STANDARD COMPLETENESS THEOREM FOR ΠMTL

ROSTISLAV HORČÍK

Abstract. ΠMTL is a schematic extension of the monoidal t-norm based logic (MTL) by the characteristic axioms of product logic. In this paper we prove that ΠMTL satisfies the standard completeness theorem. From the algebraic point of view, we show that the class of ΠMTL-algebras (bounded commutative cancellative residuated l-monoids) in the real unit interval $[0, 1]$ generates the variety of all ΠMTL-algebras.

1. Introduction

In [3], Esteva and Godo introduced a monoidal t-norm based logic (MTL for short). MTL was obtained from Hájek's basic logic (BL) [6] by omitting the axiom of divisibility. Without this axiom continuity of the t-norm representing the truth function for the conjunction is not ensured. The algebraic counterpart of MTL (algebras of truth values) are bounded commutative residuated l-monoids satisfying the pre-linearity axiom. Further the authors of [3] showed that MTL is a reasonable fuzzy logic satisfying the completeness theorem.

Recently, the question of standard completeness of MTL and its extensions has been deeply studied. Jenei and Montagna showed that MTL satisfies the standard completeness theorem (see [9]). This means that MTL can be really considered the logic of left-continuous t-norms. Further in [4], the authors studied the standard completeness of several schematic extensions of MTL. They succeeded in proving the standard completeness theorem for IMTL and SMTL, where IMTL is an extension of MTL by the axiom of double negation ($\neg\neg\varphi \Rightarrow \varphi$) and SMTL is an extension of MTL by the axiom ensuring that the negation is Gödel negation ($\varphi \wedge \neg\varphi \Rightarrow \overline{0}$). They also tried to prove the standard completeness for ΠMTL which is an axiomatic extension of MTL where the conjunction is interpreted by a left-continuous cancellative t-norm. They showed that ΠMTL is complete with respect to all ΠMTL-chains whose underlying sets are rational numbers from the unit interval, i.e., $[0, 1] \cap \mathbb{Q}$. However, they did not succeed to extend this result to the whole unit interval $[0, 1]$.

In this paper we are going to present how to overcome this problem and we show that ΠMTL is standard complete, i.e., a formula φ is provable in ΠMTL iff φ is a tautology in all ΠMTL-algebras in the real interval $[0, 1]$.

Throughout the text, $\mathbb{N}$ denotes the set of natural numbers with ordinary order including also 0, i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$, and ω^* denotes the set of natural numbers without 0 endowed with the reverse order ($1 > 2 > 3 \ldots$).

2. Preliminaries

Since ΠMTL is a schematic extension of MTL, we firstly introduce MTL. The language of MTL contains a set of propositional variables, a conjunction &, an implication $\Rightarrow$, the minimum conjunction $\wedge$, and the truth constant $\overline{0}$. Derived

The work was supported by the Grant Agency of the Czech Republic under projects GACR 201/02/1540, 401/03/H047, and by Net CEEPUS SK-042.

connectives are defined as follows:

$$\begin{array}{rll} \varphi\vee\psi & \text{is} & ((\varphi\Rightarrow\psi)\Rightarrow\psi)\wedge((\psi\Rightarrow\varphi)\Rightarrow\varphi)\,,\\ \neg\varphi & \text{is} & \varphi\Rightarrow\overline{0}\,,\\ \varphi\equiv\psi & \text{is} & (\varphi\Rightarrow\psi)\&(\psi\Rightarrow\varphi)\,,\\ \overline{1} & \text{is} & \neg\overline{0}\,. \end{array}$$

In [3], the authors introduced a Hilbert style calculus for MTL with the following axiomatization:

$$\begin{array}{ll} \text{(A1)} & (\varphi\Rightarrow\psi)\Rightarrow((\psi\Rightarrow\chi)\Rightarrow(\varphi\Rightarrow\chi))\,,\\ \text{(A2)} & \varphi\&\psi\Rightarrow\varphi\,,\\ \text{(A3)} & \varphi\&\psi\Rightarrow\psi\&\varphi\,,\\ \text{(A4)} & (\varphi\wedge\psi)\Rightarrow\varphi\,,\\ \text{(A5)} & (\varphi\wedge\psi)\Rightarrow(\psi\wedge\varphi)\,,\\ \text{(A6)} & (\varphi\&(\varphi\Rightarrow\psi))\Rightarrow(\varphi\wedge\psi)\,,\\ \text{(A7a)} & (\varphi\Rightarrow(\psi\Rightarrow\chi))\Rightarrow(\varphi\&\psi\Rightarrow\chi)\,,\\ \text{(A7b)} & (\varphi\&\psi\Rightarrow\chi)\Rightarrow(\varphi\Rightarrow(\psi\Rightarrow\chi))\,,\\ \text{(A8)} & ((\varphi\Rightarrow\psi)\Rightarrow\chi)\Rightarrow(((\psi\Rightarrow\varphi)\Rightarrow\chi)\Rightarrow\chi)\,,\\ \text{(A9)} & \overline{0}\Rightarrow\varphi\,. \end{array}$$

The deduction rule of MTL is modus ponens.

The corresponding algebras of truth values are so-called MTL-algebras. MTL-algebras form a subvariety of residuated l-monoids. A lattice-ordered monoid (or l-monoid) is a monoid which is at the same time a lattice and satisfies the isotonicity condition: $a \leq b$ implies $a*c \leq b*c$ and $c*a \leq c*b$ (for details on residuated l-monoids, see [1, 5, 8]).

DEFINITION 2.1. *An* MTL-*algebra is a bounded commutative residuated l-monoid* $(L,*,\to,\sqcap,\sqcup,\mathbf{0},\mathbf{1})$*, where* $(L,*,\sqcap,\sqcup,\mathbf{0},\mathbf{1})$ *is a bounded commutative l-monoid,* $(*,\to)$ *is a residuated pair, i.e.,* $x*y\leq z$ *iff* $x\leq y\to z$*, and the pre-linearity equation is satisfied:*

$$(x\to y)\sqcup(y\to x)=\mathbf{1}\,.$$

Throughout the text, we will also use without mentioning the alternative signature for an MTL-algebra using the lattice order $\leq$ instead of $\sqcap,\sqcup$. The symbol a^n stands for $a*\cdots*a$ (n-times).

In this paper we deal with the extension of MTL by product axioms ensuring that the conjunction is strict and cancellative. This extension was introduced by Hájek in [7].

DEFINITION 2.2. *A* ΠMTL *logic is a schematic extension of* MTL *by the following axioms:*

$$\begin{array}{ll} (\Pi1) & \neg\neg\psi\Rightarrow[((\varphi\&\psi)\Rightarrow(\chi\&\psi))\Rightarrow(\varphi\Rightarrow\chi)]\,,\\ (\Pi2) & \varphi\wedge\neg\varphi\Rightarrow\overline{0}\,. \end{array}$$

The corresponding algebras of truth values (ΠMTL-algebras) form a subvariety of MTL-algebras.

DEFINITION 2.3. *A* ΠMTL-*algebra* $\mathbf{L}=(L,*,\to,\sqcap,\sqcup,\mathbf{0},\mathbf{1})$ *is* MTL-*algebra satisfying the identities corresponding to the axioms* (Π1) *and* (Π2)*:*

(1) $[(z\to\mathbf{0})\to\mathbf{0}]\to[(x*z\to y*z)\to(x\to y)]=\mathbf{1}\,,$
(2) $x\sqcap(x\to\mathbf{0})=\mathbf{0}\,.$

A linearly ordered ΠMTL-*algebra is called a* ΠMTL-*chain.*

Having defined the algebras of truth values, we can define an evaluation assigning to each formula a truth value.

DEFINITION 2.4. *Let* $\mathbf{L} = (L, *, \to, \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ *be a* ΠMTL-*algebra. An* $\mathbf{L}$-*evaluation is a map* e *from the set of* ΠMTL *formulas into* $\mathbf{L}$ *such that for every pair* φ, ψ *of formulas the following conditions hold:*

(1) $e(\varphi \& \psi) = e(\varphi) * e(\psi)$,
(2) $e(\varphi \Rightarrow \psi) = e(\varphi) \to e(\psi)$,
(3) $e(\varphi \wedge \psi) = e(\varphi) \sqcap e(\psi)$,
(4) $e(\overline{0}) = \mathbf{0}$.

In order to study the structure of ΠMTL-chains, we have to work with congruences. Therefore we recall here the notion of a filter because of its connection to the congruence lattice (see [3]).

DEFINITION 2.5. *Let* $\mathbf{L} = (L, *, \to, \leq, \mathbf{0}, \mathbf{1})$ *be a* ΠMTL-*algebra. A filter* F *in* $\mathbf{L}$ *is a subset of* L *satisfying:*

(1) *if* $x, y \in F$, *then* $x * y \in F$,
(2) *if* $x \in F$, $x \leq y$, *then* $y \in F$.

LEMMA 2.6. *For any filter* F *in a* ΠMTL-*algebra* $\mathbf{L}$, *let us define the following equivalence relation in* L:

$$x \sim_F y \text{ iff } x \to y \in F \text{ and } y \to x \in F.$$

Then $\sim_F$ *is a congruence and the quotient* $\mathbf{L}/F$ *is a* ΠMTL-*algebra.*

We will denote the equivalence class containing an element $x \in L$ with respect to a filter F by $[x]_F = \{a \in L \mid a \sim_F x\}$. Observe also that if $\mathbf{L}$ is a ΠMTL-chain then only one of the implications in the definition of $\sim_F$ is important because for all x, y either $x \leq y$ or $y \leq x$, thus either $x \to y = \mathbf{1}$ or $y \to x = \mathbf{1}$.

As it will be seen in the next section, the filters in ΠMTL-chains are also related to so-called Archimedean classes (see [5]).

DEFINITION 2.7. *Let* $\mathbf{L}$ *be a* ΠMTL-*chain,* a, b *be elements of* L, *and* $\sim$ *be an equivalence on* L *defined as follows:*

$$a \sim b \text{ iff there exists an } n \in \mathbb{N} \text{ such that } a^n \leq b \leq a \text{ or } b^n \leq a \leq b.$$

Then for any $a \in L$ *the equivalence class* $[a]_\sim$ *is called* Archimedean class.

Archimedean classes correspond to the subsets of L where the elements behave like in an Archimedean l-monoid, i.e., for any pair of elements $x, y \in [a]_\sim$, such that $x \leq y$, there is an $n \in \mathbb{N}$ such that $y^n \leq x$.

Further, we recall several results that will be useful in the sequel. In [7, Lemma 4], Hájek proved the following result:

PROPOSITION 2.8. *An* MTL-*chain* $\mathbf{L}$ *is a* ΠMTL-*chain if and only if it is cancellative, i.e. for any* $x, y, z \in L$, $z \neq \mathbf{0}$, *if* $x * z = y * z$ *then* $x = y$.

Observe that by cancellativity we obtain for $a, b, c > \mathbf{0}$ that $a < b$ implies $a * c < b * c$, in particular $a^2 < a$ and $a * b < a$ for $b < \mathbf{1}$.

The following important result states that ΠMTL-chains possess a special significance among ΠMTL-algebras (cf. [3]).

THEOREM 2.9 (Subdirect Representation Theorem). *Each* ΠMTL-*algebra is isomorphic to a subdirect product of* ΠMTL-*chains.*

Finally, it was shown in [3] that ΠMTL logic is a reasonable logic which satisfies the completeness theorem.

THEOREM 2.10 (Completeness Theorem). ΠMTL *is sound and complete with respect to the class of* ΠMTL-*chains. In other words,* $\Pi\text{MTL} \vdash \varphi$ *if and only if* $e(\varphi) = \mathbf{1}$ *for every* ΠMTL-*chain* $\mathbf{L}$ *and every* $\mathbf{L}$-*evaluation* e.

The main result of this paper is a strengthening of the latter theorem. We will show that ΠMTL is complete with respect to a smaller class of algebras. Instead of all ΠMTL-chains, we will prove that it is sufficient to consider only ΠMTL-chains in the real unit interval $[0, 1]$ with finitely many Archimedean classes. In other words, we will prove the following result:

The main result: A formula φ is provable in ΠMTL if and only if φ is a tautology in all ΠMTL-chains in $[0, 1]$ with finitely many Archimedean classes.

Notice that one direction of this statement already follows from Theorem 2.10. The second is difficult and we will prove it in the next section.

REMARK 2.11. *From the algebraic point of view, we can rephrase the main result in the sense that* ΠMTL-*chains in the real unit interval* $[0, 1]$ *with finitely many Archimedean classes generate the variety of* ΠMTL-*algebras.*

3. Proof of the main result

The proof has several steps. We will start with a formula φ which is not valid in a ΠMTL-chain $\mathbf{L}$. Then we construct a new ΠMTL-chain $\mathbf{S}$ such that φ is not valid in $\mathbf{S}$, too, and $\mathbf{S}$ has a more transparent structure. Then we will prove that $\mathbf{S}$ is order-isomorphic to the lexicographic product $(\omega^*)^n$. The next step is to extend $\mathbf{S}$ to a continuum. Finally, we will show that this extension of $\mathbf{S}$ is order-isomorphic to $[0, 1]$.

We know from Theorem 2.10 that whenever $\Pi\text{MTL} \not\vdash \varphi$ then there exists a ΠMTL-chain $\mathbf{L} = (L, *_L, \to_L, \leq, \mathbf{0}, \mathbf{1})$ and an $\mathbf{L}$-evaluation $e_{\mathbf{L}}$ such that $e_{\mathbf{L}}(\varphi) < \mathbf{1}$. Let us denote the set of all subformulas of φ by B. Since B is finite, we can assume that $B = \{\psi_1, \dots, \psi_n\}$. Let us define $a_i = e_{\mathbf{L}}(\psi_i)$, $1 \leq i \leq n$, and the following set:

$$G = \{a_i \in L \mid 1 \leq i \leq n\} . \tag{1}$$

Let $\mathbf{S}$ be the submonoid of $\mathbf{L}$ generated by G, i.e. $\mathbf{S} = (S, *, \leq, \mathbf{0}, \mathbf{1})$, where

$$S = \{a_1^{k_1} *_L \cdots *_L a_n^{k_n} \mid a_i \in G,\ k_i \in \mathbb{N},\ 1 \leq i \leq n\} \cup \{\mathbf{0}, \mathbf{1}\} ,$$

and $*$ denotes the restriction of $*_L$ to S.

LEMMA 3.1. *Each subset* $M \subseteq S$ *has a maximum.*

Proof. The proof of this lemma is based on Dickson's lemma stating that each subset of $(\mathbb{N}, <)^n$ has only finitely many minimal elements (the proof of Dickson's lemma in a little bit different form can be found in [2]). To each element $a_1^{k_1} *_L \cdots *_L a_n^{k_n} \in M$ we can assign an n-tuple $(k_1, \dots, k_n) \in \mathbb{N}^n$. Thus we have a subset $H \subseteq \mathbb{N}^n$ such that $(k_1, \dots, k_n) \in H$ implies $a_1^{k_1} *_L \cdots *_L a_n^{k_n} \in M$. Moreover, if $(k_1, \dots, k_n) < (t_1, \dots, t_n)$, we obtain $a_1^{k_1} *_L \cdots *_L a_n^{k_n} > a_1^{t_1} *_L \cdots *_L a_n^{t_n}$ by cancellativity. Since H has only finitely many minimal elements, one of them must correspond to the maximum of M. □ □

Due to Lemma 3.1, we can introduce a residuum on $\mathbf{S}$ as follows:

$$a \to b = \max\{z \in S \mid a * z \leq b\} .$$

THEOREM 3.2. *The enriched submonoid* $\mathbf{S} = (S, *, \to, \leq, \mathbf{0}, \mathbf{1})$ *is a* ΠMTL-*chain and there exists an* $\mathbf{S}$-*evaluation* $e_{\mathbf{S}}$ *such that* $e_{\mathbf{S}}(\varphi) = e_{\mathbf{L}}(\varphi)$.

Proof. Firstly, we know that **S** is a cancellative linearly ordered monoid. Since **S** is a chain, the pre-linearity axiom is obviously satisfied. Thus the only thing which we have to check is whether $(*, \to)$ form a residuated pair, i.e. $a * b \leq c$ iff $a \leq b \to c$. But this easily follows from the definition of $\to$. Hence **S** is a ΠMTL-chain.

Secondly, let us define an evaluation $e_{\mathbf{S}}(v) = e_{\mathbf{L}}(v)$ for each propositional variable v appearing in φ and $e_{\mathbf{S}}(v)$ arbitrary otherwise. Then we show by induction on the complexity of φ that $e_{\mathbf{S}}(\psi_i) = e_{\mathbf{L}}(\psi_i)$ for all subformulas ψ_i of φ, in particular $e_{\mathbf{S}}(\varphi) = e_{\mathbf{L}}(\varphi)$. The first step is obvious by the definition of $e_{\mathbf{S}}$. Now suppose that $\psi_k = \psi_i \& \psi_j$. Then $e_{\mathbf{S}}(\psi_k) = e_{\mathbf{S}}(\psi_i) * e_{\mathbf{S}}(\psi_j) = e_{\mathbf{L}}(\psi_i) * e_{\mathbf{L}}(\psi_j) = a_i *_L a_j = a_k = e_{\mathbf{L}}(\psi_k)$ (similarly for $\psi_k = \psi_i \wedge \psi_j$). Finally, suppose that $\psi_k = (\psi_i \Rightarrow \psi_j)$. Then $e_{\mathbf{S}}(\psi_k) = a_i \to a_j = \max\{z \in S \mid a_i * z \leq a_j\}$. Let $a_k = a_i \to_L a_j$ ($a_k \in S$ because $\psi_i \Rightarrow \psi_j$ is a subformula of φ). Then $a_i * a_k \leq a_j$. Thus $a_k \leq a_i \to a_j$. Now suppose that there is an element $z' \in S$ such that $z' > a_k$ and $a_i * z' \leq a_j$. Since $z' \in L$, we get $z' \leq a_i \to_L a_j = a_k$, a contradiction. Hence $a_i \to a_j = a_k = e_{\mathbf{L}}(\psi_k)$. □ □

Note that **S** need not be a sub-ΠMTL-chain of **L** since **S** arises only from a submonoid of **L**. However, the existence of the evaluation $e_{\mathbf{S}}$ such that $e_{\mathbf{S}}(\varphi) < \mathbf{1}$ is sufficient for us.

The second step of the proof is a detailed investigation of the structure of **S**. We start with the properties of Archimedean classes of **S** and then we show their relation to filters.

LEMMA 3.3. *Let $b \in S$. Then the Archimedean classes of* **S**, *$[a]_\sim$, $a \in S$, have the following properties:*

(1) *$[a]_\sim$ is closed under $*$.*
(2) *$[a]_\sim$ is a left-open and right-closed interval for $a \neq \mathbf{0}, \mathbf{1}$.*
(3) $[a * b]_\sim = [\min\{a, b\}]_\sim$.
(4) *There are only finitely many Archimedean classes.*

Proof. (1) Suppose that $x, y \in [a]_\sim$ and $x \leq y$. Then $x^2 \leq x * y \leq x$, thus $x * y$ belongs to $[a]_\sim$.

(2) The right-closedness of $[a]_\sim$ is obvious because each subset of S has a maximum by Lemma 3.1. The left-openness follows from the fact that $x \in [a]_\sim$ implies $x^2 \in [a]_\sim$ and $x^2 < x$ from cancellativity. Finally, we have to show that there is no gap in $[a]_\sim$. Suppose that $x, y \in [a]_\sim$, $z \notin [a]_\sim$, and $x < z < y$. Then there is an n such that $y^n \leq x < z < y$. Thus $z \in [a]_\sim$, a contradiction.

(3) Without any loss of generality suppose that $a \leq b$. Then $a^2 \leq a * b \leq a$. Thus $a * b \in [a]_\sim = [\min\{a, b\}]_\sim$.

(4) Since **S** is finitely generated using only $*$ and because of (3), there must be only finitely many Archimedean classes in **S**. □

□

Note that there are always at least two Archimedean classes, $\{\mathbf{0}\}$ and $\{\mathbf{1}\}$, and $S/\sim$ is linearly ordered because of Lemma 3.3(2), i.e., $[a]_\sim < [b]_\sim$ iff $a \notin [b]_\sim$ and $a < b$. Let us denote Archimedean classes by C_i, $i = 0, \ldots, m+1$, in such a way that $C_0 = \{\mathbf{0}\}$, $C_{m+1} = \{\mathbf{1}\}$, and $C_j < C_k$ for $j < k$. Archimedean classes are important due to their relation to filters.

LEMMA 3.4. *Let $\{C_0 = \{\mathbf{0}\}, \ldots, C_i, \ldots, C_{m+1} = \{\mathbf{1}\}\}$ be the set of Archimedean classes of* **S**, *$0 \leq i \leq m+1$, and*

$$F_i = \bigcup_{i \leq j} C_j\,.$$

Then the set $\{F_0 = S, \ldots, F_i, \ldots, F_{m+1} = \{\mathbf{1}\}\}$ is the set of all filters of **S**.

Proof. Firstly, given an index i, we have to show that F_i is a filter of $\mathbf{S}$. The set F_i is closed under $*$ because of Lemma 3.3(1),(3). The fact that F_i is also a lattice-filter is obvious because $S/\sim$ is linearly ordered.

Secondly, let F be a filter of $\mathbf{S}$ and $a \in F$. Then $a \in C_i$ for some $0 \leq i \leq m+1$. We show that $C_i \subseteq F$. Let $b \in C_i$. Obviously, if $b \geq a$ then $b \in F$. Suppose that $b < a$. Then there is an $n \in \mathbb{N}$ such that $a^n \leq b$. Since $a^n \in F$, it follows that $b \in F$. Thus F must be a union of Archimedean classes. Now, let us take the minimal i such that $C_i \cap F \neq \emptyset$. Since $C_i < C_j$ for any $i < j$, it follows that $C_j \subseteq F$. Thus $F = \bigcup_{i \leq j} C_j$. □ □

From now on we will denote by F_i the filter corresponding to C_i. Observe that $F_0 = S$, $F_1 = S \setminus \{\mathbf{0}\}$, and $F_{m+1} = \{\mathbf{1}\}$. Further note that each $[x]_{F_i}$ has a maximum by Lemma 3.1. We denote this maximum by $m_x^{F_i}$.

LEMMA 3.5. *Let $x \in S$, $x > \mathbf{0}$, and F_i be a non-trivial filter, i.e., $0 < i < m+1$. Then the equivalence class $[x]_{F_i}$ is a left-open and right-closed interval.*

Proof. We show that $[x]_{F_i}$ is an interval. Suppose that $a, b \in [x]_{F_i}$ and $a \leq c \leq b$ for some $c \in S$. Then $b \to c \geq b \to a \in F_i$. Hence $c \in [x]_{F_i}$. Moreover, as the equivalence class $[x]_{F_i}$ has a maximum $m_x^{F_i}$, $[x]_{F_i}$ is right-closed.

Finally, we will show that there is no minimum. Suppose that $z \in [x]_{F_i}$ and $z \neq m_x^{F_i}$. There must be such z. Since $F_i \neq \{\mathbf{1}\}$, there is an $s \in F_i$ such that $s < \mathbf{1}$. Let $z = m_x^{F_i} * s \neq m_x^{F_i}$, then $m_x^{F_i} \to z = s$ and $z \in [x]_{F_i}$. Now take the element $m_x^{F_i} \to z \in F_i$. Since F_i is closed under $*$, $(m_x^{F_i} \to z)^2 \in F_i$ and $m_x^{F_i} * (m_x^{F_i} \to z)^2 \in [x]_{F_i}$ because $m_x^{F_i} \to [m_x^{F_i} * (m_x^{F_i} \to z)^2] = (m_x^{F_i} \to z)^2$. Now from the fact that $m_x^{F_i} * (m_x^{F_i} \to z) \leq z$, it follows that $m_x^{F_i} * (m_x^{F_i} \to z)^2 < z$. □ □

LEMMA 3.6. *Let F_i, F_j be filters in $\mathbf{S}$ and $i < j$. Then S/F_j is a refinement of S/F_i, i.e., $[y]_{F_j} \subseteq [x]_{F_i}$ for any $y \in [x]_{F_i}$.*

Proof. Since $F_j \subseteq F_i$, it follows that the congruences corresponding to F_j, F_i fulfill $\sim_{F_j} \subseteq \sim_{F_i}$. □ □

We are going to introduce the vector notation of the elements of $\mathbf{S}$. Let G be the set of generators of $\mathbf{S}$ defined in Equation (1), $G_i = C_i \cap G$, and $\mathbf{S}_i$ be the submonoid of $\mathbf{S}$ generated by G_i, i.e.,

$$S_i = \{g_1^{k_1} * \cdots * g_r^{k_r} \mid g_j \in G_i,\ k_j \in \mathbb{N},\ j = 1, \ldots, r\} \cup \{\mathbf{0}, \mathbf{1}\}.$$

Observe that $S_i \setminus \{\mathbf{0}, \mathbf{1}\} \subseteq C_i$ and $S_i \setminus \{\mathbf{0}\} \subseteq F_i$.

LEMMA 3.7. *Let $i \in \{1, \ldots, m\}$. Then $S_i \setminus \{\mathbf{0}\}$ is order-isomorphic to ω^*.*

Proof. Since $S_i \setminus \{\mathbf{0}, \mathbf{1}\} \subseteq C_i$, $\mathbf{S}_i \setminus \{\mathbf{0}\}$ is Archimedean, i.e.,

$$(\forall x \in S_i \setminus \{\mathbf{0}\})(\forall g \in G_i)(\exists k \in \mathbb{N}):\ g^k \leq x.$$

Thus there is a $k \in \mathbb{N}$ such that $g_1^k * \cdots * g_r^k \leq x$. Hence $M_x = \{z \in S_i \mid z \geq x\}$ is finite and the desired order-isomorphism assigns to x the cardinality of M_x. □ □

COROLLARY 3.8. *Let $H \subseteq S_i$ and H be infinite. Then for any $x \in S_i \setminus \{\mathbf{0}\}$ there is an element $w \in H$ such that $w \leq x$.*

Due to commutativity and associativity, each element $x \in S$ can be expressed in the form:

$$x = p_1 * p_2 * \cdots * p_m\,, \quad p_i \in S_i\,.$$

Thus we can assign to each element x a vector $\bar{x} = (p_1, \ldots, p_m)$ and find a function h such that $x = h(\bar{x}) = p_1 * \cdots * p_m$. Note that $\mathbf{1} = h(\mathbf{1}, \ldots, \mathbf{1})$ and $\mathbf{0} = h(p_1, \ldots, p_m)$, if $p_i = \mathbf{0}$ for at least one $i \in \{1, \ldots, m\}$. The projection to the i-th coordinate is denoted by π_i. Observe that if $p_i < 1$ and $p_j > 0$ for $j = i, \dots, m$, then the element represented by $(\mathbf{1}, \dots, \mathbf{1}, p_i, \dots, p_m)$ belongs to C_i and F_i. To each subset $M \subseteq S$ there exists a set of vectors $\bar{M}$ such that $h(\bar{M}) = M$ and $|M| \leq |\bar{M}|$.

Let us denote by $\langle G_1, \dots, G_k \rangle$ the universe of the submonoid of $\mathbf{S}$ generated by $\bigcup_{1 \leq j \leq k} G_j$.

LEMMA 3.9. *Let $x \in S$, $x > \mathbf{0}$, $i > 1$, and F_i be a filter. Then $m_x^{F_i} \in \langle G_1, \dots, G_{i-1} \rangle$, i.e., $m_x^{F_i} = p_1 * p_2 * \dots * p_{i-1} = h(p_1, \dots, p_{i-1}, \mathbf{1}, \dots, \mathbf{1})$ for some $p_j \in S_j$, $j = 1, \dots, i-1$.*

Proof. If $i = m + 1$, then $\langle G_1, \dots, G_m \rangle = S$ and obviously $m_x^{F_{m+1}} \in S$. Thus assume that $i < m + 1$. If $x \in F_i$ then $m_x^{F_i} = \mathbf{1}$ and $\mathbf{1} \in \langle G_1, \dots, G_{i-1} \rangle$. Finally, suppose that $x \notin F_i$ and $m_x^{F_i} = p_1 * p_2 * \dots * p_{i-1} * z$, $z < \mathbf{1}$, $z \in F_i$. Then $p_1 * p_2 * \dots * p_{i-1} \to m_x^{F_i} = z$. Thus $p_1 * p_2 * \dots * p_{i-1} \in [x]_{F_i}$, a contradiction with the condition that $m_x^{F_i}$ is maximal. □ □

LEMMA 3.10. *Let $x \in S$, $x > \mathbf{0}$, $i > 0$, and F_i be a filter. Then the set $M = [x]_{F_i} \cap \langle G_1, \dots, G_{i-1} \rangle$ is finite.*

Proof. Firstly, if $i = 1$ then $\langle G_1, \dots, G_{i-1} \rangle$ is a subuniverse generated by the empty set, i.e., $\langle G_1, \dots, G_{i-1} \rangle = \{\mathbf{0}, \mathbf{1}\}$. Since $[x]_{F_1} = [\mathbf{1}]_{F_1} = F_1$, the intersection $[x]_{F_i} \cap \langle G_1, \dots, G_{i-1} \rangle = \{\mathbf{1}\}$.

Secondly, let $i > 1$. Note that the elements from $\bar{M}$ are of the form $(p_1, \dots, p_{i-1}, \mathbf{1}, \dots, \mathbf{1})$ for some $p_j \in S_j$, $j = 1, \dots, i-1$. Suppose that M is infinite. Then $\bar{M}$ is also infinite and there exists a minimal k, $k \leq i - 1$, such that $\pi_k(\bar{M})$ is infinite and $\pi_j(\bar{M})$ is finite for all $j < k$. Thus there must be a subset $\bar{H} \subseteq \bar{M}$ such that $\pi_k(\bar{H})$ is infinite and $\pi_j(\bar{H}) = \{q_j\}$ for all $j < k$ and some $q_j \in S_j$.

Let us take an element $p_k \in \pi_k(\bar{H})$ and $p_k < \mathbf{1}$. Since $\pi_k(\bar{H}) \subseteq S_k$ is infinite, there must be an element $w \in \pi_k(\bar{H})$ such that $w \leq p_k^2$ by Corollary 3.8 and the fact that $p_k^2 \in S_k$.

Now let us take two vectors $\bar{a}, \bar{b} \in \bar{H}$ such that $\pi_k(\bar{a}) = p_k$ and $\pi_k(\bar{b}) = w$. Then $h(\bar{a}) \to h(\bar{b}) = a \to b \in F_i$ because $a, b \in [x]_{F_i}$. Let us denote $q = q_1 * \dots * q_{k-1}$, $z_a = \pi_{k+1}(\bar{a}) * \dots * \pi_{i-1}(\bar{a})$, and $z_b = \pi_{k+1}(\bar{b}) * \dots * \pi_{i-1}(\bar{b})$. Then $a \to b = q * p_k * z_a \to q * w * z_b = p_k * z_a \to w * z_b \leq p_k * z_a \to p_k^2 * z_b = z_a \to p_k * z_b$. Thus $z_a \to p_k * z_b \in F_i$. From $z_a \to p_k * z_b \in F_i$ and $z_a \in F_{k+1}$ it follows that $p_k * z_b \in F_i$, a contradiction with the fact that $p_k * z_b \in C_k$. □ □

LEMMA 3.11. *Let $x \in S$, $x > \mathbf{0}$, $i > 0$, F_i be a filter, $a, b \in [x]_{F_i}$, and $a \leq b$. Then there exists an element $w \in S_i$ such that $b * w \leq a$ and $b * w \in [x]_{F_i}$.*

Proof. Firstly, if $a = b$, then take $w = \mathbf{1}$. Secondly, if $a < b$, let $z = b \to a$. Then $z < \mathbf{1}$, $z \in F_i$, and $b * z \leq a$. We can write $z = h(\mathbf{1}, \dots, \mathbf{1}, p_i, \dots, p_m)$ for some $p_j \in S_j$, $j = i, \dots, m$. There are two cases. In the first case, let $p_i < \mathbf{1}$. Then $p_i < p_{i+1} * \dots * p_m$ because $p_i \in C_i$ and $p_{i+1} * \dots * p_m \in C_{i+1}$ by Lemma 3.3(3). Let us take $w = p_i^2 < z$. Then $b * w \leq b * z \leq a$ and $w \in S_i$. Moreover, as $b \to b * w = w \in S_i \subseteq F_i$, $b * w$ belongs to $[x]_{F_i}$. In the second case, let $p_i = \mathbf{1}$. Then we can take any element $w \in S_i$, $\mathbf{0} < w < \mathbf{1}$. Since $w < z$ and $b * w \in [x]_{F_i}$, the proof is done. □ □

Using Lemmas 3.9, 3.10, and 3.11, we are going to prove the crucial structural lemma. This lemma describes the behaviour of the equivalence classes w.r.t. F_{i+1} which are subsets of one equivalence class w.r.t. F_i. Since S/F_{i+1} is a refinement of S/F_i by Lemma 3.6, such subsets form the set $\{[y]_{F_{i+1}} \mid y \in [x]_{F_i}\}$.

LEMMA 3.12. *Let $x \in S$, $x > \mathbf{0}$, $0 < i < m + 1$, and F_i be a filter. Then the set $\{[y]_{F_{i+1}} \mid y \in [x]_{F_i}\}$ is order-isomorphic to ω^*.*

Proof. Let $M = [x]_{F_i} \cap \langle G_1, \dots, G_{i-1} \rangle$. We will show that each element $z \in [x]_{F_i}$ can be expressed in the form $z = b * s$ for some $b \in M$ and some $s \in F_i$. Firstly, if $i = 1$, then $M = \{\mathbf{1}\}$. Since $z > \mathbf{0}$, $z \in F_1$ and $z = \mathbf{1} * z$. Secondly, assume that $i > 1$. Then $z = p_1 * \cdots * p_{i-1} * p_i * \cdots * p_m$ and we can write $z = p_1 * \cdots * p_{i-1} * s$ for $s = p_i * \cdots * p_m \in F_i$. Further, $p_1 * \cdots * p_{i-1} \in [x]_{F_i}$ because $p_1 * \cdots * p_{i-1} \to z = s$. Since $p_1 * \cdots * p_{i-1} \in \langle G_1, \dots, G_{i-1} \rangle$, $p_1 * \cdots * p_{i-1} \in M$.

Thus for each maximum $m_y^{F_{i+1}} \in [x]_{F_i}$, we can write $m_y^{F_{i+1}} = b * s$ for some $b \in M$ and some $s \in F_i$. Since $m_y^{F_{i+1}} \in \langle G_1, \dots, G_i \rangle$ by Lemma 3.9, it follows that s must belong to S_i.

By Lemma 3.11 we can find for each $m_y^{F_{i+1}} \in [x]_{F_i}$ and for each $b \in M$, an element $w \in S_i$ such that $b * w \leq m_y^{F_{i+1}}$ and $b * w \in [x]_{F_i}$. Since $S_i \setminus \{\mathbf{0}\}$ is order-isomorphic to ω^* by Lemma 3.7 and M is finite by Lemma 3.10, we get that the set

$$H = \{b * s \mid s \in S_i,\ b \in M,\ b * s \geq m_y^{F_{i+1}}\}$$

is finite. Since $\{m_u^{F_{i+1}} \mid m_u^{F_{i+1}} \geq m_y^{F_{i+1}},\ u \in [x]_{F_i}\} \subseteq H$, the desired order-isomorphism $\#$ can be defined as follows:

$$\#[y]_{F_{i+1}} = |\{m_u^{F_{i+1}} \mid m_u^{F_{i+1}} \geq m_y^{F_{i+1}},\ u \in [x]_{F_i}\}| .$$

It is obvious that $\#[y]_{F_{i+1}} \leq |H|$. In other words, the natural number $\#[y]_{F_{i+1}}$ represents the position of $[y]_{F_{i+1}}$ within $[x]_{F_i}$. □ □

Now we define a mapping $t : S \setminus \{\mathbf{0}\} \to (\omega^*)^m$ as follows:

$$t(x) = (\#[x]_{F_2}, \#[x]_{F_3}, \dots, \#[x]_{F_{m+1}}) .$$

THEOREM 3.13. *The mapping $t : S \setminus \{\mathbf{0}\} \to (\omega^*)^m$ is an order-isomorphism, where $(\omega^*)^m$ denotes the lexicographic product of m copies of ω^*.*

Proof. Firstly, we have to show that t is one-to-one. Consider two different elements $x, y \in S \setminus \{\mathbf{0}\}$. Then there exists a minimal i such that $[x]_{F_i} \neq [y]_{F_i}$. Thus $\#[x]_{F_i} \neq \#[y]_{F_i}$ and $t(x) \neq t(y)$.

Secondly, we have to show that the function t is onto. Consider an m-tuple $(n_2, n_3, \dots, n_{m+1})$. By Lemma 3.12 we know that equivalence classes $[y]_{F_2}$ which are subsets of $[x]_{F_1} = [\mathbf{1}]_{F_1} = S \setminus \{\mathbf{0}\}$ are order-isomorphic to ω^*. Thus we can find an equivalence class $[x_2]_{F_2}$ such that $\#[x_2]_{F_2} = n_2$. Then again by Lemma 3.12 we can find an equivalence class $[x_3]_{F_3} \subseteq [x_2]_{F_2}$ such that $\#[x_3]_{F_3} = n_3$. Repeating this procedure we finally find $[x_{m+1}]_{F_{m+1}}$ such that $\#[x_{m+1}]_{F_{m+1}} = n_{m+1}$. Since $F_{m+1} = \{\mathbf{1}\}$ is the trivial filter, $[x_{m+1}]_{F_{m+1}} = \{x_{m+1}\}$ and $t(x_{m+1}) = (n_2, n_3, \dots, n_{m+1})$ because $[x_2]_{F_2} \supseteq [x_3]_{F_3} \supseteq \cdots \supseteq [x_{m+1}]_{F_{m+1}} = \{x_{m+1}\}$.

Finally, we prove that t is an order-isomorphism. Consider two elements such that $x < y$. Then there exists a minimal i such that $[x]_{F_i} < [y]_{F_i}$. Thus $\#[x]_{F_j} = \#[y]_{F_j}$ for all $j = 2, \dots, i-1$, and $\#[x]_{F_i} < \#[y]_{F_i}$. Thus $t(x) < t(y)$. □ □

Now we have the ΠMTL-chain $\mathbf{S}$ which is order-isomorphic to the lexicographic product $(\omega^*)^m$ and the evaluation $e_{\mathbf{S}}$ such that $e_{\mathbf{S}}(\varphi) < \mathbf{1}$. The next step is to build a new ΠMTL-chain $\mathbf{S}'$ order-isomorphic to $[0,1]$ in which $\mathbf{S}$ can be embedded. The new universe is defined as follows:

$$S' = \{(s, r) \mid s \in S \setminus \{\mathbf{0}\},\ r \in]0,1]\} \cup \{(\mathbf{0}, 1)\} .$$

This construction is the same as in [9], except for the fact that we use reals as second components in the definition of S' instead of rationals.

The order $\leq'$ on S' is lexicographic, i.e., $(s_1, r_1) \leq' (s_2, r_2)$ iff $s_1 \leq s_2$ or $s_1 = s_2$ and $r_1 \leq r_2$. The operations are defined by the following formulas:

$$\begin{aligned}(a,x) *' (b,y) &= (a * b, xy)\,,\\ (a,x) \to' (b,y) &= \begin{cases}(a \to b, 1) & \text{if } a * (a \to b) < b\,,\\ (a \to b, \min\{1, y/x\}) & \text{otherwise}\,.\end{cases}\end{aligned}$$

It is easy to check that $\mathbf{S}' = (S', *', \to', \leq', (\mathbf{0}, 1), (\mathbf{1}, 1))$ is a ΠMTL-chain.

Finally, the mapping $\Psi : S \to S'$ defined by $\Psi(x) = (x, 1)$ is a ΠMTL-homomorphism since it satisfies the following equalities:

$$\Psi(x * y) = (x * y, 1) = (x, 1) *' (y, 1) = \Psi(x) *' \Psi(y)\,,$$

and

$$\Psi(x \to y) = (x \to y, 1) = (x, 1) \to' (y, 1) = \Psi(x) \to' \Psi(y)\,.$$

Moreover, Ψ obviously preserves the order, i.e., $x \leq y$ implies $\Psi(x) \leq' \Psi(y)$.

The remaining step is to find an order-isomorphism $\Phi : S' \to [0,1]$. Let us define the mapping Φ as follows:

$$\begin{aligned}\Phi(\mathbf{0}, 1) &= 0\,,\\ \Phi(a, x) &= \frac{1}{2^{k_1}} + \frac{1}{2^{k_1+k_2}} + \cdots + \frac{1}{2^{k_1+\cdots+k_{m-1}}} + \frac{1+x}{2^{k_1+\cdots+k_m}}\,,\end{aligned}$$

where $(k_1, k_2, \ldots, k_m) = t(a)$.

THEOREM 3.14. *The mapping Φ is an order-isomorphism between S' and the real unit interval* $[0,1]$.

Proof. Since the elements of S' are lexicographically ordered and so are the elements of S, it is easy to see that Φ is an order-isomorphism. □ □

Finally, we define the operations in $[0,1]$ as usual:

$$a \odot b = \Phi(\Phi^{-1}(a) *' \Phi^{-1}(b))\,, \quad a \to_{\odot} b = \Phi(\Phi^{-1}(a) \to' \Phi^{-1}(b))\,.$$

Then $[\mathbf{0}, \mathbf{1}] = ([0,1], \odot, \to_{\odot}, \leq, 0, 1)$ is a ΠMTL-chain and $[\mathbf{0}, \mathbf{1}] \not\models \varphi$, i.e., $\Phi(\Psi(e_{\mathbf{S}}(\varphi))) < 1$. Thus the proof of the standard completeness theorem is done. □

THEOREM 3.15 (Standard Completeness Theorem). *A formula φ is provable in* ΠMTL *if and only if φ is a tautology in all* ΠMTL*-chains in* $[0,1]$ *with finitely many Archimedean classes.*

4. Finite Strong Standard Completeness

In the previous section we proved that provable formulas are valid in $[0,1]$ and vice versa. Nevertheless from the logical point of view, it is desirable to extend Theorem 3.15 also for theories. Before that we have to extend also Theorem 2.10 to its strong version.

We recall here several needful notions. Let T be a theory over ΠMTL. Then T is called complete if either $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$ for any pair of formulas φ, ψ. Further, let $\mathbf{L}$ be a ΠMTL-algebra. Then an $\mathbf{L}$-evaluation e such that for each formula $\varphi \in T$, $e(\varphi) = \mathbf{1}$, is called $\mathbf{L}$-model of T.

LEMMA 4.1. *For each theory T over* ΠMTL *such that $T \not\vdash \varphi$, there is a complete supertheory $T' \supseteq T$ such that $T' \not\vdash \varphi$.*

Proof. This can be done in the same way as the proof of [6, Lemma 2.4.2] because ΠMTL has the same deduction theorem as BL and all needed formulas are provable in ΠMTL (see [3, Section 2.1]). □ □

Although the authors of [3] did not state the following theorem, it follows easily from the facts in their paper.

THEOREM 4.2 (Strong Completeness Theorem). *Let T be a theory over* ΠMTL *and φ be a formula. Then $T \vdash \varphi$ if and only if for each* ΠMTL*-chain* $\mathbf{L}$ *and each* $\mathbf{L}$*-model e of T, $e(\varphi) = \mathbf{1}$.*

Proof. The first direction follows easily from soundness and the definition of $\mathbf{L}$-model. The second one is proved by the classical construction of Lindenbaum-Tarski algebra of a complete supertheory $T' \supseteq T$ such that $T' \not\vdash \varphi$. The existence of T' follows from Lemma 4.1 and [3, Lemma 1] shows that the construction of Lindenbaum-Tarski algebra is correct. □ □

THEOREM 4.3 (Finite Strong Standard Completeness). *Let T be a finite theory over* ΠMTL *and φ be a formula. Then $T \vdash \varphi$ if and only if for each* ΠMTL*-chain* $\mathbf{L}$ *in* $[0, 1]$ *with finitely many Archimedean classes and each* $\mathbf{L}$*-model e of T, $e(\varphi) = \mathbf{1}$.*

Proof. We will prove only the non-trivial direction. Suppose that $T \not\vdash \varphi$. Then Theorem 4.2 gives us a ΠMTL-chain $\mathbf{L}$ and an $\mathbf{L}$-model e of T such that $e(\varphi) < \mathbf{1}$. We will proceed similarly as in Section 3. Let us define the following set:

$$M = \{\psi \mid \psi \text{ is a subformula of } \tau,\ \tau \in T \cup \{\varphi\}\}\,.$$

Then we construct a submonoid $\mathbf{S}$ of $\mathbf{L}$ generated by the set:

$$G = \{a \in L \mid a_i = e(\psi),\ \psi \in M\}\,.$$

Since $\mathbf{S}$ is finitely generated as in Section 3, we can define a residuum and show that $\mathbf{S}$ is a ΠMTL-chain. Further, if we define $e_{\mathbf{S}}(v) = e(v)$ for each propositional variable v, we obtain an $\mathbf{S}$-evaluation such that $e_{\mathbf{S}}(\psi) = e(\psi)$ for all $\psi \in M$. This can be proved by a straightforward modification of the proof of Theorem 3.2. Moreover, since $e_{\mathbf{S}}(\tau) = e(\tau) = \mathbf{1}$ for all $\tau \in T$, $e_{\mathbf{S}}$ is an $\mathbf{S}$-model of T.

Finally, $\mathbf{S}$ can be embedded into a ΠMTL-chain in $[0, 1]$ with finitely many Archimedean classes in the same way as in the proof of Theorem 3.15. Thus there exists an embedding $\Phi : \mathbf{S} \to [\mathbf{0}, \mathbf{1}]$ such that $\Phi(e_{\mathbf{S}}(\varphi)) < 1$ and $\Phi(e_{\mathbf{S}}(\tau)) = 1$ for all $\tau \in T$. □ □

References

[1] G. Birkhoff: *Lattice Theory.* Amer. Math. Soc. Colloquium Publications (3rd edition), 1995.
[2] D. Cox, J. Little, D. O'Shea: *Ideals, Varieties, and Algorithms.* Springer (2nd edition), 1996.
[3] F. Esteva, L. Godo: *Monoidal t-norm Based Logic: Towards a logic for left-continuous t-norms.* Fuzzy Sets and Systems 124(3):271–288, 2001.
[4] F. Esteva, J. Gispert, L. Godo, F. Montagna: *On the Standard Completeness of some Axiomatic Extensions of the Monoidal T-norm Logic.* Studia Logica 71(2):199–226, 2002.
[5] L. Fuchs: *Partially Ordered Algebraic Systems.* Pergamon Press, Oxford, 1963.
[6] P. Hájek: *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.
[7] P. Hájek: *Observations on the Monoidal T-norm Logic.* Fuzzy Sets and Systems 132(1):107–112, 2002.
[8] U. Höhle: *Commutative, Residuated l-monoids.* In Non–Classical Logics and Their Applications to Fuzzy Subsets, U. Höhle, E.P. Klement (eds.), Kluwer Academic Publisher, Dordrecht, pp. 53–106, 1995.
[9] S. Jenei, F. Montagna: *A Proof of Standard Completeness for Esteva and Godo's Logic MTL.* Studia Logica 70(2):183–192, 2002.

Center for Machine Perception, Department of Cybernetics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague 6, Czech Republic.