DENSIFICATION VIA POLYNOMIALS, LANGUAGES, AND FRAMES

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ABSTRACT. It is known that every countable totally-ordered set can be embedded into a countable dense one. We extend this result to totally ordered commutative monoids and to totally-ordered commutative residuated lattices (the latter result fails in the absence of commutativity). The latter has applications to density elimination of semilinear substructural logics. In particular we obtain as a corollary a purely algebraic proof of the standard completeness of uninorm logic; the advantage over the known proof-theoretic proof and the semantical proof is that it is extremely short and transparent and all details can be verified easily using standard algebraic constructions. As another corollary we obtain a modified form of amalgamation.

1. Introduction

1.1. **Presentation style and modularity of the paper.** One of the benefits of working in areas that touch on different research subjects is that the main ideas of certain results can be adopted and presented so as to be of interest to a range of readers. We therefore opt for a modular presentation, as we do not want to assume or impose knowledge of other fields to the reader. At the same time, for the sake of interested readers, we include comments that illustrate the connections, in the context of our special setting, between different research areas such as polynomial rings in classical algebra, syntactic congruences in language theory and semigroup theory, Dedekind-MacNeille completions in order-theory, syntactic arguments for density eliminations in proof theory, as well as the theory of nuclei and of residuated frames in residuated lattices and substructural logics. In particular, we give a very accessible and economical presentation of each result using the tools from the corresponding area in order to avoid mentioning any general theory, and only in later sections we explain our design choices (for definitions that seem to be ad hoc) and the connections with related constructions using different tools, in order to make the paper useful for future research attempts in related topics. A side goal of the paper is to demonstrate that methods and ideas from these distinct areas are actually very much related.

1.2. **Results and layout.** A feature of all of our constructions is that they produce a dense structure in steps by inserting (at least) one new element in a given gap in the original totally-ordered structure. A gap is simply a pair of elements $g < h$, such that no other element is between them. Each of the next three sections has two parts. The first part presents a proof of the corresponding construction in a spartan fashion, with no discussion of the choices of the definitions or any motivation. The second part has a detailed discussion and justification of the definition by connecting to a more general theory. The final section aims at connecting these three sections and discusses the similarities and differences of the methods, essentially giving to our results case-study status in a more general project of linking different areas.

In detail, we prove that every totally-ordered commutative monoid can be embedded into a dense one, by considering the natural syntactic congruence given by languages that separate the points of the given monoid from each other and from the additional point used in the densification. Readers with background in language theory or semigroup theory and interested in densification of totally-ordered monoids should start by reading Section 2.

Then we show that also every totally-ordered commutative residuated lattice can be embedded into a dense one. The dense totally-ordered monoid given above may not include the residuals of the new elements, hence the need of a modification of the construction. For this we use linear polynomials, where the indeterminate X essentially plays the role of the element filling the given gap. Readers with background in classical algebra and ring theory should start their reading from

Section 3. As an aside we show that a modification of our construction leads to the standard approach of adding square roots to (totally ordered) abelian groups.

An alternative approach which actually produces an order-complete residuated lattice is by using the tool of residuated frames. By analyzing the proof using linear polynomials, we can extract a very simplified version of the construction given in [1]; see Section 4.

An application of densification to logic, and in particular to the standard completeness of certain semilinear logics, is discussed in Section 1.4. A second application of our results is discussed in Section 5 and provides a variant of the amalgamation property for totally-ordered structures.

En route to proving the above results, we introduce novel constructions. In particular, we present a construction of a residuated lattice based on the polynomial semiring over a residuated lattice. Also, we elaborate on the connections between modules over idempotent semirings and residuated lattices. In particular, we explain how we can obtain residuated lattices as images of module morphisms. Both of these constructions work in a more algebraic setting and have more algebraic flavor than those employing residuated frames, as they do not force order-completeness of the resulting algebra.

1.3. **Connections to existing results.** Part of our motivation for considering densification of totally-ordered algebras comes from substructural logic. More specifically, uninorm logic UL is the semilinear extension of full Lambek calculus with exchange FL_e [6], i.e., it is a logic complete with respect to the class of all FL_e-chains (namely, totally-ordered bounded commutative residuated lattices) ; see [9]. In [9, 3] it is shown that the logic UL is standard complete, i.e., it is complete with respect to the class of all bounded FL_e-chains whose universe is the real unit interval $[0, 1]$. The proof is based on a proof-theoretical elimination of the density rule and the inductive arguments involve many cases, thus making it hard to verify them all.

A semantical proof inspired by this proof-theoretic argument is given in [1]. The semantical proof simplifies the verification, but still involves a lengthy case analysis for checking the communication rule in order to establish the linearity of the resulting structure; the communication rule directly corresponds to the proof-theoretic rendering of (pre)linearity. We provide a purely algebraic and also very short proof of densification (and thus also standard completeness). At the same time, and more importantly, we provide an explanation of why the construction really works, using basic tools from classical algebra. In particular, we believe that our explanations justify some of the ad hoc design choices in [3] and [1]. Furthermore, our approach provides a simpler methodology which can be used for checking densifiability of other logics while avoiding a convoluted proof-theoretic case analysis or verification of the communication rule. Actually, the simplified tools provided in this paper have already allowed us to prove densifiability for a large class of subvarieties axiomatized by equations in the signature of multiplication and join; as the inclusion of these results would make the current paper too lengthy and also distract from its explanatory nature, we present them in a separate forthcoming paper.

1.4. **The one-step densification approach.** In order to densify a countable totally ordered algebra $\mathbf{A}$, we first present a construction that given a *gap* $g < h$ in $\mathbf{A}$, namely a pair $g < h$ of elements of A for which there is no element of A between them, produces a new totally ordered algebra $\bar{\mathbf{A}}$ into which $\mathbf{A}$ embeds and which contains an element strictly between the images in $\bar{A}$ of g and h, thus *filling*, or resolving, the gap; we call $\bar{\mathbf{A}}$ a *one-step densification* or *one-step linear extension* of $\mathbf{A}$. Of course this process could result in $\bar{\mathbf{A}}$ having new gaps, even if $\mathbf{A}$ had a single gap.

However, having established the above, given a countable totally-ordered algebra $\mathbf{A}$, we consider all countable totally-ordered algebras that have $\mathbf{A}$ as a subalgebra and we order them by the subalgebra relation. By an easy application of Zorn's Lemma to this poset we obtain the existence of a maximal superalgebra $\mathbf{B}$ of $\mathbf{A}$, which has to be dense by maximality (otherwise, if it has a gap, an isomorphic copy of $\bar{\mathbf{B}}$ has $\mathbf{B}$ as a proper subalgebra).

Unfortunately, $\mathbf{B}$ need not be countable. Nevertheless, using the downward Löwenheim-Skolem theorem (see [8, Cor. 3.1.4]), there is a countable elementary substructure $\mathbf{C}$ of $\mathbf{B}$ containing A. Since the property of being dense is a first-order property, $\mathbf{C}$ has to be dense as well. As $\mathbf{C}$ is countable and dense its order reduct $(C, \leq)$ is isomorphic to the rationals. It is well known that the Dedekind-MacNeille completion of $(C, \leq)$ is isomorphic to the interval $[0,1]$. Moreover, in case $\mathbf{A}$, and thus $\mathbf{C}$, is a totally-ordered commutative monoid or residuated lattice, we can define on the Dedekind-MacNeille completion of $(C, \leq)$ the same algebraic structure (see e.g. [6]), so it becomes a totally-ordered commutative monoid or residuated lattice based on $[0,1]$. In the case of totally ordered commutative residuated lattices (or *commutative residuated chains*), these algebras are known as *standard* and our result implies that they generate (even via quasivariety generation) the variety of all *semilinear commutative residuated lattices*, namely the variety generated by all commutative residuated chains.

2. Totally ordered monoids and syntactic congruences

Before presenting the construction of one-step densification we recall the definition of a commutative totally ordered monoid. A *partially ordered monoid* (or *pomonoid*) is a structure $(M, \cdot, 1, \leq)$ where $(M, \cdot, 1)$ is a monoid, $\leq$ is partial order on M and $a \leq b$ implies $cad \leq cbd$ for all $a, b, c, d \in M$; it is called commutative if the monoid reduct is commutative. If the order is total/linear, then $\mathbf{M}$ is said to be a *totally ordered monoid* (or *tomonoid*). Also, we define ${\downarrow} a = \{m \in M : m \leq a\}$.

2.1. **The construction.**

Now given a commutative tomonoid $\mathbf{A}$ with a gap $g < h$ we will construct a one-step linear extension of it, namely a commutative tomonoid $\bar{\mathbf{A}}$ such that $\mathbf{A}$ is embeddable into $\bar{\mathbf{A}}$ via an embedding $\phi \colon A \to \bar{A}$. Moreover, there is an element $p \in \bar{A}$ such that $\phi(g) < p < \phi(h)$.

The first and naive way to create a one-step densification is of course to add a single new element p to the algebra $\mathbf{A}$. The operations of the new algebra based on $A \cup \{p\}$ should extend those of $\mathbf{A}$, as we require an embedding. In the new algebra, by order preservation and $g \leq p \leq h$, we have $ag \leq ap \leq ah$, for all $a \in A$. Therefore, a natural way to define multiplication by p is to set $a \cdot p = ah$ (or similarly $a \cdot g = ag$), for $a \in A$; of course we would have to exclude $a = 1$, as we surely want $1 \cdot p = p$. However, this causes complications with associativity. In particular, if $ab = 1$ for some $a, b \in A$, then $p = 1 \cdot p = (ab) \cdot p = a(b \cdot p) = a(bh) = (ab)h = h$, a contradiction. So we have to proceed differently.

First we define a monoid extension of $\mathbf{A}$ by the new element p and then look for a congruence which does not collapse the elements of $A \cup \{p\}$ and is large enough to produce a totally-ordered quotient. We explain in Section 2.2.2 why it suffices to consider the following small extension. Let $\mathbf{A} \cup \mathbf{A}p$ be the commutative monoid, where $Ap = \{ap \mid a \in A\}$ (formally speaking we write ap for (a, p) and Ap for $A \times \{p\}$) and multiplication is given by stipulating that for all $a, b \in A$:

$$a \cdot b = ab \quad a \cdot (bp) = (ab)p, \quad (ap) \cdot (bp) = (abh)p.$$

Note that the order on $\mathbf{A}$ can be naturally extended to Ap by setting $ap \leq bp$ iff $a \leq b$. Then $A \cup Ap$ can be ordered as a disjoint union of posets (see Figure 1). Further, we can define a total order $\sqsubseteq$ on the subset $A \cup \{p\}$ of $A \cup Ap$, by simply stipulating that $\sqsubseteq$ extends the order of $\mathbf{A}$ by the conditions $g \sqsubseteq p \sqsubseteq h$. In that sense $\mathbf{A} \cup \mathbf{A}p$ contains an ordered part, namely the chain $(A \cup \{p\}, \sqsubseteq)$; see Figure 1.

For every $a, b \in A$ we define the set $a \Rightarrow b = \{c \in A \mid ac \leq b\}$. The following lemma states some simple properties of such sets.

Lemma 2.1. *Let $a, b, c \in A$. Then the following hold:*

(1) *$a \Rightarrow b$ is downward closed.*
(2) *$1 \Rightarrow b = \{c \in A \mid c \leq b\} = {\downarrow} b$.*
(3) *$b \leq c$ implies $a \Rightarrow b \subseteq a \Rightarrow c$.*
(4) *$b \leq c$ implies $c \Rightarrow a \subseteq b \Rightarrow a$.*
(5) *The collection $\{a \Rightarrow b \mid a, b \in A\}$ of subsets of A forms a chain under inclusion, as downsets of a chain.*

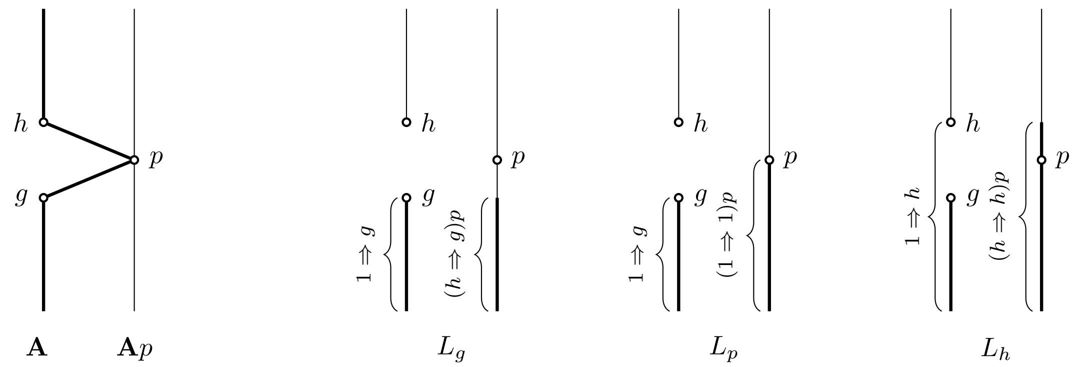

FIGURE 1. (Left) The monoid extension $\mathbf{A} \cup \mathbf{A}p$ together with the chain $(A \cup \{p\}, \sqsubseteq)$ depicted by the thick line; (Right) The subsets L_g, L_p and L_h depicted by the thick lines.

We define the following subsets of $A \cup Ap$ in an ad hoc way and in Section 2.2.1 we justify the definition:

$$L_a = (1 \Rightarrow a) \cup (h \Rightarrow a)p \text{ for } a \in A,$$
$$L_p = (1 \Rightarrow g) \cup (1 \Rightarrow 1)p,$$

where $(b \Rightarrow c)p = \{dp \in Ap \mid d \in b \Rightarrow c\}$ for $b, c \in A$. The subsets L_g, L_p and L_h are depicted in Figure 1 (note that $h \Rightarrow g \subsetneq 1 \Rightarrow 1 \subseteq h \Rightarrow h$).

Given $L \subseteq \mathbf{A} \cup \mathbf{A}p$, which we will call a *language* of $\mathbf{A} \cup \mathbf{A}p$, and $u \in \mathbf{A} \cup \mathbf{A}p$, we define the *quotient* of L by u as follows:

$$u^{-1}L = \{x \in A \cup Ap \mid ux \in L\}.$$

We first show that there are only two types of sets of the form $u^{-1}L_z$, for $u \in A \cup Ap$.

Lemma 2.2. *Let $u \in A \cup Ap$ and $z \in A \cup \{p\}$. Then $u^{-1}L_z$ is of the form*

$$b^{-1}L_a = (b \Rightarrow a) \cup (bh \Rightarrow a)p \quad \text{or} \quad b^{-1}L_p = (b \Rightarrow g) \cup (b \Rightarrow 1)p$$

for some $a, b \in A$.

Proof. Let $a, b \in A$. Then we have

$$\begin{aligned}
b^{-1}L_a &= \{x \in A \cup Ap \mid bx \in L_a\} = \{c \in A \mid bc \in 1 \Rightarrow a\} \cup \{cp \in Ap \mid bcp \in (h \Rightarrow a)p\} \\
&= \{c \in A \mid bc \leq a\} \cup \{cp \in Ap \mid bch \leq a\} = (b \Rightarrow a) \cup (bh \Rightarrow a)p, \\
(bp)^{-1}L_a &= \{c \in A \mid bcp \in (h \Rightarrow a)p\} \cup \{cp \in Ap \mid bchp \in (h \Rightarrow a)p\} = (bh \Rightarrow a) \cup (bh^2 \Rightarrow a)p \\
&= (bh)^{-1}L_a, \\
b^{-1}L_p &= \{c \in A \mid bc \in 1 \Rightarrow g\} \cup \{cp \in Ap \mid bcp \in (1 \Rightarrow 1)p\} = (b \Rightarrow g) \cup (b \Rightarrow 1)p, \\
(bp)^{-1}L_p &= \{c \in A \mid bcp \in (1 \Rightarrow 1)p\} \cup \{cp \in Ap \mid bchp \in (1 \Rightarrow 1)p\} = (b \Rightarrow 1) \cup (bh \Rightarrow 1)p \\
&= b^{-1}L_1.
\end{aligned}$$

□

Lemma 2.3. *For all $z \in A \cup \{p\}$ and $u \in A \cup Ap$, we have that $u^{-1}L_z$ is a downset for $\sqsubseteq$.*

Proof. By Lemma 2.2, it suffices to prove that for all $a, b \in A$ the sets $b^{-1}L_a$ and $b^{-1}L_p$ are downsets for $\sqsubseteq$. To that goal let $x, y \in A \cup \{p\}$ with $x \sqsubseteq y$. Downset closure is trivially true for $x = y$, so we assume that $x \sqsubset y$ for the rest of the proof. Note that the following implications hold and also their hypotheses exhaust all cases:

- if $x, y \in A$ then $x \leq y$,
- if $x = p$ then $h \leq y$,
- if $y = p$ then $x \leq g$.

From $y \in b^{-1}L_a$ we will derive $x \in b^{-1}L_a$. If $x, y \in A$ then $y \in b \Rightarrow a$. Since $b \Rightarrow a$ is downward closed, we have $x \in (b \Rightarrow a) \subseteq b^{-1}L_a$. If $x = p$ then $h \leq y$ and $y \in b \Rightarrow a$, so $bh \leq by \leq a$. Thus $1 \in bh \Rightarrow a$, i.e., $p \in (bh \Rightarrow a)p \subseteq b^{-1}L_a$. Finally, if $y = p$ then $1 \in bh \Rightarrow a$. Thus $bx \leq bg \leq bh \leq a$ and $x \in (b \Rightarrow a) \subseteq b^{-1}L_a$.

Similarly from $y \in b^{-1}L_p$ we will obtain $x \in b^{-1}L_p$. If $x, y \in A$ we can use the same argument as above in order to show that $x \in b^{-1}L_p$. If $x = p$ then $h \leq y$ and $y \in b \Rightarrow g$, so $bh \leq by \leq g$. Thus $b \leq 1$ (otherwise we would have $h \leq bh \leq g < h$). Consequently, $1 \in b \Rightarrow 1$, i.e., $p \in (b \Rightarrow 1)p \subseteq b^{-1}L_p$. Finally, if $y = p$ then $1 \in b \Rightarrow 1$. Thus $b \leq 1$ which implies $bx \leq g$. Consequently, $x \in b \Rightarrow g \subseteq b^{-1}L_p$. □

For any pomonoid $\mathbf{M}$, the *syntactic preorder* $\preceq_L$ (introduced in [13], see also [11]) given by a language $L \subseteq M$ is defined as follows: for $x, y \in M$,

$$x \preceq_L y \quad \text{iff} \quad \forall u \in M(y \in u^{-1}L \implies x \in u^{-1}L).$$

The symmetrization $\sim_L \,=\, \preceq_L \cap (\preceq_L)^{-1}$ is the *syntactic congruence* corresponding to the set L (also known as *Leibniz congruence* in Abstract Algebraic Logic; see e.g. [4]). It is easy to see that the syntactic congruence $\sim_L$ is the largest monoid congruence which *saturates* the set L (i.e., $a \in L$ and $a \sim_L b$ implies $b \in L$). Consequently, $\sim_L$ *separates* the elements of L from those in the complement L^c of L, that is if $a \in L$ and $b \in L^c$ then $a \not\sim_L b$; here complements are computed relative to M. Then the quotient monoid $\mathbf{M}/{\sim_L}$, endowed with the ordering $\leq$ given by

$$[x]_L \leq [y]_L \quad \text{iff} \quad x \preceq_L y,$$

becomes a pomonoid; here $[x]_L$ denotes the congruence class of x with respect to $\sim_L$. Given an indexed system of languages $\{L_i \mid i \in I\}$, the intersection $\theta = \bigcap_{i \in I} \sim_{L_i}$ is again a monoid congruence and the quotient $\mathbf{M}/\theta$ forms a pomonoid where the order is given by

$$[x]_\theta \leq [y]_\theta \quad \text{iff} \quad x \preceq_{L_i} y \text{ for all } i \in I.$$

Here we consider the collection $\{L_z \mid z \in A \cup \{p\}\}$ of languages and the congruence $\theta = \bigcap_{z \in A \cup \{p\}} \sim_{L_z}$.

Lemma 2.4. *$(\mathbf{A} \cup \mathbf{A}p)/\theta$ is a pomonoid in which $\mathbf{A}$ is embeddable via $a \mapsto [a]_\theta$. In addition, $[g]_\theta < [p]_\theta < [h]_\theta$.*

Proof. Clearly, $(\mathbf{A} \cup \mathbf{A}p)/\theta$ is a pomonoid. Thanks to Lemma 2.3, for all $x, y \subseteq A \cup \{p\}$ we have $[x]_\theta \leq [y]_\theta$ whenever $x \sqsubseteq y$. Moreover, if $x \sqsubset y$ then $x \in L_x$ but $y \notin L_x$. Indeed, if $x, y \in A$ then $x < y$. Thus $y \notin (1 \Rightarrow x) \cup (h \Rightarrow x)p = L_x$. If $x = p$ then $h \leq y$. Thus $y \notin (1 \Rightarrow g) \cup (1 \Rightarrow 1)p = L_x$. Finally, if $y = p$ then $x \leq g$. Thus $h \nleq x$, i.e., $p \notin (1 \Rightarrow x) \cup (h \Rightarrow x)p = L_x$. Consequently, $x \not\sim_{L_x} y$. Thus $[x]_\theta < [y]_\theta$. In particular, the map $\phi\colon A \to (A \cup Ap)/\theta$ defined by $a \mapsto [a]_\theta$ is a pomonoid embedding. □

It remains to prove that $(\mathbf{A} \cup \mathbf{A}p)/\theta$ is totally ordered. In order to prove it, we will use the following lemma.

Lemma 2.5. *The subsets $\{b^{-1}L_a \mid a, b \in A\} \cup \{b^{-1}L_p \mid b \in A\}$ of $A \cup Ap$ form a chain under inclusion.*

Proof. We first prove that $b^{-1}L_p$ and $c^{-1}L_p$ are always comparable by showing that $b \leq c$ implies $c^{-1}L_p \subseteq b^{-1}L_p$ for all $b, c \in A$. If $b \leq c$ then $c \Rightarrow g \subseteq b \Rightarrow g$ and also $c \Rightarrow 1 \subseteq b \Rightarrow 1$ by Lemma 2.1. Thus $c^{-1}L_p \subseteq b^{-1}L_p$.

Second we check that for all $a, b, c, d \in A$ the sets $b^{-1}L_a$ and $d^{-1}L_c$ can be compared. Since the sets $b \Rightarrow a$ and $d \Rightarrow c$ are comparable by Lemma 2.1, it suffices to prove that $b \Rightarrow a \subseteq d \Rightarrow c$ implies $bh \Rightarrow a \subseteq dh \Rightarrow c$ for all $a, b, c, d \in A$. Let $x \in bh \Rightarrow a$, i.e., $bhx \leq a$. Then $hx \in b \Rightarrow a \subseteq d \Rightarrow c$. Consequently, $dhx \leq c$, i.e., $x \in dh \Rightarrow c$.

Finally, we check that $b^{-1}L_a$ and $c^{-1}L_p$ are comparable for all $a, b, c \in A$. Suppose that $b \Rightarrow a \subseteq c \Rightarrow g$. Then we have to show that $bh \Rightarrow a \subseteq c \Rightarrow 1$. Let $x \in bh \Rightarrow a$, i.e., $bhx \leq a$. Thus $hx \in b \Rightarrow a \subseteq c \Rightarrow g$. Consequently, $chx \leq g$. It follows that $cx \leq 1$ (otherwise $h \leq chx \leq g < h$). Hence $x \in c \Rightarrow 1$. Conversely, assume that $b \Rightarrow a \supsetneq c \Rightarrow g$. Then we have to show that

$c \Rightarrow 1 \subseteq bh \Rightarrow a$. By the assumption there is $y \in b \Rightarrow a$ such that $y \notin c \Rightarrow g$ (i.e., $by \leq a$ and $g < h \leq cy$). Let $x \in c \Rightarrow 1$, i.e., $cx \leq 1$. Then $bhx \leq bcyx \leq by \leq a$, i.e., $x \in bh \Rightarrow a$. □

Theorem 2.6. *$(\mathbf{A} \cup \mathbf{A}p)/\theta$ is a tomonoid in which $\mathbf{A}$ is embeddable via $a \mapsto [a]_\theta$. In addition, $[g]_\theta < [p]_\theta < [h]_\theta$.*

Proof. In view of Lemma 2.4, it remains to show that $(\mathbf{A} \cup \mathbf{A}p)/\theta$ is totally ordered. Let $x, y \in A \cup Ap$. Assume that $[x]_\theta \not\leq [y]_\theta$. By definition of θ there is $w \in A \cup \{p\}$ such that $x \not\preceq_{L_w} y$. This means that there exists $u \in A \cup Ap$ such that $y \in u^{-1}L_w$ and $x \notin u^{-1}L_w$. We have to show that $[y]_\theta \leq [x]_\theta$. Again by definition of θ we have to prove that for all $z \in A \cup \{p\}$ we have $[y]_{L_z} \leq [x]_{L_z}$ (i.e., $y \preceq_{L_z} x$). Let $x \in v^{-1}L_z$. Since $x \notin u^{-1}L_w$, we must have $u^{-1}L_w \subseteq v^{-1}L_z$ by Lemma 2.5. Thus $y \in v^{-1}L_z$. Consequently, $y \preceq_{L_z} x$. □

2.2. **Explanation of design choices.** Having given the full proof, here we explain the ad hoc choice of languages that we considered, as well as the choice of the monoid extension of $\mathbf{A}$.

2.2.1. *The definition of the sets.* To solve the one-step densification problem, we wish to find a congruence θ on $\mathbf{A} \cup \mathbf{A}p$ satisfying the conditions:

- $\mathbf{A}/\theta$ is totally ordered and
- $[x]_\theta < [y]_\theta$ for all $x, y \in A \cup \{p\}$ such that $x \sqsubset y$.

The congruence θ will be the intersection of syntactic congruences of languages in the collection $\{L_z \mid z \in A \cup \{p\}\}$. In order to satisfy at least the version of the second condition above that has to do with weak inclusions ($\leq$ and $\sqsubseteq$), we want the syntactic congruences $\sim_{L_z}$ to preserve the order $\sqsubseteq$ on $A \cup \{p\}$, i.e., the following condition must be satisfied for all $x, y \in A \cup \{p\}$:

$$x \sqsubseteq y \implies [x]_{L_z} \leq [y]_{L_z}. \tag{1}$$

It is further desirable for $\sim_{L_z}$ to separate z from as many elements in $A \cup \{p\}$ as possible, so as to obtain the strict version of the second condition in a smooth way. Thus we will define L_z to be the smallest set containing z and satisfying (1). At this stage, for the first, linearity, property we can only hope that Lemma 2.1(5) will be of some use.

Condition (1), which simply states that each $u^{-1}L_z$ is downward closed with respect to $\sqsubseteq$, can be expressed as a system of closure rules using the definition of the syntactic preorder. For every $x, y \in A \cup \{p\}$ such that $x \sqsubseteq y$ we have the following rule:

$$\forall u \in A \cup Ap \; (y \in u^{-1}L_z \implies x \in u^{-1}L_z). \tag{2}$$

In particular, for $u = 1$ we get that L_z has to be downward closed with respect to $\sqsubseteq$. Since $z \in L_z$, it follows that $1 \Rightarrow a \subseteq L_a$ for all $a \in A$ and $1 \Rightarrow g \subseteq L_p$. Furthermore, for $b \in A$ with $b \leq 1$ we have $b \sqsubseteq 1$. By (2) for $u = p$ we obtain $1p \in L_p \implies bp \in L_p$. Since $p \in L_p$, we infer $bp \in L_p$. Consequently, $(1 \Rightarrow 1)p \subseteq L_p$.

Let $a \in A$. Since $p \sqsubseteq h$, we must have $uh \in L_a$ implies $up \in L_a$ for all $u \in A \cup Ap$. In particular, for $u = b \in A$ we have $bh \in L_a$ implies $bp \in L_a$. Hence, if $bh \leq a$ (i.e., $b \in h \Rightarrow a$) then $bh \in L_a$ and therefore $bp \in L_a$. Consequently, $(h \Rightarrow a)p \subseteq L_a$.

As mentioned above we want L_z's to be smallest possible. Thus we define

$$L_a = (1 \Rightarrow a) \cup (h \Rightarrow a)p \text{ for } a \in A,$$
$$L_p = (1 \Rightarrow g) \cup (1 \Rightarrow 1)p.$$

The above languages define the congruence θ which gives us the final linear extension $(\mathbf{A} \cup \mathbf{A}p)/\theta$. There is also another illuminating way of viewing the congruence θ. Namely, $\theta = \Theta(p, h) \cap \sim_{L_p}$ where $\Theta(p, h)$ is the principal congruence on $\mathbf{A} \cup \mathbf{A}p$ generated by the pair (p, h). To see this, it suffices to show that $\Theta(p, h) = \bigcap_{a \in A} \sim_{L_a}$. First we prove that $\Theta(p, h) \subseteq \sim_{L_a}$ for all $a \in A$. Since $p \sqsubseteq h$, we have $p \preceq_{L_a} h$. Conversely, we need to show that $h \preceq_{L_a} p$. This by definition boils down to the validity of the implication $p \in b^{-1}L_a \implies h \in b^{-1}L_a$ for all $b \in A$. If $p \in b^{-1}L_a$ then $1 \in bh \Rightarrow a$ (i.e., $bh \leq a$). Consequently, $h \in b \Rightarrow a \subseteq b^{-1}L_a$. Thus every $\sim_{L_a}$ contains the principal congruence $\Theta(p, h)$, i.e., $\Theta(p, h) \subseteq \bigcap_{a \in A} \sim_{L_a}$. Assume that this inclusion would be strict. Then at least two congruence classes of $\Theta(p, h)$ have to be collapsed in the quotient by $\bigcap_{a \in A} \sim_{L_a}$. Note that every congruence class of $\Theta(p, h)$ has a representative $c \in A$. Consequently,

$\bigcap_{a\in A} \sim_{L_a}$ would collapse congruence classes $[c]_{\Theta(p,h)}$ and $[d]_{\Theta(p,h)}$ for some $c, d \in A$ such that $c < d$. However, this is impossible since $c \not\sim_{L_c} d$. Hence $\Theta(p,h) = \bigcap_{a\in A} \sim_{L_a}$.

This way of viewing θ as $\Theta(p,h) \cap \sim_{L_p}$ offers another explanation of how the linear extension $(\mathbf{A} \cup \mathbf{A}p)/\theta$ was constructed. Namely, the principal congruence $\Theta(p,h)$ has almost all desired properties. It preserves the order $\sqsubseteq$ on $A \cup \{p\}$ and it separates all elements in A and also g from p. Nevertheless, it collapses p and h. That is why we consider a finer congruence, namely the intersection $\Theta(p,h) \cap \sim_{L_p}$. This step is reasonable because the syntactic congruence $\sim_{L_p}$ separates p from h and preserves the order $\sqsubseteq$ on $A \cup \{p\}$.

2.2.2. *Small monoid extension.* The choice of the monoid extension $\mathbf{A} \cup \mathbf{A}p$ seems ad hoc. A more reasonable candidate for the monoid extension is the free extension $\mathbf{A}p^*$, where $Ap^* = \{ap^n : a \in A, n \in \mathbb{N}\}$ under the multiplication given by $ap^n \cdot bp^m = (ab)p^{n+m}$. It turns out that such a large extension is not necessary because if we would choose the free monoid extension $\mathbf{A}p^*$ and follow the same strategy as above, the resulting congruence θ would contain the congruence η generated by $p^2 = ph$. Consequently, we can start immediately with $\mathbf{A}p^*/\eta \cong \mathbf{A} \cup \mathbf{A}p$ without changing the final result.

To see that η is contained in θ, consider the same strategy as before looking for the smallest sets $L_z \subseteq Ap^*$ such that they separate all element from $A \cup \{p\}$ and satisfy (1). Using the analogous arguments replacing $\mathbf{A} \cup \mathbf{A}p$ with $\mathbf{A}p^*$, we would end up with the following sets:

$$L_a = \bigcup_{n\in\mathbb{N}} (h^n \Rightarrow a)p^n \text{ for } a \in A,$$

$$L_p = (1 \Rightarrow g) \cup \bigcup_{n\in\mathbb{N}} (h^n \Rightarrow 1)p^{n+1}.$$

By the definition of θ, it suffices to prove that $\eta \subseteq \sim_{L_z}$ for every $z \in A \cup \{p\}$. We clearly have $[p^2]_{L_z} \leq [ph]_{L_z}$ (i.e., $p^2 \preceq_{L_z} ph$) since $p \sqsubseteq h$. Conversely, we have to show $ph \preceq_{L_z} p^2$, i.e., $p^2 \in u^{-1}L_z$ implies $ph \in u^{-1}L_z$ for all $u \in Ap^*$. Suppose that $p^2 \in u^{-1}L_z$. Then $up^2 \in L_z$ for some $u = ap^k$, $a \in A$ and $k \in \mathbb{N}$. Consequently, a is either in $h^{k+2} \Rightarrow b$, if $z = b \in A$, or in $h^{k+1} \Rightarrow 1$, if $z = p$. In the first case $ah \in h^{k+1} \Rightarrow b$. Thus $uph = ahp^{k+1} \in (h^{k+1} \Rightarrow b)p^{k+1} \subseteq L_b$. In the second case $ah \in h^k \Rightarrow 1$. Thus $uph = ahp^{k+1} \in (h^k \Rightarrow 1)p^{k+1} \subseteq L_p$.

3. Densification of commutative residuated chains, via linear polynomials

We first describe the construction in an economical and relatively ad hoc way so as to give a very short proof. In the next section we explain why these design choices are natural.

3.1. **The construction.** A *commutative residuated lattice-ordered semigroup* is a structure $\mathbf{A} = (A, \wedge, \vee, \cdot, \to)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot)$ is a commutative semigroup and
- the residuation condition holds: $x \cdot y \leq z \Leftrightarrow y \leq x \to z$, for all $x, y, z \in A$.

If there is an element $1 \in A$ such that $1x = x$ for all $x \in A$ then we call $\mathbf{A}$ a *commutative residuated lattice*. If, in addition, the lattice order is total, $\mathbf{A}$ is said to be a *commutative residuated chain*. It follows easily from the residuation condition that the multiplication $\cdot$ distributes over finite joins (i.e., $x(y \vee z) = xy \vee xz$). In particular, it is monotone in both arguments. Similarly, we have $(x \vee y) \to z = (x \to z) \wedge (y \to z)$ and $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$. Consequently the residuum $\to$ is antitone in the first argument and monotone in second one. For details on residuated structures see [6].

Let $\mathbf{A}$ be a commutative residuated lattice and let $A(X)$ be the set $\{a \vee bX : a, b \in A\}$ of linear polynomials over the $\{\vee, \cdot\}$ reduct of $\mathbf{A}$ in one indeterminate X. This can be ordered by the cartesian product ordering (as $A(X)$ is isomorphic to $A \times A$) and it becomes a lattice. Clearly, $A \vee AX$ is not closed under the usual product of polynomials. However, given an element $h \in A$, we define in an ad hoc way the product of two linear polynomials $p = p_0 \vee p_1X$ and $q = q_0 \vee q_1X$ by:

$$(p_0 \vee p_1X) \cdot_h (q_0 \vee q_1X) = p_0q_0 \vee (p_0q_1 \vee p_1q_0 \vee p_1q_1h)X.$$

We further define implication $\to_h$ on $A(X)$ by

$$(p_0 \vee p_1X) \to_h (q_0 \vee q_1X) = ((p_0 \to q_0) \wedge (p_1 \to q_1)) \vee ((p_0 \to q_1) \wedge (p_1h \to q_1))X.$$

Lemma 3.1. *The set $A(X)$ forms a commutative residuated lattice-ordered semigroup $\mathbf{A}(X)$ under the above operations.*

Proof. The operation $\cdot_h$ is clearly commutative and it is straightforward to check that it is associative. To show that $\mathbf{A}(X)$ is residuated we have

$$\begin{array}{l}
(p_0 \vee p_1X) \cdot_h (q_0 \vee q_1X) \leq r_0 \vee r_1X \Leftrightarrow \\
p_0q_0 \vee (p_0q_1 \vee p_1q_0 \vee p_1q_1h)X \leq r_0 \vee r_1X \Leftrightarrow \\
p_0q_0 \leq r_0, \ \ p_0q_1 \leq r_1, \ \ p_1q_0 \leq r_1, \ \ p_1q_1h \leq r_1 \Leftrightarrow \\
q_0 \vee q_1X \leq ((p_0 \to r_0) \wedge (p_1 \to r_1)) \vee ((p_0 \to r_1) \wedge (p_1h \to r_1))X \Leftrightarrow \\
q_0 \vee q_1X \leq (p_0 \vee p_1X) \to_h (r_0 \vee r_1X).
\end{array}$$

As it is a lattice, it follows that it is a commutative residuated lattice-ordered semigroup. □

For all $a \in A$, we consider in an ad hoc fashion the linear polynomials

$$\hat{a} = a \vee (h \to a)X \text{ and } \tilde{a} = (a \to g) \vee (a \to 1)X.$$

We denote by $\bar{A}$ the set of these two types of polynomials, for all $a \in A$, and we write $a^\wedge$ for $\hat{a}$ and $a^\sim$ for $\tilde{a}$ if a long expression takes the place of a.

Lemma 3.2. *$\bar{A}$ is a totally-ordered subset of $A(X)$.*

Proof. Indeed, if $a \leq b$, then $\hat{a} \leq \hat{b}$ and $\tilde{b} \leq \tilde{a}$. We now compare $\hat{a}$ and $\tilde{b}$.

If $ab \leq g$, then $a \leq b \to g$. Also, $b \leq a \to g$, so $b(h \to a) \leq (h \to a)(a \to g) \leq h \to g < 1$, hence $h \to a \leq b \to 1$.

If $g < ab$ (i.e., $h \leq ab$ as $g < h$ is a gap), then $a \not\leq b \to g$, so $b \to g < a$. Also, $h(b \to 1) \leq ab(b \to 1) \leq a$, so $b \to 1 \leq h \to a$. □

For any linear polynomial $P = p_0 \vee p_1X$ we define $\gamma(P) = \widehat{P(h)} \wedge ((p_0 \to g) \wedge (p_1 \to 1))^\sim$, where $P(h) = p_0 \vee p_1h$.

Lemma 3.3. *Given a linear polynomial P, $\gamma(P)$ is the smallest element of $\bar{A}$ that is above P. Thus, γ is a closure operator on $\mathbf{A}(X)$ with image $\bar{A}$.*

Proof. We first find the smallest $\hat{a}$ above P by considering $p_0 \vee p_1X \leq a \vee (h \to a)X$. Then $p_0 \leq a$ and $p_1h \leq a$, namely, $p_0 \vee p_1h \leq a$. Clearly, the minimal a is then $p_0 \vee p_1h = P(h)$.

To find the smallest $\tilde{a}$ above P, we consider $p_0 \vee p_1X \leq (a \to g) \vee (a \to 1)X$. Then $p_0 \leq a \to g$ and $p_1 \leq a \to 1$, namely $a \leq (p_0 \to g) \wedge (p_1 \to 1)$. The largest a (yielding the smallest $\tilde{a}$) is then $(p_0 \to g) \wedge (p_1 \to 1)$. □

We define the algebra $\bar{\mathbf{A}} = (\bar{A}, \wedge, \vee_\gamma, \cdot_\gamma, \to_h, \hat{1})$, where $x \cdot_\gamma y = \gamma(x \cdot_h y)$, $x \vee_\gamma y = \gamma(x \vee y)$.

Lemma 3.4. *The algebra $\bar{\mathbf{A}}$ is a commutative residuated chain and $\hat{g} < \tilde{1} < \hat{h}$.*

Proof. We first check that for all $P \in A(X)$ and $Q \in \bar{A}$, we have $P \to_h Q \in \bar{A}$.

$$\begin{aligned}
P \to_h \hat{a} &= (p_0 \vee p_1X) \to_h (a \vee (h \to a)X) \\
&= ((p_0 \to a) \wedge (p_1 \to (h \to a))) \vee ((p_0 \to (h \to a)) \wedge (p_1h \to (h \to a)))X \\
&= ((p_0 \to a) \wedge (p_1h \to a)) \vee ((h \to (p_0 \to a)) \wedge (h \to (p_1h \to a)))X \\
&= ((p_0 \vee p_1h) \to a)^\wedge \\
&= (P(h) \to a)^\wedge
\end{aligned}$$

$$\begin{aligned}
P \to_h \tilde{a} &= (p_0 \vee p_1 X) \to_h ((a \to g) \vee (a \to 1)X) \\
&= ((p_0 \to (a \to g)) \wedge (p_1 \to (a \to 1))) \vee ((p_0 \to (a \to 1)) \wedge (p_1 h \to (a \to 1)))X \\
&= ((p_0 a \to g) \wedge (p_1 a \to 1)) \vee ((p_0 a \to 1) \wedge (p_1 ha \to 1))X \\
&= ((p_0 a \to g) \vee (p_0 a \to 1)X) \wedge ((p_1 a \to 1) \vee (p_1 ha \to 1)X) \\
&= \widetilde{p_0 a} \wedge \widehat{p_1 a \to 1}
\end{aligned}$$

Since γ is a closure operator by Lemma 3.3 whose image $\bar{A}$ satisfies $P \to_h Q \in \bar{A}$ for all $P \in A(X)$ and $Q \in \bar{A}$, it follows that γ is a *nucleus* on $\mathbf{A}(X)$, namely a closure operator satisfying $\gamma(x)\gamma(y) \leq \gamma(xy)$, so $\bar{\mathbf{A}}$ is a commutative residuated lattice-ordered semigroup; see e.g. [6, Section 3.4.11] for details.

Next we have to show that $\hat{1}$ is a multiplicative unit. We have

$$\hat{1} \cdot_\gamma \hat{a} = \gamma(\hat{1} \cdot_h \hat{a}) = \gamma(a \vee ((h \to a) \vee a(h \to 1) \vee h(h \to 1)(h \to a))X) = \gamma(\hat{a}) = \hat{a},$$

because $a(h \to 1), h(h \to 1)(h \to a) \leq h \to a$. Similarly,

$$\hat{1} \cdot_\gamma \tilde{a} = \gamma((a \to g) \vee ((a \to 1) \vee (a \to g)(h \to 1) \vee h(h \to 1)(a \to 1))X) = \gamma(\tilde{a}) = \tilde{a}.$$

By Lemma 3.2 $\bar{\mathbf{A}}$ is a chain, so it is a commutative residuated chain.

We also have $\hat{g} < \tilde{1} < \hat{h}$, namely $g \vee (h \to g)X < g \vee X < h \vee (h \to h)X$, because $g \leq g < h$ and $h \to g < 1 \leq h \to h$ (if we had $1 \leq h \to g$, then $h \leq g$, a contradiction). □

Theorem 3.5. *The map $\phi : \mathbf{A} \to \bar{\mathbf{A}}$, given by $\phi(a) = \hat{a}$ is a residuated lattice embedding. Thus, $\bar{\mathbf{A}}$ serves as a one-step densification of $\mathbf{A}$.*

Proof. It is clear that $a \leq b$ iff $\hat{a} \leq \hat{b}$, so ϕ is an order embedding. For implication we have

$$\begin{aligned}
\hat{a} \to_h \hat{b} &= (a \vee (h \to a)X) \to_h (b \vee (h \to b)X) \\
&= ((a \to b) \wedge ((h \to a) \to (h \to b))) \vee ((a \to (h \to b)) \wedge ((h \to a)h \to (h \to b)))X \\
&= ((a \to b) \wedge ((h \to a)h \to b)) \vee ((ah \to b) \wedge ((h \to a)h^2 \to b))X \\
&= (a \to b) \vee (ah \to b)X \\
&= \widehat{a \to b}.
\end{aligned}$$

Next we show that ϕ preserves multiplication. We have $\hat{a} \cdot_h \hat{b} = ab \vee (a(h \to b) \vee b(h \to a))X$. Thus we have $\hat{a} \cdot_h \hat{b} \leq \widehat{ab}$. Consequently, $\hat{a} \cdot_\gamma \hat{b} = \gamma(\hat{a} \cdot_h \hat{b}) \leq \widehat{ab}$ as $\widehat{ab}$ is γ-closed. For the converse inequality it suffices to show that $\hat{a} \cdot_h \hat{b} \leq Q$ implies $\widehat{ab} \leq Q$ for all $Q \in \bar{A}$ showing that the least γ-closed element above $\hat{a} \cdot_h \hat{b}$ is $\widehat{ab}$. Assume that $\hat{a} \cdot_h \hat{b} \leq \hat{c}$. Then $ab \leq c$ and so $h \to ab \leq h \to c$. Thus $\widehat{ab} \leq \hat{c}$. Analogously, if $\hat{a} \cdot_h \hat{b} \leq \tilde{c}$ then $ab \leq c \to g$. Hence $h \to ab \leq h \to (c \to g) = c \to (h \to g) \leq c \to 1$. Thus $\widehat{ab} \leq \tilde{c}$.

Finally, by Lemma 3.4 the gap $g < h$ is filled in $\bar{\mathbf{A}}$ by $\tilde{1}$, so $\bar{\mathbf{A}}$ serves as a one-step densification of $\mathbf{A}$. □

3.2. **Intuition and explanation of ad hoc constructions.** We would like to point out that readers who are only interested in the results may simply skip this section and read only the previous section with the construction (or even just the statement of the theorem for that matter). On the other hand, readers who want to use the real ideas behind the proof and the constructions in order to modify them and apply them to related questions may benefit from the explanations and constructions in this section.

Throughout the rest of Section 3 we will assume that the commutative residuated chain $\mathbf{A}$ which we want to extend by a new element has a bottom element $\bot$. It follows that $\mathbf{A}$ has also a top element $\top$ because $\bot \to \bot = \top$. This assumption is not necessary (we have not assumed it in the previous section) but it simplifies our notation so that we can introduce a semiring of (formal) polynomials over $\mathbf{A}$ in the usual way and $\mathbf{A}(X)$ will become a residuated lattice (not only a residuated lattice-ordered semigroup). Moreover, this assumption does not harm much the generality of our results because one can easily extend $\mathbf{A}$ by an additional bottom and a top element if $\mathbf{A}$ does not have them. For elements $\bot, \top \notin A$, it is well known [6, Section 3.6.7] that there is a unique commutative residuated lattice $\mathbf{A}_\perp$ on the set $A_\perp = A \cup \{\perp, \top\}$ specified by the following conditions

- $\mathbf{A}$ is a subalgebra of $\mathbf{A}_\perp$,
- $\perp < a < \top$ for all $a \in A$,
- $a \cdot \top = \top = \top \cdot a$, for all $a \in A \cup \{\top\}$,
- $b \cdot \perp = \perp = \perp \cdot b$, for all $b \in A \cup \{\perp, \top\}$,
- $c \to \top = \top$, for all $c \in A \cup \{\perp, \top\}$,
- $\perp \to \perp = \top$ and $b \to \perp = \perp$, for $b \in A \cup \{\top\}$.

As we wish to extent $\mathbf{A}$ by a new element that fills the gap $g < h$, it is natural to consider an extension, as in classical algebra, similar to the construction of field extensions through the ring of polynomials. It turns out that our approach above works partly because $\mathbf{A}$, although not a field or ring, can be viewed as an (idempotent) semiring with respect to the operations $\vee$ and $\cdot$; see Section 3.2.1. This is what allows us to construct the semiring $\mathbf{A}[X]$ of polynomials; although $\mathbf{A}[X]$ does not form in general a residuated lattice as the multiplication of polynomials is not residuated, the restriction of this multiplication on the set of at most linear polynomials is residuated as we will see in Section 3.2.2. This allows us to introduce in Section 3.2.3 a residuated lattice $\mathbf{A}(X)$ on the set of at most linear polynomials with the operations $\cdot_h$ and $\to_h$ on $\mathbf{A}(X)$.

It is also interesting that, unlike in field extensions, the final extension $\bar{\mathbf{A}}$ is not a quotient of $\mathbf{A}[X]$ (or even $\mathbf{A}(X)$ in our case) by some principal congruence, since $\mathbf{A}[X]$ is nothing as simple as a principal ideal domain. Rather, the correct way to think about $\bar{\mathbf{A}}$ is actually as an $\mathbf{A}(X)$-module which then happens to be a residuated lattice. The actual reason that it happens to be linear is a consequence of the fact that it is the module image of another natural module $\mathbf{A}^X$ which is totally ordered. The module $\mathbf{A}^X$ is obtained by minimalistic considerations of asking that it contains A and an additional element X, and it is a module. We explain the full details of the module approach to residuated lattices in Sections 3.2.4 and 3.2.5, and this takes away a lot of the guesswork that goes behind the above slick, but unmotivated construction.

Finally, it is not even clear that the semiring $\mathbf{A}[X]$ of polynomials will be a natural enough free extension, as apart from semiring operations we also have the implications. The more natural and general extension is that of powerseries $\mathbf{A}[[X]]$ and this is what we used in our original proof of the result, before distilling it down to the (even linear) polynomials.

Also, in Section 3.3 we provide a slight modification that yields the construction of adding square roots to totally ordered abelian groups.

3.2.1. *Residuated lattices as semirings and modules.* For every commutative residuated lattice $\mathbf{L}$ we have that $a(b \vee c) = ab \vee ac$, for all $a, b, c \in L$, which together with the fact that $(L, \cdot, 1)$ is a commutative monoid and $(L, \vee)$ is a commutative idempotent semigroup shows that the $\{\cdot, \vee, 1\}$ reduct of $\mathbf{L}$ is an idempotent semiring. We denote this semiring by $\mathbf{L}^\vee$. (If $\mathbf{L}$ has a bottom element $\perp$, then we also have that $(L, \vee, \perp)$ is a monoid and $a \cdot \perp = \perp$, for all $a \in L$.)

Moreover, we have the conditions

- $1 \to x = x$,
- $(ab) \to x = a \to (b \to x)$,
- $a \to (x \wedge y) = (a \to x) \wedge (a \to y)$ and
- $(a \vee b) \to x = (a \to x) \wedge (b \to x)$,

which together with the fact that $(L, \wedge)$ is a commutative semigroup shows that $(L, \wedge)$ is a module (whose elements we think of as *vectors*) over the semiring $(L, \vee, 1)$ (whose elements we think of as *scalars*) where the action (or *scalar multiplication*) is $\to$, the residuum of $\mathbf{L}$. We denote this $\mathbf{L}^\vee$-module by $\mathbf{L}^\wedge$. (If $\mathbf{L}$ has a top element $\top$, then we also have that $(L, \wedge, \top)$ is a monoid and $a \to \top = \top$ for all $a \in L$.)

We recall the notion of a nucleus and its relation with semiring congruences and submodules. A nucleus on a commutative residuated lattice $\mathbf{L}$ is a closure operator $\gamma\colon L \to L$ (i.e., $a \leq \gamma(a)$, $\gamma(\gamma(a)) = \gamma(a)$ and $a \leq b$ implies $\gamma(a) \leq \gamma(b)$) such that $\gamma(a)\gamma(b) \leq \gamma(ab)$ for all $a, b \in L$. Given a nuclues on $\mathbf{L}$, one can introduce a residuated lattice $\mathbf{L}_\gamma = (L_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \to, \gamma(1))$ called a *nuclear retraction* of $\mathbf{L}$, where $L_\gamma = \gamma[L]$ is the set of γ-closed elements and $a \vee_\gamma b = \gamma(a \vee b)$, $a \cdot_\gamma b = \gamma(ab)$;

see e.g. [6, Section 3.4.11]. The following are equivalent characterizations of nuclei (see [6] and [12, Section 8.3]):

(1) γ is a nucleus on a commutative residuated lattice $\mathbf{L}$,
(2) the kernel $\sim_\gamma$ of γ is a semiring congruence on $\mathbf{L}^\vee$ such that L_γ are precisely maxima of congruence classes; in particular we have $\gamma(a) = \max\,[a]_{\sim_\gamma}$,
(3) L_γ forms a closure system (i.e., an image of a closure operator) and an $\mathbf{L}^\vee$-submodule of $\mathbf{L}^\wedge$.

Thus one can view $\mathbf{L}_\gamma$ as a semiring quotient of $\mathbf{L}^\vee$ being at the same time an $\mathbf{L}^\vee$-submodule of $\mathbf{L}^\wedge$. The fact that L_γ forms a submodule follows because the condition $\gamma(a)\gamma(b) \leq \gamma(ab)$ is equivalent to $\gamma(a \to \gamma(b)) = a \to \gamma(b)$. The elements of a congruence class $[a]_{\sim_\gamma}$ act in the same way on L_γ. To see that note that $a \to \gamma(b) = \gamma(a) \to \gamma(b)$. Indeed, we have $\gamma(a) \to \gamma(b) \leq a \to \gamma(b)$ since $a \leq \gamma(a)$. Conversely, we have

$$\gamma(a)(a \to \gamma(b)) \leq \gamma(a)\gamma(a \to \gamma(b)) \leq \gamma(a(a \to \gamma(b))) \leq \gamma(\gamma(b)) = \gamma(b).$$

Thus every a acts as the maximum $\gamma(a)$ of its equivalence class $[a]_{\sim_\gamma}$.

The semiring congruence $\sim_\gamma$ can be equivalently described as follows:

$$a \sim_\gamma b \quad \text{iff} \quad a \to m = b \to m \text{ for all } m \in L_\gamma\,.$$

Indeed, $a \sim_\gamma b$ (i.e., $\gamma(a) = \gamma(b)$) implies $a \to m = \gamma(a) \to m = \gamma(b) \to m = b \to m$. Conversely, assume that $a \to m = b \to m$ for all $m \in L_\gamma$. Then we have $\gamma(b) \leq \gamma(a)$ since $1 \leq \gamma(a) \to \gamma(a) = \gamma(b) \to \gamma(a)$. Further we have $\gamma(a) \leq \gamma(b)$ by the symmetric argument. Thus $a \sim_\gamma b$.

3.2.2. *Semiring of polynomials.* Thinking of $\mathbf{A}$ as a semiring, we denote by $\mathbf{A}[X]$ the set of all polynomials with coefficients from $\mathbf{A}$ over the indeterminate X. We use the notation $P = p_0 \vee p_1X \vee p_2X^2 \vee \ldots \vee p_nX^n = \bigvee_{i=0}^n p_iX^i$ for polynomials of degree at most n. As usual, every polynomial of degree at most n can be considered as a polynomial of degree at most m, for $m \geq n$, by appending terms of the form $\bot X^i$.

Clearly $\mathbf{A}[X]$ is a semiring (see e.g. [7]) with operations given by

$$\left(\bigvee_{i=0}^n p_iX^i\right) \vee \left(\bigvee_{i=0}^n q_iX^i\right) = \bigvee_{i=0}^n (p_i \vee q_i)X^i,$$

$$\left(\bigvee_{i=0}^n p_iX^i\right) \cdot \left(\bigvee_{j=0}^m q_jX^j\right) = \bigvee_{k=0}^{n+m} \left(\bigvee_{i+j=k} p_iq_j\right) X^k.$$

However, since $\mathbf{A}$ is a lattice, we can further define

$$\left(\bigvee_{i=0}^n p_iX^i\right) \wedge \left(\bigvee_{i=0}^n q_iX^i\right) = \bigvee_{i=0}^n (p_i \wedge q_i)X^i.$$

Thus $\mathbf{A}[X]$ forms also a lattice (namely, the direct sum of ω copies of $\mathbf{A}$). Nevertheless, $\mathbf{A}[X]$ does not form a residuated lattice (for instance because it does not possess a top element). Still multiplication of polynomials is residuated in a certain restricted sense. We recall the notion of a residuated binary commutative map (see e.g. [6]). Let U, V be posets and $\circ\colon U \times U \to V$ a commutative map (i.e., $u \circ v = v \circ u$ for all $u, v \in U$). The map $\circ$ is said to be residuated if there is a map $\to\colon U \times V \to U$ such that

$$u \circ v \leq w \quad \text{iff} \quad v \leq u \to w.$$

Denote the set of polynomials of degree at most n by $A_n[X]$. If we restrict the multiplication in $\mathbf{A}[X]$ to $A_n[X] \times A_n[X]$ we obtain a map from $A_n[X] \times A_n[X]$ to $A_{2n}[X]$. This restriction is residuated in the above sense. Indeed, it suffices to find a map $\to\colon A_n[X] \times A_{2n}[X] \to A_n[X]$ such that for all $P, Q \in A_n[X]$ and $R \in A_{2n}[X]$ we have $P \cdot Q \leq R$ iff $Q \leq P \to R$. Let $P = \bigvee_{i=0}^n p_iX^i$,

$Q = \bigvee_{i=0}^{n} q_i X^i$ and $R = \bigvee_{i=0}^{2n} r_i X^i$. Then we have

$$
\begin{aligned}
P \cdot Q = \left(\bigvee_{i=0}^{n} p_i X^i\right) \cdot Q = \bigvee_{i=0}^{n} (p_i X^i Q) \leq R \quad & \text{iff} \quad p_i X^i Q \leq R \text{ for all } i = 0, \ldots, n, \\
& \text{iff} \quad p_i q_j \leq r_{i+j} \text{ for all } i, j = 0, \ldots, n, \\
& \text{iff} \quad q_j \leq p_i \to r_{i+j} \text{ for all } i, j = 0, \ldots, n, \\
& \text{iff} \quad q_j \leq \bigwedge_{i=0}^{n} (p_i \to r_{i+j}) \text{ for all } j = 0, \ldots, n, \\
& \text{iff} \quad Q \leq \bigvee_{i=0}^{n} \left(\bigwedge_{i=0}^{n} (p_i \to r_{i+j})\right) X^j.
\end{aligned}
$$

Thus the desired operation $\to$ is the following one:

$$
P \to R = \bigvee_{j=0}^{n} \left(\bigwedge_{i=0}^{n} (p_i \to r_{i+j})\right) X^j.
$$

We want to point out that the semiring of formal powerseries over $\mathbf{A}$ can be defined in an analogous way. If, in addition, $\mathbf{A}$ is complete as a lattice, we obtain a residuated lattice $\mathbf{A}[[X]]$ with almost identical operations (actually the operations for polynomials are the restrictions of the ones for powerseries).

We note that for $X \to P = \bigvee_{i=0}^{n-1} p_{i+1} X^i$, which we call the *derivative* or *shift* of P and we denote it by P'.

3.2.3. *From polynomials to linear polynomials.* The next step is to introduce a residuated lattice $\mathbf{A}(X)$ on the set $A_1[X]$ of all polynomials of degree at most 1. We define the semiring structure of $\mathbf{A}(X)$ as the semiring quotient of $\mathbf{A}[X]$ given by the congruence generated by $X^2 = hX$; this congruence is the kernel of the semiring homomorphism $\phi_h : \mathbf{A}[X] \to \mathbf{A}(X)$ given by $\phi_h(P) = p_0 \vee P'(h)X$. This map induces the operations on $\mathbf{A}(X)$: $P \vee_h Q = \phi_h(P \vee Q) = P \vee Q$, $P \cdot_h Q = \phi_h(PQ)$, $\phi_h(1) = 1$, for polynomials P and Q of degree at most 1. The semiring $\mathbf{A}(X)$ forms clearly a lattice but even a residuated lattice due to the following general fact.

Let U, V be partially ordered sets and $\circ \colon U \times U \to V$ a residuated commutative map. Thus there is a map $\to \colon U \times V \to U$ such that $x \circ y \leq z \Leftrightarrow y \leq x \to z$. Also, assume that we have maps $q : U \to V$, $\ell : V \to U$ such that $\ell(x) \leq y \Leftrightarrow x \leq q(y)$, i.e., they form an adjunction. In particular, $q(y)$ is the largest element of U whose image by ℓ is less than or equal to y. Then one can define a residuated operation $\star \colon L \times L \to L$ by setting $x \star y = \ell(x \circ y)$ and the corresponding residual by $x \to q(y)$. Indeed, we have

$$
x \star y = \ell(x \circ y) \leq z \Leftrightarrow x \circ y \leq q(z) \Leftrightarrow y \leq x \to q(z).
$$

Now we can use the above general fact in order to show that $\mathbf{A}(X)$ forms a residuated lattice. First, we know that the restriction of multiplication in $\mathbf{A}[X]$ to $A_1[X] \times A_1[X]$ is residuated, i.e., there is an operation $\to \colon A_1[X] \times A_2[X] \to A_1[X]$ such that $PQ \leq R \Leftrightarrow Q \leq P \to R$. Further there is a map $q \colon A_2[X] \to A_1[X]$ such that for all $P \in A_2[X]$ and $R \in A_1[X]$ we have $\phi_h(P) \leq R \Leftrightarrow P \leq q(R)$. Indeed, the largest quadratic polynomial $q(r_0 \vee r_1 X)$ whose image by ϕ_h is below $r_0 \vee r_1 X$ is $r_0 \vee r_1 X \vee (h \to r_1)X^2$ as is shown below:

$$
\begin{aligned}
\phi_h(p_0 \vee p_1 X \vee p_2 X^2) \leq r_0 \vee r_1 X \quad & \Leftrightarrow \quad p_0 \vee (p_1 \vee p_2 h) X \leq r_0 \vee r_1 X \\
& \Leftrightarrow \quad (p_0 \leq r_0 \;\&\; p_1 \leq r_1 \;\&\; p_2 h \leq r_1) \\
& \Leftrightarrow \quad p_0 \vee p_1 X \vee p_2 X^2 \leq r_0 \vee r_1 X \vee (h \to r_1) X^2 \\
& \Leftrightarrow \quad p_0 \vee p_1 X \vee p_2 X^2 \leq q(r_0 \vee r_1 X).
\end{aligned}
$$

Consequently, for $P, Q, R \in A_1[X]$ we have

$$
P \cdot_h Q = \phi_h(PQ) \leq R \Leftrightarrow PQ \leq q(R) \Leftrightarrow Q \leq P \to q(R).
$$

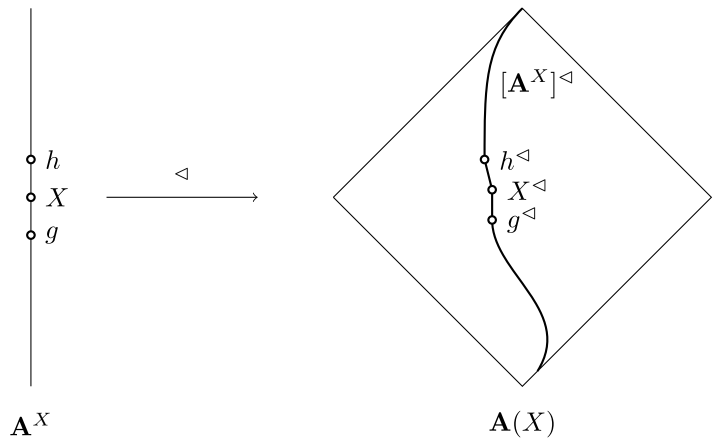

FIGURE 2. Module morphism ${}^\lhd\colon \mathbf{A}^X \to \mathbf{A}(X)$.

3.2.4. *The plan.* Recall that we assume that $\mathbf{A}$ is a commutative residuated chain, namely a totally ordered commutative residuated lattice. We also assume that there are elements $g, h \in A$ such that $g < h$ and there is no element of A between g and h, namely $g < h$ forms a *gap*. Our goal is to construct a new commutative residuated chain $\bar{\mathbf{A}}$ into which $\mathbf{A}$ will embed, say under an embedding that we call ${}^\lhd : \mathbf{A} \to \bar{\mathbf{A}}$, such that $\bar{\mathbf{A}}$ will contain at least one more element between $g^\lhd$ and $h^\lhd$.

This can be alternatively rephrased as saying that we actually want to embed into $\bar{\mathbf{A}}$ the partial algebra $\mathbf{A} \cup \{X\}$, namely ${}^\lhd : \mathbf{A} \cup \{X\} \to \bar{\mathbf{A}}$, where X is a new element such that $g < X < h$, in order to have $g^\lhd < X^\lhd < h^\lhd$ in $\bar{\mathbf{A}}$. (In viewing $\mathbf{A} \cup \{X\}$ as a partial algebra we consider the set $A \cup \{X\}$ under the partially defined operations that extend the operations on $\mathbf{A}$ with $a \wedge X = a$ and $a \vee X = X$, for $a \le g$, and with $a \wedge X = X$ and $a \vee X = a$ for $h \le a$.)

It will turn out that $\bar{\mathbf{A}}$ will be a nuclear retraction of $\mathbf{A}(X)$, i.e., $\bar{\mathbf{A}} \cong \mathbf{A}(X)_\gamma$ for some nucleus γ. Thus $\bar{\mathbf{A}}$ will be a semiring quotient of $\mathbf{A}(X)$ and at the same time an $\mathbf{A}(X)^\vee$-submodule of $\mathbf{A}(X)^\wedge$. In order to find this submodule, we will look for a totally ordered $\mathbf{A}(X)^\vee$-module $\mathbf{A}^X$ containing the set $A \cup \{X\}$ and a module morphism ${}^\lhd\colon \mathbf{A}^X \to \mathbf{A}(X)$ which will be injective on $A \cup \{X\}$ and whose image $[\mathbf{A}^X]^\lhd$ will form the desired nuclear retraction $\bar{\mathbf{A}}$ (see Figure 2). In addition, we will require that $z^\lhd = \gamma(z)$ for any $z \in A \cup \{X\}$ so that the element z in our partial algebra $\mathbf{A} \cup \{X\}$ is mapped to the maximum of congruence class $[z]_{\sim_\gamma}$ of the corresponding element in $\mathbf{A}(X)$.

3.2.5. *The module.* Based on our requirements, we look for a totally ordered $\mathbf{A}(X)^\vee$-module $\mathbf{A}^X$ and a module morphism ${}^\lhd\colon \mathbf{A}^X \to \mathbf{A}(X)^\wedge$ whose image $[\mathbf{A}^X]^\lhd$ forms a nuclear retraction induced by a nucleus γ, i.e., it is an $\mathbf{A}(X)^\vee$-submodule of $\mathbf{A}(X)^\wedge$ and at the same time a closure system. In order to define the $\mathbf{A}(X)^\vee$-module $\mathbf{A}^X$, we will look for a totally ordered set containing $A \cup \{X\}$ which at the same time admits a module action[1] $\mathbin{\backslash\!\!\backslash}$ from $\mathbf{A}(X)^\vee$. Summing up our requirements, we look for

(1) a totally ordered $\mathbf{A}(X)^\vee$-module $\mathbf{A}^X$ containing the ordered set $A \cup \{X\}$,
(2) a module morphism ${}^\lhd\colon \mathbf{A}^X \to \mathbf{A}(X)^\wedge$, i.e., it preserves $\wedge$ and $(P \mathbin{\backslash\!\!\backslash} z)^\lhd = P \to_h z^\lhd$,
(3) for all $z \in A \cup \{X\}$ we require $z^\lhd = \gamma(z)$ (hence $u \to_h z^\lhd = u^\lhd \to_h z^\lhd$ for all $u \in A \cup \{X\}$),
(4) ${}^\lhd$ is injective on $A \cup \{X\}$, i.e., $u \le z$ iff $u^\lhd \le z^\lhd$ for $u, z \in A \cup \{X\}$,
(5) the restriction of ${}^\lhd$ to $\mathbf{A}$ becomes a residuated lattice embedding, i.e., $(ab)^\lhd = a^\lhd \cdot_h b^\lhd$ and $(a \to b)^\lhd = a^\lhd \to_h b^\lhd$.

[1]Recall that a module action of a semiring $(R, \vee, \cdot, 1)$ to a semigroup $(M, \wedge)$ is a map $\mathbin{\backslash\!\!\backslash} : R \times M \to M$ that satisfies the usual axioms of a vector space, where R is the set of scalars, M the set of vectors and $\mathbin{\backslash\!\!\backslash}$ is scalar multiplication: $1 \mathbin{\backslash\!\!\backslash} m = m$, $r_1 \mathbin{\backslash\!\!\backslash} (r_2 \mathbin{\backslash\!\!\backslash} m) = (r_1 r_2) \mathbin{\backslash\!\!\backslash} m$, $r \mathbin{\backslash\!\!\backslash} (m_1 \wedge m_2) = (r \mathbin{\backslash\!\!\backslash} m_1) \wedge (r \mathbin{\backslash\!\!\backslash} m_2)$ and $(r_1 \vee r_2) \mathbin{\backslash\!\!\backslash} m = (r_1 \mathbin{\backslash\!\!\backslash} m) \wedge (r_2 \mathbin{\backslash\!\!\backslash} m)$.

Note that for every module action from $\mathbf{A}(X)^\vee$ the following has to hold for all $z \in A^X$:

$$(p_0 \vee p_1 X) \backslash\!\backslash z = p_0 \backslash\!\backslash z \wedge (p_1 X) \backslash\!\backslash z = p_0 \backslash\!\backslash z \wedge p_1 \backslash\!\backslash (X \backslash\!\backslash z).$$

So it is enough to define the action by elements of A and by X.

First we define $a \backslash\!\backslash b$ for $a, b \in A$. Using the requirements (2), (3) and (5), the following has to hold:

$$(a \backslash\!\backslash b)^\lhd = a \to_h b^\lhd = a^\lhd \to_h b^\lhd = (a \to b)^\lhd.$$

Thus it makes sense to define $a \backslash\!\backslash b = a \to b$ for $a, b \in A$ since then the above equality holds.

For the elements $X \backslash\!\backslash a$ we note that for all $b \in A$ we have by requirements (2), (3), (5), (4) and the fact that h is the least element of A above X in A^X the following chain of equivalences:

$$\begin{aligned} b^\lhd \leq (X \backslash\!\backslash a)^\lhd = X^\lhd \to_h a^\lhd \quad &\text{iff} \quad b^\lhd \cdot_h X^\lhd \leq a^\lhd \\ &\text{iff} \quad X^\lhd \leq b^\lhd \to_h a^\lhd = (b \to a)^\lhd \\ &\text{iff} \quad X \leq b \to a \\ &\text{iff} \quad h \leq b \to a \\ &\text{iff} \quad b \leq h \to a. \end{aligned}$$

In an attempt to be economical and define A^X to be as small as possible, we explore the possibility of defining $X \backslash\!\backslash a = h \to a$, namely $X \backslash\!\backslash a$ does not yield a new element of $\mathbf{A}^X$ but an old element of $\mathbf{A}$, which then has to be $h \to a$ by the above calculation.

Further, by the requirement (3) we have $X \leq X^\lhd = 1 \to_h X^\lhd = 1^\lhd \to_h X^\lhd$. Consequently, $1^\lhd \leq X \to_h X^\lhd = (X \backslash\!\backslash X)^\lhd$. Again trying to be economical we define $X \backslash\!\backslash X = 1$ so that $X \backslash\!\backslash X$ is an old element of $\mathbf{A}$.

Concerning the definition of $a \backslash\!\backslash X$ for $a \in A$, we note that for all $b \in A$ the following holds:

$$\begin{aligned} b^\lhd \leq (a \backslash\!\backslash X)^\lhd = a^\lhd \to_h X^\lhd \quad &\text{iff} \quad (ab)^\lhd = b^\lhd \cdot_h a^\lhd \leq X^\lhd \\ &\text{iff} \quad ab \leq g \\ &\text{iff} \quad b \leq a \to g. \end{aligned} \tag{*}$$

Unlike the above case, we cannot hope to define $a \backslash\!\backslash X = a \to g$, as for example for $a = 1$ this would collapse X with g, but also because it turns out that this will not yield an action. Namely, $a \to 1 = a \backslash\!\backslash (X \backslash\!\backslash X) = X \backslash\!\backslash (a \backslash\!\backslash X) = X \backslash\!\backslash (a \to g) = h \to (a \to g) = ha \to g$ which does not hold in general. This means that the elements of the form $a \backslash\!\backslash X$ need to be new elements of $\mathbf{A}^X$; formally we could define them as pairs (a, X), but we use the more suggestive notation $a \backslash\!\backslash X$. So we actually define $A^X := A \cup (A \backslash\!\backslash X)$, and note that it contains $A \cup \{X\}$ if we identify X with $1 \backslash\!\backslash X$. This means that with our method it is impossible to define a commutative residuated chain structure on just the set $A \cup \{X\}$, but we need to add new elements that will materialize the implications $a \to X$. In summary we have motivated the definition

- $b \backslash\!\backslash a := b \to a$,
- $X \backslash\!\backslash a = h \to a$,
- $b \backslash\!\backslash (a \backslash\!\backslash X) = (ab) \backslash\!\backslash X$,
- $X \backslash\!\backslash X = 1$,

which uniquely extends to the action of $\mathbf{A}(X)^\vee$ to $A^X = A \cup A \backslash\!\backslash X$ according to the definition:

$$(p_0 \vee p_1 X) \backslash\!\backslash a = (p_0 \to a) \wedge (p_1 h \to a), \qquad (p_0 \vee p_1 X) \backslash\!\backslash (a \backslash\!\backslash X) = (p_0 a \backslash\!\backslash X) \wedge (p_1 a \to 1).$$

The final step is to define the order structure on A^X, which is also used in the computation of the above meets. This order extends the order on $\mathbf{A}$. Further, we set $a \backslash\!\backslash X \leq b \backslash\!\backslash X$ iff $b \leq a$ for $a, b \in A$. This makes sense because $b \leq a$ implies $(a \backslash\!\backslash X)^\lhd = a \to_h X^\lhd \leq b \to_h X^\lhd = (b \backslash\!\backslash X)^\lhd$. Finally following (*), we set $b \leq a \backslash\!\backslash X$ iff $b \leq a \to g$.

Clearly this is a total ordering so $\mathbf{A}^X$ is a totally-ordered module. We observe that for this ordering the set of positive elements (elements greater or equal to 1) of $\mathbf{A}^X$ is $(A^X)^+ = A^+ \cup (\downarrow g) \backslash\!\backslash X$, where A^+ is the set of *positive elements* of A.

Concluding we define ${}^{\lhd}: \mathbf{A}^X \to \mathbf{A}(X)^{\wedge}$ by setting $z^{\lhd}$ to be the maximum of the set $\{P \in \mathbf{A}(X) : 1 \leq P \mathbin{\backslash\!\backslash} z\}$. This is motivated by the fact that $P \leq z^{\lhd} = 1^{\lhd} \to_h z^{\lhd}$ iff $1^{\lhd} \leq P \to_h z^{\lhd} = (P \mathbin{\backslash\!\backslash} z)^{\lhd}$. The following theorem shows that $a^{\lhd} = \hat{a}$ and $(a \mathbin{\backslash\!\backslash} X)^{\lhd} = \tilde{a}$.

Theorem 3.6. *Assume that $\mathbf{A}$ is a commutative residuated chain with a gap $g < h$.*

1. *For all $a \in A$, we have $a^{\lhd} = a \vee (h \to a)X$ and $(a \mathbin{\backslash\!\backslash} X)^{\lhd} = (a \to g) \vee (a \to 1)X$.*
2. *The map ${}^{\lhd}: \mathbf{A}^X \to \mathbf{A}(X)^{\wedge}$ is an $\mathbf{A}(X)^{\vee}$-module morphism that is injective on $A \cup \{X\}$.*
3. *The image $\bar{\mathbf{A}} = [\mathbf{A}^X]^{\lhd}$ forms a closure system and the least closed element above $P = p_0 \vee p_1 X \in A(X)$ is $P(h)^{\lhd} \wedge (((p_0 \to g) \wedge (p_1 \to 1)) \mathbin{\backslash\!\backslash} X)^{\lhd}$.*
4. *$\bar{\mathbf{A}}$ is a commutative residuated chain and $g^{\lhd} < X^{\lhd} < h^{\lhd}$.*
5. *The restriction of ${}^{\lhd}$ to $\mathbf{A}$ is a residuated lattice embedding.*

Proof. (1) We compute $a^{\lhd}$ by finding the maximum linear polynomial $P = p_0 \vee p_1 X$ such that $1 \leq P \mathbin{\backslash\!\backslash} a$, namely such that $1 \leq (p_0 \to a) \wedge (p_1 h \to a)$, which is equivalent to $p_0 \leq a$ and $p_1 \leq h \to a$. We also compute $(a \mathbin{\backslash\!\backslash} X)^{\lhd}$ by finding the maximum linear polynomial $P = p_0 \vee p_1 X$ such that $1 \leq P \mathbin{\backslash\!\backslash} (a \mathbin{\backslash\!\backslash} X)$. The last condition is equivalent to $1 \leq (p_0 a \mathbin{\backslash\!\backslash} X) \wedge (p_1 a \to 1)$, namely to $1 \leq (p_0 a \mathbin{\backslash\!\backslash} X)$ and $1 \leq (p_1 a \to 1)$. From the second we get $p_1 \leq a \to 1$ and from the first we get, based on the definition of $(A^X)^+$, that $p_0 a \leq g$, namely $p_0 \leq a \to g$.

(2) To show that ${}^{\lhd}$ is a $\wedge$-homomorphism it is enough to show that it preserves the order, as its domain is totally ordered. To that direction note that if $a \leq b$ then $a \vee (h \to a)X \leq b \vee (h \to b)X$. Also, if $a \mathbin{\backslash\!\backslash} X \leq b \mathbin{\backslash\!\backslash} X$ then $b \leq a$ (by definition) and then $(a \to g) \vee (a \to 1)X \leq (b \to g) \vee (b \to 1)X$. Since $a \leq b \mathbin{\backslash\!\backslash} X$ iff $ab \leq g$, the order preservation is exactly the content of the proof of Lemma 3.2.

To show that it preserves the action by elements of $\mathbf{A}(X)^{\vee}$ we have

$$(P \mathbin{\backslash\!\backslash} a)^{\lhd} = ((p_0 \vee p_1 X) \mathbin{\backslash\!\backslash} a)^{\lhd} = ((p_0 \to a) \wedge (p_1 h \to a))^{\lhd} = (P(h) \to a)^{\lhd} = P \to_h a^{\lhd},$$

where the last equality is from the proof of Lemma 3.4 and the fact that $a^{\lhd} = \hat{a}$. Also, we have

$$(P \mathbin{\backslash\!\backslash} (a \mathbin{\backslash\!\backslash} X))^{\lhd} = ((p_0 \vee p_1 X) \mathbin{\backslash\!\backslash} (a \mathbin{\backslash\!\backslash} X))^{\lhd} = ((p_0 a \mathbin{\backslash\!\backslash} X) \wedge (p_1 a \to 1))^{\lhd} = \widetilde{p_0 a} \wedge \widehat{p_1 a \to 1} = P \to_h (a \mathbin{\backslash\!\backslash} X)^{\lhd},$$

where the last equality follows from the proof of Lemma 3.4 and the fact that $(a \mathbin{\backslash\!\backslash} X)^{\lhd} = \tilde{a}$, while for the next to the last equation we used that ${}^{\lhd}$ is a $\wedge$-homomorphism.

(3) This follows from Lemma 3.3.

(4) We know that $\bar{\mathbf{A}}$ is an $\mathbf{A}(X)^{\vee}$-submodule of $\mathbf{A}(X)^{\wedge}$ being an image of a module morphism ${}^{\lhd}$. Also it is actually a closure system in $\mathbf{A}(X)$, namely for every linear polynomial P there is a smallest element $\gamma(P)$ of $\bar{A}$ that is above the polynomial P. This means that $\bar{A}$ supports a residuated lattice structure being a *nuclear retraction* of $\mathbf{A}(X)$ by γ. The fact that $g^{\lhd} < X^{\lhd} < h^{\lhd}$ follows from Lemma 3.4.

(5) This follows from Theorem 3.5. ☐

3.3. **Totally ordered abelian groups.** If the commutative residuated chain $\mathbf{A}$ is a group, then $\bar{\mathbf{A}}$ will be an commutative residuated chain, but may not be (actually it is not) a group.[2] However, we can slightly modify the construction to produce a group. We consider the congruence on $\mathbf{A}[X]$ given by $X^2 = h$ and this yields the following multiplication on $\mathbf{A}(X)$.

$$(p_0 \vee p_1 X) \cdot (q_0 \vee q_1 X) = (p_0 q_0 \vee p_1 q_1 h) \vee (p_0 q_1 \vee p_1 q_0)X.$$

By analogy to the above we consider the elements $a^{\lhd} = a \vee (h \to a)X$ and $(a \mathbin{\backslash\!\backslash} X)^{\lhd} = (a \to g) \vee (a \to 1)X$. It turns out that if we apply this construction at an arbitrary gap $g < h$, we may not get a totally ordered algebra. However, such a gap exists in a totally-ordered group iff there is also a gap $1 < g \to h$. It turns out that filling the latter gap also results in filling the former, so without loss of generality we may assume that $g = 1$ and that h is a cover of 1. Then we will see that the construction produces a totally-ordered group, which fills the gap.

The elements we obtain are $a^{\lhd} = a \vee (h^{-1}a)X$ and $(a \mathbin{\backslash\!\backslash} X)^{\lhd} = (a^{-1}) \vee (a^{-1})X$. Since all elements are invertible the latter type of elements can be written as $a \vee aX$. Since h is an upper cover of 1, h^{-1} is a lower cover of 1, and $h^{-1}a$ is a lower cover of a. This means that

[2]For the case where $\mathbf{A}$ is the ℓ-group of integers, then $\bar{\mathbf{A}}$ ends up being an commutative residuated chain isomorphic to $\mathbb{Z} \cup \{n^- : n \in \mathbb{Z}\}$, where $n \cdot m^- = (nm)^-$ and $n^- \cdot m^- = (nm)^-$.

$h^{-1}a \vee h^{-1}aX < a \vee h^{-1}aX < a \vee aX$ are coverings and thus we have an interweaving of the two types of elements in a transparent way, exactly as the odd and the even integers are ordered within the set of all integers.

We can also compute the multiplication of these types of elements according to the above definition.

$$(a \vee h^{-1}aX) \cdot (b \vee h^{-1}bX) = (ab \vee h^{-1}ah^{-1}bh) \vee (ah^{-1}b \vee bh^{-1}a)X = ab \vee h^{-1}abX.$$

$$(a \vee h^{-1}aX) \cdot (b \vee bX) = (ab \vee h^{-1}abh) \vee (ab \vee bh^{-1}a)X = ab \vee abX.$$

$$(a \vee aX) \cdot (b \vee bX) = (ab \vee abh) \vee (ab \vee ba)X = abh \vee h^{-1}abhX.$$

If we denote $a \vee (h^{-1}a)X$ simply by a and $a \vee aX$ by $a\sqrt{h}$, the above computations show that we extend the multiplication of $\mathbf{A}$ and also $a \cdot b\sqrt{h} = ab\sqrt{h}$ and $a\sqrt{h} \cdot b\sqrt{h} = ab$.

So this provides exactly the usual construction of adding square roots of h. For example, repeating this construction on the totally-ordered group of integers and taking the direct limit produces the dyadic numbers.

4. Complete and dense commutative residuated chain, via residuated frames

We follow the patterns of the previous two sections of first providing a short proof of the result followed by a discussion of the ad hoc choices made in the construction.

4.1. **The construction.** Given an commutative residuated chain $\mathbf{A}$ and an element $p \notin A$, its one-step densification will consist of subsets of some monoid extension of $\mathbf{A}$ as in Section 2. The monoid extension of $\mathbf{A}$ is again taken to be $W = A \cup Ap$ under the operation that extends the multiplication on $\mathbf{A}$ and satisfies $p^2 = ph$; in detail the operation is given by $a \cdot b = ab$, $a \cdot bp = ap \cdot b = (ab)p$ and $ap \cdot bp = (abh)p$. Recall that the order $\leq$ on $\mathbf{A}$ extends naturally to W so that we can view it as a disjoint union of two chains.

We refer to the subsets of W that we will be taking as *closed subsets*, and they will all be downsets with respect to above-mentioned extended order. More specifically, they will be described with the help of a new set W' of elements, which will serve as formal names for the closed subsets, with the understanding that multiple names may end up describing the same closed subset. We define $W' = A \cup (A \times \{p\})$.

We also define an action of W to W', namely a function $\mathbin{\backslash\!\!\backslash} : W \times W' \to W'$ that satisfies $1 \mathbin{\backslash\!\!\backslash} z = z$ and $x \mathbin{\backslash\!\!\backslash} (y \mathbin{\backslash\!\!\backslash} z) = (xy) \mathbin{\backslash\!\!\backslash} z$, for all $x, y \in W$ and $z \in W'$; in the following we will write simply p for $(1, p)$. The action will be generated by the conditions:

$$a \mathbin{\backslash\!\!\backslash} b = a \to b, \quad p \mathbin{\backslash\!\!\backslash} b = h \to b, \quad p \mathbin{\backslash\!\!\backslash} p = 1.$$

In detail the action is given by the following definition, which covers all combinations of types of elements in W and W', and can be derived by the above generating conditions and the properties of an action. It is then straightforward to check that this definition indeed yields an action.

$$a \mathbin{\backslash\!\!\backslash} b = a \to b, \quad a \mathbin{\backslash\!\!\backslash} (b, p) = (ab) \mathbin{\backslash\!\!\backslash} p = (ab, p), \quad ap \mathbin{\backslash\!\!\backslash} b = ah \to b, \quad ap \mathbin{\backslash\!\!\backslash} (b, p) = ab \to 1.$$

Note that $(a, p) = a \mathbin{\backslash\!\!\backslash} (1, p) = a \mathbin{\backslash\!\!\backslash} p$, so we obtain the more intuitive description $W' = A \cup A \mathbin{\backslash\!\!\backslash} p$, where of course $A \mathbin{\backslash\!\!\backslash} p := \{a \mathbin{\backslash\!\!\backslash} p : a \in A\}$.

We consider the subset $D = \{a : 1 \leq a\} \cup \{a \mathbin{\backslash\!\!\backslash} p : a \leq g\}$ of W' and we define the relation $N \subseteq W \times W'$ by: $x \mathrel{N} z$ iff $x \mathbin{\backslash\!\!\backslash} z \in D$. In detail,

$$a \mathrel{N} b \Leftrightarrow a \leq b, \quad a \mathrel{N} b \mathbin{\backslash\!\!\backslash} p \Leftrightarrow ab \leq g, \quad ap \mathrel{N} b \Leftrightarrow ah \leq b, \quad ap \mathrel{N} b \mathbin{\backslash\!\!\backslash} p \Leftrightarrow ab \leq 1.$$

Note that for all $x, y \in W$ and $z \in W'$, we have

$$x \cdot y \mathrel{N} z \Leftrightarrow (x \cdot y) \mathbin{\backslash\!\!\backslash} z \in D \Leftrightarrow y \mathbin{\backslash\!\!\backslash} (x \mathbin{\backslash\!\!\backslash} z) \in D \Leftrightarrow y \mathrel{N} x \mathbin{\backslash\!\!\backslash} z.$$

We have verified that $\mathbf{W} = (W, W', N, \cdot, 1, \mathbin{\backslash\!\!\backslash})$ is a commutative *residuated frame*, in the sense of [5].[3] It follows from results in [5] that the *dual algebra* $\mathbf{W}^+$ is a commutative residuated lattice,

[3] The definition in [5] is fairly general, but for our purposes we can define a commutative *residuated frame* to be a structure of the form $(W, W', N, \circ, \varepsilon, \mathbin{\backslash\!\!\backslash})$, where $(W, \circ, \varepsilon)$ is a commutative monoid, W' is a set, $N \subseteq W \times W'$ is a relation, $\mathbin{\backslash\!\!\backslash} : W \times W' \to W'$, and the nuclearity condition holds: $x \circ y \mathrel{N} z \Leftrightarrow y \mathrel{N} x \mathbin{\backslash\!\!\backslash} z$.

where $\mathbf{W}^+ = (\gamma[\mathcal{P}(W)], \cap, \cup_\gamma, \cdot_\gamma, \to, \gamma\{1\})$; for $X, Y \subseteq W$ and $Z \subseteq W'$ we define

$$X^\rhd = \{z \in W' \mid \forall x \in X : x\ N\ z\}, \quad Z^\lhd = \{x \in W \mid \forall z \in Z : x\ N\ z\}, \quad \gamma(X) = X^{\rhd\lhd},$$
$$X \cdot Y = \{xy \mid x \in X, y \in Y\}, \quad X \cdot_\gamma Y = \gamma(X \cdot Y),$$
$$X \cup_\gamma Y = \gamma(X \cup Y), \quad X \to Y = \{w \in W \mid X \cdot \{w\} \subseteq Y\}.$$

In fact, $(\mathcal{P}(W), \cap, \cup, \cdot, \to, \{1\})$ forms a commutative residuated lattice and $\mathbf{W}^+$ is its nuclear retraction induced by the above-defined nucleus γ.

It also follows that every *closed set*, namely every element of W^+, is an intersection of *basic closed sets*, namely sets of the form $\{z\}^\lhd$, for $z \in W'$ (see e.g.[5]). We have only two types of basic closed sets

$$\{a\}^\lhd = \{w \in W \mid w\ N\ a\} = \{b \in A \mid b \leq a\} \cup \{bp \in Ap \mid bh \leq a\} = {\downarrow} a \cup {\downarrow}(h \to a) \cdot \{p\},$$

$$\{a \backslash\!\backslash p\}^\lhd = \{w \in W \mid w\ N\ a\backslash\!\backslash p\} =$$
$$= \{b \in A \mid b \leq a \to g\} \cup \{bp \in Ap \mid ba \leq 1\} = {\downarrow}(a \to g) \cup ({\downarrow}(a \to 1)) \cdot \{p\}.$$

Note that the basic closed sets are really downward closed. Compare them also with the languages L_a, $a^{-1}L_p$ from Section 2 and polynomials $\hat{a}$, $\tilde{a}$ from Section 3.

Lemma 4.1. *If $\mathbf{A}$ is an commutative residuated chain, then so is $\mathbf{W}^+$.*

Proof. It suffices to show that the basic closed sets are totally ordered by inclusion.

Clearly, if $a \leq b$, then $\{a\}^\lhd \subseteq \{b\}^\lhd$ and $\{b\backslash\!\backslash p\}^\lhd \subseteq \{a\backslash\!\backslash p\}^\lhd$. Consider $\{a\}^\lhd$ and $\{b\backslash\!\backslash p\}^\lhd$.

If $ab \leq g$, then $a \leq b \to g$, so ${\downarrow} a \subseteq {\downarrow}(b \to g)$. Also, $b \leq a \to g$, so $b(h \to a) \leq (h \to a)(a \to g) \leq h \to g < 1$, hence $h \to a \leq b \to 1$ and ${\downarrow}(h \to a) \subseteq {\downarrow}(b \to 1)$.

If $h \leq ab$, then $a \not\leq b \to g$, so $b \to g < a$ and ${\downarrow}(b \to g) \subseteq {\downarrow} a$. Also, $h(b \to 1) \leq ab(b \to 1) \leq a$, so ${\downarrow}(b \to 1) \subseteq {\downarrow}(h \to a)$. □

A *Gentzen frame* is a pair $(\mathbf{W}, \mathbf{S})$ of a residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \backslash\!\backslash)$ and a partial algebra $\mathbf{S}$ (in the language of residuated lattices) such that S that can be viewed as a common subset of W and W' and under this identification the following conditions (called *logical rules*) hold for all $x, y \in W$, $z \in W'$ and $a, b \in S$; we consider a logical rule only in case the operation on the elements of S in the conclusion is defined.

$$\frac{x\ N\ a \quad a\ N\ z}{x\ N\ z}\ (cut) \qquad \frac{}{a\ N\ a}\ (id) \qquad \frac{\varepsilon\ N\ z}{1\ N\ z}\ (L1) \qquad \frac{}{\varepsilon\ N\ 1}\ (R1)$$
$$\frac{a\ N\ z}{a \wedge b\ N\ z}\ (L\wedge) \qquad \frac{b\ N\ z}{a \wedge b\ N\ z}\ (L\wedge) \qquad \frac{x\ N\ a \quad x\ N\ b}{x\ N\ a \wedge b}\ (R\wedge)$$
$$\frac{a\ N\ z \quad b\ N\ z}{a \vee b\ N\ z}\ (L\vee) \qquad \frac{x\ N\ a}{x\ N\ a \vee b}\ (R\vee) \qquad \frac{x\ N\ b}{x\ N\ a \vee b}\ (R\vee)$$
$$\frac{a \circ b\ N\ z}{a \cdot b\ N\ z}\ (L\cdot) \qquad \frac{x\ N\ a \quad y\ N\ b}{x \circ y\ N\ a \cdot b}\ (R\cdot) \qquad \frac{x\ N\ a \quad b\ N\ z}{a \to b\ N\ x\backslash\!\backslash z}\ (L\to) \qquad \frac{x\ N\ a\backslash\!\backslash b}{x\ N\ a \to b}\ (R\to)$$

Note that $\mathbf{A} \cup \{p\}$ can be considered as a partial algebra where we extend the operations of $\mathbf{A}$ with $a \vee p = p$ and $a = a \wedge p$ if $a \leq g$, and $a \vee p = a$ and $p = a \wedge p$ if $h \leq a$. Thus $\mathbf{A} \cup \{p\}$ forms a chain ordered by the order $\sqsubseteq$ defined in Section 2 (see Figure 1). The partial algebra $\mathbf{A} \cup \{p\}$ will play the role of $\mathbf{S}$ above. By results of [5] it follows from the next theorem that the function mapping $z \in A \cup \{p\}$ to $\{z\}^\lhd$ is an embedding of this partial algebra into $\mathbf{W}^+$, so indeed the latter can serve as a one-step densification at $g < h$.

Theorem 4.2. *The pair $(\mathbf{W}, \mathbf{A} \cup \{p\})$ forms an antisymmetric Gentzen frame.*

Proof. We first show antisymmetry of the Gentzen frame, namely that $q\ N\ r$ and $r\ N\ q$ implies $q = r$, for all $q, r \in A \cup \{p\}$. This is clear for elements of A and for $q = r = p$. It is also true for $q = p$ and $r \in A$, as $h \leq r$ and $r \leq g$ is impossible.

Next we have to verify that the relation N satisfies all the logical rules. The identity rule (id) holds since $p\backslash\!\backslash p = 1 \in D$ and $a \leq a$ for all $a \in A$. The rule $(L1)$ and $(R1)$ clearly hold as ε and 1 coincide in our case. The validity of $(L\vee)$ and $(R\wedge)$ follows since $\mathbf{A} \cup \{p\}$ forms a chain. The validity of $(L\wedge)$ amounts to the fact that the basic closed sets are downward closed with respect to $\sqsubseteq$ (cf. Lemma 2.3). As the basic closed sets are totally ordered by inclusion according to the proof of Lemma 4.1, the rule $(R\vee)$ holds as well.

The cut rule holds if we prove that $u \in \{z\}^{\lhd}$ implies $\{u\}^{\lhd} \subseteq \{z\}^{\lhd}$ for $u \in A \cup \{p\}$ and $z \in W'$. If $u = a \in A$ then $a \leq b$ or $a \leq b \to g$ depending whether $z = b$ or $z = b \mathbin{\backslash\!\backslash} p$ for some $b \in A$. In the first case $\{a\}^{\lhd} \subseteq \{b\}^{\lhd}$. In the second we have $h \to a \leq bh \to g \leq b \to 1$. Hence $\{a\}^{\lhd} \subseteq \{b \mathbin{\backslash\!\backslash} p\}^{\lhd}$ also in this case. If $u = p$ then $1 \leq h \to b$ or $1 \leq b \to 1$ again depending whether $z = b$ or $z = b \mathbin{\backslash\!\backslash} p$ for some $b \in A$. The first case follows since $g \leq h \leq b$ which further implies $\{p\}^{\lhd} \subseteq \{b\}^{\lhd}$. In the second case, $g \leq b \to g$ since $b \leq 1$. Thus $\{p\}^{\lhd} \subseteq \{b \mathbin{\backslash\!\backslash} p\}^{\lhd}$.

The rules $(L\cdot)$, $(R\cdot)$, $(L{\to})$ and $(R{\to})$ have to be checked only for $a, b \in A$ because $a \cdot p$, $p \cdot b$, $a \to p$ and $p \to b$ are not defined. The rules $(L\cdot)$ and $(R{\to})$ are trivial because for $a, b \in A$ we have $a \circ b = a \cdot b$ and $a \mathbin{\backslash\!\backslash} b = a \to b$ in our setting. Concerning the right logical rule $(R\cdot)$, assume that $x = cp^m \in \{a\}^{\lhd}$ and $y = dp^n \in \{b\}^{\lhd}$ for some $c, d \in A$ and $m, n \in \{0, 1\}$. Then $ch^m \leq a$ and $dh^n \leq b$. Thus $cdh^{m+n} \leq ab$ which is equivalent to $x \cdot y = cdh^{\max\{0, m+n-1\}} p^{\min\{1, m+n\}} \in \{ab\}^{\lhd}$. Finally, for the validity of $(L{\to})$, assume that $b \in \{z\}^{\lhd}$ and $x = cp^k \in \{a\}^{\lhd}$ for some $c \in A$ and $k \in \{0, 1\}$. From the second assumption we have $ch^k \leq a$. Consequently, $ch^k(a \to b) \leq a(a \to b) \leq b \in \{z\}^{\lhd}$. Since $\{z\}^{\lhd}$ is downward closed, we have $ch^k(a \to b) \ N \ z$ which is equivalent to $h^k \ N \ c(a \to b) \mathbin{\backslash\!\backslash} z$. It follows that $p^k \ N \ c(a \to b) \mathbin{\backslash\!\backslash} z$ because the basic closed sets are also downward closed with respect to $\sqsubseteq$ and $p \sqsubseteq h$. Consequently, $a \to b \ N \ cp^k \mathbin{\backslash\!\backslash} z$.

□

4.2. **Explanations of design choices.** In this section we again explain our design choices made in the previous section. We also mention how the construction of our residuated frame is related to the one given in [1].

4.2.1. *Residuated frames as* $\mathbf{M}$*-sets with a designated subset.* We first note that in most residuated frames considered in applications we have that $\mathbin{\backslash\!\backslash}$ is an action. Recall that given a monoid $(W, \circ, \varepsilon)$, the pair $(W', \mathbin{\backslash\!\backslash})$ is called a $(W, \circ, \varepsilon)$*-set* if $\mathbin{\backslash\!\backslash}$ is an action. So, such residuated frames are simply $(W, \circ, \varepsilon)$-sets $(W', \mathbin{\backslash\!\backslash})$ together with an arbitrary subset D of W'. This subset D is interdefinable with the relation N by: $D = \{z \in W' : \varepsilon \ N \ z\} = \{\varepsilon\}^{\rhd}$ and $x \ N \ z \Leftrightarrow x \mathbin{\backslash\!\backslash} z \in D$. Also the nuclearity property of N follows directly from properties of the action.

4.2.2. *Residuated frame construction.* To produce a one-step densification of $\mathbf{A}$, we essentially start from the standard frame associated with $\mathbf{A}$, which is $(A, A, \leq, \cdot, 1, \to)$. Then it is natural to extend this frame by a new element p in a free fashion. This involves taking the free extension of the monoid $(A, \cdot, 1)$ with the new element p, namely the monoid $W = Ap^* = \{ap^n : a \in A, n \in \mathbb{N}\}$, where $ap^n \cdot bp^m = abp^{n+m}$. Furthermore, we need to extent the right universe A to at least $A \cup \{p\}$. Since we want an action from Ap^* to the right universe, we can take the free action and choose $W' = W \times (A \cup \{p\})$, where $x \mathbin{\backslash\!\backslash} (y, q) = (x \circ y, q)$. This is exactly the approach used in [1]. However we have seen that in Section 4.1 both W and W' were chosen much smaller. Nevertheless, we actually obtain exactly the same dual algebra as in [1].

The reasons why we can consider a smaller monoid extension W follows from the fact that the nucleus constructed in [1] identifies p^2 with ph, in the sense that $\gamma(\{p^2\}) = \gamma(\{ph\})$, so we can make this identification much earlier at the level of the initial W.

Our choice of $W' = A \cup A \mathbin{\backslash\!\backslash} p$ is also smaller than $W \times (A \cup \{p\})$ Although the big $W' = W \times (A \cup \{p\})$ yields the same dual algebra, it produces seemingly many more basic closed sets $\{z\}^{\lhd}$, $z \in W'$ simply because W' is bigger. It turns out that a lot of these basic closed sets coincide, but if one checks properties at the frame level directly the large W' creates an explosion of cases, making the verification of conditions, such as linearity, extremely cumbersome; see [1]. Of course W' needs to be a natural set of names for the basic closed sets; this is important as one could otherwise choose as elements of W' all closed sets (usually a very hard task anyway), thus beating the purpose of even using frames to construct the dual algebra. However, we want to stress that it is a wise idea to reduce W' to a point where it still remains natural, but has no obvious redundancies. To explain our point we show how to calculate the reduction from $Ap^* \times (A \cup \{p\})$ to $A \cup (A \times \{p\})$, or in more intuitive notation, and using the identification $q \equiv (1, q)$, the reduction from $Ap^* \mathbin{\backslash\!\backslash} (A \cup \{p\})$ to $A \cup (A \mathbin{\backslash\!\backslash} p)$.

Note that $p^* = \{1\} \cup p^+$ and also $p^+ = pp^*$, where $p^+ = \{p^n : n > 0\}$, so $Ap^* = Ap^+ \cup A = Ap^*p \cup A$. Using properties of the action and lifting $\Vert$ to sets elementwise we have

$$\begin{aligned} Ap^* \Vert (A \cup \{p\}) &\equiv (Ap^* \Vert A) \cup (Ap^* \Vert \{p\}) \\ &\equiv (Ap^* \Vert A) \cup ((Ap^*p \cup A) \Vert \{p\}) \\ &\equiv (Ap^* \Vert A) \cup (Ap^*p \Vert \{p\}) \cup (A \Vert \{p\}) \\ &\equiv (p^* \Vert (A \Vert A)) \cup (p^* \Vert (A \Vert (\{p\} \Vert \{p\}))) \cup (A \Vert \{p\}) \end{aligned}$$

Defining $Y \overset{\mathcal{P}}{\to} Z = \{y \to z : y \in Y, z \in Z\}$ and making the identifications $p^* \Vert (A \Vert A) \equiv p^* \Vert (A \overset{\mathcal{P}}{\to} A) \equiv p^* \Vert A \equiv h^* \overset{\mathcal{P}}{\to} A \subseteq A$ and $p^* \Vert (A \Vert (\{p\} \Vert \{p\})) \equiv p^* \Vert (A \overset{\mathcal{P}}{\to} \{1\}) \equiv h^* \overset{\mathcal{P}}{\to} (A \overset{\mathcal{P}}{\to} \{1\}) \subseteq A$, stemming from the definition of $\Vert$ in the frame, we conclude with the set $A \cup A \Vert \{p\}$. In the above we wrote $z \equiv z'$ for $\{z\}^{\lhd} = \{z'\}^{\lhd}$, and we lifted $\equiv$ to sets of elements in an elementwise fashion.

4.2.3. *Adding square roots to an abelian group.* To obtain the construction of Section 3.3, we modify the multiplication of $W = A \cup Ap$ so that $p^2 = h$. Then picking W' as above does not yield an action in general, so we define $W' = W \times (A \cup \{p\})$. However, it turns out that the basic closed sets again fall under two categories if $g = 1$ and h is its cover.

5. Mixed amalgamation

The one-step densification described in the previous sections can be also viewed as a form of amalgamation. More precisely, given a commutative residuated chain $\mathbf{A}$ with a gap $g < h$ and its one-step densification $\bar{\mathbf{A}}$, one can view $\bar{\mathbf{A}}$ as an amalgam of $\mathbf{A}$ and the three element chain $\{g < p < h\}$ over the two element chain $\{g < h\}$ (see Figure 3).

Here we will show that the above result can be further generalized. In particular, any commutative residuated chain $\mathbf{B}$ can be amalgamated with any chain C over any chain A to produce a commutative residuated chain $\mathbf{D}$. The analogous result also holds for commutative tomonoids. Note that usual amalgamation for commutative residuated chains neither follows from nor implies the above mixed amalgamation property. In particular, densification is not really related to the usual amalgamation property as there is no a priori prescribed structure of the subalgebra generated by the new element(s) in the densified algebra.

We will make use of the following auxiliary constructions. Given a commutative residuated chain $\mathbf{A}$, one can define a residuated chain on the set $\mathcal{U}(A)$ of all upward closed subsets of A. Indeed, it is easy to see that the map $\gamma\colon \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $\gamma(S) = {\uparrow}S$, where ${\uparrow}S$ is the upset generated by S, is a nucleus on the residuated lattice $\mathcal{P}(\mathbf{A}) = (\mathcal{P}(A), \cap, \cup, \cdot, \to, \{1\})$. Thus $\mathcal{U}(\mathbf{A}) = \mathcal{P}(\mathbf{A})_\gamma$ is a residuated chain. Moreover, the map $a \mapsto {\uparrow}\{a\}$ is an order-reversing injection preserving multiplication in $\mathbf{A}$. If we apply this construction twice, we obtain a residuated chain $\mathcal{U}(\mathcal{U}(\mathbf{A}))$ and an order-embedding $\tau\colon A \to \mathcal{U}(\mathcal{U}(A))$ defined by $\tau(a) = \mathcal{U}_a$, where $\mathcal{U}_a = {\uparrow}\{{\uparrow}\{a\}\} = \{U \in \mathcal{U}(A) \mid a \in U\}$. In addition, it is a residuated lattice homomorphism. It clearly preserves multiplication so it suffices to check that it preserves $\to$. We have

$$\begin{aligned} \mathcal{U}_a \to \mathcal{U}_b &= \{V \in \mathcal{U}(A) \mid \mathcal{U}_a \cdot_\gamma V \subseteq \mathcal{U}_b\} \\ &= \{V \in \mathcal{U}(A) \mid \forall U \in \mathcal{U}_a : {\uparrow}(U \cdot V) \in \mathcal{U}_b\} \\ &= \{V \in \mathcal{U}(A) \mid \forall U \in \mathcal{U}_a : b \in {\uparrow}(U \cdot V)\} \\ &= \{V \in \mathcal{U}(A) \mid b \in {\uparrow}({\uparrow}\{a\} \cdot V)\} \\ &= \{V \in \mathcal{U}(A) \mid \exists v \in V : av \le b\} \\ &= \{V \in \mathcal{U}(A) \mid \exists v \in V : v \le a \to b\} = \mathcal{U}_{a \to b}. \end{aligned}$$

In fact, the above construction is just the canonical extension of $\mathbf{A}$ (details on canonical extensions for residuated lattices can be found in [6, Chapter 6]).

Now we are ready to describe a construction which plugs a new element into any place within a given commutative residuated chain $\mathbf{A}$. Let $G \subseteq A$ be a downset, we will define an extension $\mathbf{A}_G$ of $\mathbf{A}$ that inserts a new element just above all the elements of G and below all elements in $A \setminus G$. If G has a maximum g which is covered by h (i.e., $g < h$ forms a gap) then $\mathbf{A}_G$ is defined as the one-step densification of $\mathbf{A}$. Otherwise G has no maximum element, or it has a maximum element

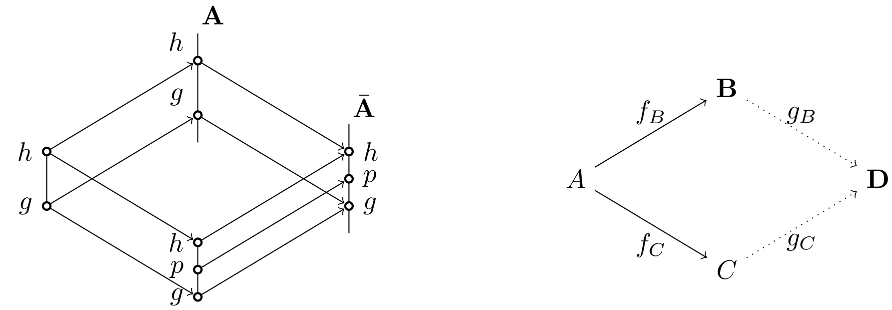

FIGURE 3. (Left) The one-step densification as amalgamation; (Right) The V-formation and its completion.

but its complement has no minimum element (in other words G does not define a gap $g < h$). Then we define $\mathbf{A}_G$ to be $\mathcal{U}(\mathcal{U}(\mathbf{A})$. This construction creates new elements for each infinitary join and infinitary meet, so $\mathbf{A}_G$ will contain one or two new elements between G and its complement.

In all of the above cases we have a construction which extends $\mathbf{A}$ by at least one element which is above all elements in G and below all elements in $A \setminus G$. We will denote by ϕ the embedding of $\mathbf{A}$ into $\mathbf{A}_G$ and actually we replace the image of ϕ by the original elements of $\mathbf{A}$, thus converting ϕ into an inclusion map and $\mathbf{A}$ into a subalgebra of $\mathbf{A}_G$.

Theorem 5.1. *Let A, C be chains, let $\mathbf{B}$ be a commutative residuated chains and let $f_B : A \to B$ and $f_C : A \to C$ be order embeddings. (The tuple $(A, \mathbf{B}, f_B, C, f_C)$ is often called a V-formation.) Then there is a commutative residuated chain $\mathbf{D}$, an order embedding $g_C : C \to D$ and a residuated lattice embedding $g_B : \mathbf{B} \to \mathbf{D}$ such that the right diagram in Figure 3 commutes, i.e., $g_B f_B = g_C f_C$.*

Proof. Without any loss of generality we identify the elements of A with their images in C under f_C. So we assume that $A \subseteq C$ and f_C is just the inclusion map. Using the axiom of choice, we can index the elements of $C \setminus A$ by an ordinal α, i.e., $C \setminus A = \{c_\beta \in C \setminus A \mid \beta < \alpha\}$. We will construct the algebra $\mathbf{D}$ starting from $\mathbf{B}$ and inserting the elements of $C \setminus A$ by transfinite recursion.

We define $C_0 = A$, $\mathbf{D}_0 = \mathbf{B}$ and $g_0 = f_B$. Further, we consider the chain $\{C_\beta \mid \beta \leq \alpha\}$ ordered by inclusion, where $C_\beta = A \cup \{c_\delta \in C \setminus A \mid \delta < \beta\}$, i.e., C_β contains all elements from A plus the elements in $C \setminus A$ indexed by ordinals less than β. The minimum of this chain is $C_0 = A$ and the maximum is $C_\alpha = C$.

Now we define algebras $\mathbf{D}_\beta$ and the maps $g_\beta \colon C_\beta \to D_\beta$ for ordinals $0 < \beta \leq \alpha$. If $\beta = \delta + 1$ for some ordinal δ. Then define $\mathbf{D}_\beta = (\mathbf{D}_\delta)_G$ for $G = {\downarrow} g_\delta[{\downarrow}\{c_\beta\} \cap C_\delta]$. The map g_β is an extension of g_δ mapping c_β to an element in $(D_\delta)_G$ which is above all elements in G and below all elements in $D_\delta \setminus G$. Thus g_β is an order-embedding and $g_\beta|_{C_\delta} = g_\delta$ so the corresponding inner diagram commutes in Figure 4.

If β is a limit ordinal then we define $\mathbf{D}_\beta = \bigcup_{\delta<\beta} \mathbf{D}_\delta$ and $g_\beta = \bigcup_{\delta<\beta} g_\delta$. It is obvious that $\mathbf{D}_\beta$ is a commutative residuated chain and g_β an order-embedding from C_β to D_β. Moreover $g_\beta|_{C_\delta} = g_\delta$ for every $\delta < \beta$.

Finally, the algebra $\mathbf{D} = \mathbf{D}_\alpha$ is the desired amalgam since $\mathbf{B}$ is its subalgebra and $g_\alpha|_A = g_0 = f_B$. So the outer diagram in Figure 4 commutes. □

6. Comparison of methods and viewpoints

Finally, we compare and relate the three approaches we described in this paper.

6.1. **From residuated frames to modules.** First we observe that the construction via residuated frames is related to that one using modules.

Assume that $\mathbf{W} = (W, \circ, \varepsilon)$ is a commutative monoid and W' a $\mathbf{W}$-set with action $\mathbin{\backslash\!\!\backslash}$. Then $(\mathcal{P}(W), \cup, \circ, \{\varepsilon\})$ is a complete semiring and $(\mathcal{P}(W'), \bigcup^\partial)$, namely with reverse inclusion, is a module over $(\mathcal{P}(W), \bigcup, \circ, \{\varepsilon\})$ with lifted action $X \mathbin{\backslash\!\!\backslash} Z = \{x \mathbin{\backslash\!\!\backslash} z : x \in X, z \in Z\}$; we use the

$$
\begin{tikzcd}
 & \mathbf{B} = \mathbf{D}_0 \arrow[rr, hook] \arrow[rrrrrrr, hook, bend left] & & \mathbf{D}_1 \arrow[rr, hook] & & \mathbf{D}_2 \cdots & & & \mathbf{D} = \mathbf{D}_\alpha \\
A = C_0 \arrow[ur, "g_0"] \arrow[rr, hook] \arrow[rrrrrrr, hook, bend right] & & C_1 \arrow[ur, "g_1"] \arrow[rr, hook] & & C_2 \cdots \arrow[ur, "g_2"] & & & C = C_\alpha \arrow[ur, "g_\alpha"] &
\end{tikzcd}
$$

FIGURE 4. The proof of the mixed amalgamation.

nonstandard notation $\bigcup^\partial$ to denote the union operation, but considered as the meet of the module. Indeed, $\bigcup X_i \bbslash \bigcup^\partial Z_j = \bigcup^\partial (X_i \bbslash Z_j)$ and $Y \bbslash (X \bbslash Z) = (X \circ Y) \bbslash Z$.

We also know that since $\mathcal{P}(W)$ is a residuated lattice, $(\mathcal{P}(W), \bigcap)$ is also a module over $(\mathcal{P}(W), \bigcup, \circ, \{\varepsilon\})$, where the action is $\to$.

Note that the map ${}^{\lhd} : (\mathcal{P}(W'), \bigcup^\partial) \to (\mathcal{P}(W), \bigcap)$ is a module morphism. Indeed, we have $X \circ Y \ N \ Z$ iff $Y \ N \ X \bbslash Z$. Namely, $Y \subseteq X \to Z^{\lhd}$ iff $X \circ Y \subseteq Z^{\lhd}$ iff $Y \subseteq (X \bbslash Z)^{\lhd}$. So, $(X \bbslash Z)^{\lhd} = X \to Z^{\lhd}$. Also, $(\bigcup^\partial Z_i)^{\lhd} = \bigcap Z_i^{\lhd}$. The image of this map is an $(\mathcal{P}(W), \cup, \circ, \{\varepsilon\})$-submodule of $(\mathcal{P}(W), \bigcap)$ and actually an intersection-closed structure in $\mathcal{P}(W)$, so it forms a residuated lattice and is exactly the dual algebra $\mathbf{W}^+$ of the frame $\mathbf{W}$.

6.2. **From residuated frames to polynomials.** Given a (commutative) residuated lattice $\mathbf{A}$, we consider the residuated frame $\mathbf{W} = (W, W', N, \cdot, 1, \bbslash)$ we defined in Section 4.1. To show the relationship between this residuated frame and the residuated lattice $\mathbf{A}(X)$, we consider the subset $X = \{p\}$ of W. By collecting all the monomials cp^n for the same n together, we note that each subset C of W can be written as $C = C_0 \cup C_1 X$, where $C_n = \{c \in A : cp^n \in C\}$.

Note that multiplication in the powerset $\mathcal{P}(W)$ is given by the formula

$$(C_0 \cup C_1 X)(D_0 \cup D_1 X) = C_0 D_0 \cup (C_0 D_1 \cup C_1 D_0 \cup C_1 D_1 \{h\}) X$$

and join is defined as the componentwise union. The resulting structure is a semiring which is similar to $\mathbf{A}(X)$ in the sense that we replaced the coefficient semiring $\mathbf{A}$ by $(\mathcal{P}(A), \cup, \cdot, \{1\})$ and X is not an abstract indeterminate this time but the set $\{p\}$.

We also note that in the case when $\mathbf{A}$ is complete, for every principal downset $D \subseteq A$ and arbitrary subset $C \subseteq A$, we have that $C \to D \in \mathcal{P}(A)$ is also a principal downset; also the intersection of principal downsets is also principal. In other words the principal downsets form a nuclear retraction of $(\mathcal{P}(A), \cup, \cap, \cdot, \to, \{1\})$, where the nucleus is given by $S \mapsto {\downarrow}(\bigvee S)$.

We extend this function to the powerset of $W = A \cup Ap$, under the direct product ordering, by setting $C_0 \cup C_1 X \mapsto {\downarrow} \bigvee C_0 \cup ({\downarrow} \bigvee C_1) X$. This defines also a nucleus now on the residuated lattice $\mathcal{P}(\mathbf{W}) = (\mathcal{P}(W), \cup, \cap, \cdot, \to, \{1\})$, whose image is isomorphic to the semiring (actually residuated lattice) $\mathbf{A}(X)$.

Since $\mathbf{A}(X)$ is a nuclear retraction of the residuated lattice $\mathcal{P}(\mathbf{W})$, it is also a homomorphic image of the semiring $\mathcal{P}(\mathbf{W})$ and also a $\mathcal{P}(\mathbf{W})^{\vee}$-submodule of $\mathcal{P}(\mathbf{W})^{\wedge}$. It turns out that the final dual algebra $\mathbf{W}^+$ is a further submodule of it. Thus in case that $\mathbf{A}$ is complete the approach of residuated frames and the approach using polynomials coincide by simply taking $X = \{p\}$.

6.3. **From residuated frames to languages.** Finally, we compare the approaches using residuated frames and syntactic congruences. Assume that we have a commutative residuated chain $\mathbf{A}$ with a gap $g < h$, θ is the pomonoid congruence on $\mathbf{A} \cup \mathbf{A}p$ defined in Section 2 and γ is the nucleus induced by the residuated frame $\mathbf{W}$ defined in Section 4.1. We claim that $x \ \theta \ y$ iff $\gamma(\{x\}) = \gamma(\{y\})$ for all $x, y \in A \cup Ap$.

Indeed, first note that if $\mathbf{A}$ is residuated then the language $b^{-1}L_a$ for $a, b \in A$ is just the language $L_{b\to a}$. Further note that for $a, b \in A$ we have $\{a\}^{\lhd} = L_a$ and $\{b\backslash\!\backslash p\}^{\lhd} = b^{-1}L_p$. Thus $b^{-1}L_a = \{b \to a\}^{\lhd}$. Consequently, we have (cf. Lemma 2.2)

$$\begin{aligned}
x \bigcap_{a\in A} \sim_{L_a} y \quad &\text{iff} \quad x \sim_{L_a} y \text{ for all } a \in A \\
&\text{iff} \quad x \in b^{-1}L_a \Leftrightarrow y \in b^{-1}L_a \text{ for all } a, b \in A \\
&\text{iff} \quad x \in \{b \to a\}^{\lhd} \Leftrightarrow y \in \{b \to a\}^{\lhd} \text{ for all } a, b \in A \\
&\text{iff} \quad x \in \{a\}^{\lhd} \Leftrightarrow y \in \{a\}^{\lhd} \text{ for all } a \in A.
\end{aligned}$$

Similarly, we have (cf. Lemma 2.2)

$$\begin{aligned}
x \sim_{L_p} y \quad &\text{iff} \quad x \in u^{-1}L_p \Leftrightarrow y \in u^{-1}L_p \text{ for all } u \in A \cup Ap \\
&\text{iff} \quad x \in b^{-1}L_p \Leftrightarrow y \in b^{-1}L_p \text{ and } x \in b^{-1}L_1 \Leftrightarrow y \in b^{-1}L_1 \text{ for all } b \in A \\
&\text{iff} \quad x \in \{b\backslash\!\backslash p\}^{\lhd} \Leftrightarrow y \in \{b\backslash\!\backslash p\}^{\lhd} \text{ and } x \in \{b \to 1\}^{\lhd} \Leftrightarrow y \in \{b \to 1\}^{\lhd} \text{ for all } b \in A.
\end{aligned}$$

Using the above facts, we finally obtain the following chain of equivalences:

$$\begin{aligned}
x\, \theta\, y \quad &\text{iff} \quad x \bigcap_{a\in A} \sim_{L_z} y \text{ and } x \sim_{L_p} y \\
&\text{iff} \quad x \in \{a\}^{\lhd} \Leftrightarrow y \in \{a\}^{\lhd} \text{ and } x \in \{b\backslash\!\backslash p\}^{\lhd} \Leftrightarrow y \in \{b\backslash\!\backslash p\}^{\lhd} \text{ for all } a, b \in A \\
&\text{iff} \quad x \in \{z\}^{\lhd} \Leftrightarrow y \in \{z\}^{\lhd} \text{ for all } z \in A \cup A\backslash\!\backslash p \\
&\text{iff} \quad \gamma(\{x\}) = \gamma(\{y\}).
\end{aligned}$$

6.4. **Combinations with completions.** Given a commutative residuated chain $\mathbf{A}$ with a gap $g < h$, the approach with residuated frames produces a complete residuated chain, the one with polynomials produces a residuated chain (which may not be complete) and the approach with languages yields simply a tomonoid, and the three results sit one inside the other. However, the last two sit as appropriate substructures of the first, and to be precise the first complete residuated chain is the Dedekind-MacNeille completion of each of the other two. In that sense the approach by residuated frames subsumes the other two, as one can then take the residuated sublattice or subtomonoid of the complete residuated chain and obtain the desired result.

The approaches via residuated frames and polynomials suppose that the original algebra $\mathbf{A}$ is a commutative residuated chain. Thus it is not clear whether we can use it in order to obtain a one-step densification of a commutative tomonoid. Nevertheless, every commutative tomonoid can be embedded into a commutative residuated chain using a nuclear completion. Namely, one can take the nuclear retraction of $\mathcal{P}(\mathbf{A})$ generated by downsets $\{a \Rightarrow b \subseteq A \mid a, b \in A\}$ (cf. [2, 10]).

Thus any of the three approaches can be applied, in combination with completions at appropriate places, to subsume the others. All these are of course not clear in the beginning as the tools and methods come from different disciplines and this is why we felt compelled to describe in detail the three approaches and their connections hoping to initiate a partial unification of the three subjects.

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