}_j$. (ii) $y_j = \bar{x}_j$.
- Case 3 – $x_j, \neg x_j \notin D_i$: then $x_j^e \leq x_j \vee \bar{x}_j = y_j$.

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- We prove that $e^* = x_1^e \cdots x_n^e \leq y_1 \cdots y_n = D_i^*$ in each $\mathbf{A} \in \mathsf{V}$ by showing $x_j^e \leq y_j$ for every j.
- Case 1 – $x_j \in D_i$: then (i) $e(x_j) = 1$, i.e., $x_j^e = x_j$. (ii) $y_j = x_j$.
- Case 2 – $\neg x_j \in D_i$: then (i) $e(x_j) = 0$, i.e., $x_j^e = \bar{x}_j$. (ii) $y_j = \bar{x}_j$.
- Case 3 – $x_j, \neg x_j \notin D_i$: then $x_j^e \leq x_j \vee \bar{x}_j = y_j$.
- Thus $\mathbf{A} \models 1 \leq e^* \setminus B^*$ for each $\mathbf{A} \in \mathsf{V}$.

Sketch of the proof ($\Leftarrow$)

- Let $\mathbf{A} \in \mathsf{V}$ be a nontrivial algebra. We have $a \in A$ such that $a < 1$. Let v be the $\mathbf{A}$-evaluation s.t.

$$v(x_j) = \begin{cases} 1 & \text{if } e(x_j) = 1, \\ a & \text{if } e(x_j) = 0, \end{cases} \qquad v(\bar{x}_j) = \begin{cases} a & \text{if } e(x_j) = 1, \\ 1 & \text{if } e(x_j) = 0. \end{cases}$$

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- Thus we have a polynomial-time translation of $Q_n x_n \cdots Q_1 x_1 B(x_1, \ldots, x_n)$ to the FL-formula A_n^*.

Main lemma

Lemma

Let L *be a consistent substructural logic having SDP,* $0 \leq i \leq n$*, and* e *be a classical evaluation of* $x_{i+1}, \ldots, x_n$*. Then* $e(Q_i x_i \cdots Q_1 x_1 B) = 1$ *iff* $\vdash_{\mathrm{L}} e^* \setminus A_i^*$*.*

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- Note that $e_0^* = \bar{x}_i \cdot e^*$ and $e_1^* = x_i \cdot e^*$.

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- Thus $\bar{x}_i \cdot e^* = e_0^* \leq A_{i-1}^*$, i.e., $e^* \leq \bar{x}_i \setminus A_{i-1}^*$.
- By transitivity of the implication we get

$$e^* \leq \bar{x}_i \setminus A_{i-1}^* \leq (\bar{x}_i \setminus q_i)/(A_{i-1}^* \setminus q_i) \leq ((x_i \setminus q_i) \vee (\bar{x}_i \setminus q_i))/(A_{i-1}^* \setminus q_i \wedge 1) = A_i^* .$$

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- Assume that $e(Q_i x_i \cdots Q_1 x_1 B) = 1$ and $Q_i = \exists$.
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- Consequently, $1 \leq e^* \setminus A_i^*$ for each algebra $\mathbf{A} \in \mathsf{V}(\mathsf{L})$.

Proof (⇐)

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- By induction hypotheses there are $\mathbf{A}_0, \mathbf{A}_1 \in \mathrm{V(L)}$ such that $\mathbf{A}_0 \not\models e_0^* \setminus A_{i-1}^*$ and $\mathbf{A}_1 \not\models e_1^* \setminus A_{i-1}^*$.

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e_0^* \setminus A_{i-1}^* &= \bar{x}_i \cdot e^* \setminus A_{i-1}^* = e^* \setminus (\bar{x}_i \setminus A_{i-1}^*)\,, \\
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- By SDP there is $\mathbf{A} \in \mathrm{V(L)}$ such that $\mathbf{A} \not\models 1 \leq e^* \setminus ((\bar{x}_i \setminus A_{i-1}^*) \vee (x_i \setminus A_{i-1}^*))$, i.e., there is an $\mathbf{A}$-evaluation v such that $v(e^*) > v((\bar{x}_i \setminus A_{i-1}^*) \vee (x_i \setminus A_{i-1}^*))$.

Proof ($\Leftarrow$)

- Assume that $e(Q_i x_i \cdots Q_1 x_1 B) = 0$ and $Q_i = \exists$.
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- W.l.o.g. assume that $v(q_i) = v(A_{i-1}^*)$. Then

$$v(e^*) > v((\bar{x}_i \setminus A_{i-1}^*) \vee (x_i \setminus A_{i-1}^*)) = v(((x_i \setminus q_i) \vee (\bar{x}_i \setminus q_i))/(A_{i-1}^* \setminus q_i \wedge 1)) = v(A_i^*)\,.$$

Consequently, $\mathbf{A} \not\models 1 \leq e^* \setminus A_i^*$.

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- We have now a method for proving PSPACE-hardness of a substructural logic just by showing SDP.
- Is there an algebraic proof of SDP for basic substructural logics?
- We need even less than SDP, namely

  if $\vdash_L \alpha \backslash((x \backslash \varphi) \vee (\bar{x} \backslash \varphi))$ then $\vdash_L \alpha \backslash(x \backslash \varphi)$ or $\vdash_L \alpha \backslash(\bar{x} \backslash \varphi)$,

  where φ is a formula, α is a product of atoms, and $x, \bar{x}$ are atoms.