On the Failure of Standard Completeness in ΠMTL for Infinite Theories

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Abstract

It is well-known that Hájek's basic fuzzy logic (BL), Łukasiewicz logic, and product logic are not strongly standard complete. On the other hand Esteva and Godo's monoidal t-norm based logic (MTL) and its involutive extension IMTL are strongly standard complete. In this paper we show that ΠMTL (an extension of MTL by the axioms characteristic of product logic) does not enjoy the strong standard completeness theorem like BL, Łukasiewicz, and product logic.

Key words: strong standard completeness, monoidal t-norm based logic (MTL), basic fuzzy logic (BL), product logic, Łukasiewicz logic, ΠMTL, IMTL

1 Introduction

It is well-known that Hájek's basic fuzzy logic (BL) is not strongly standard complete, i.e. there is a theory T and a formula φ such that in each standard algebra $\mathbf{L}$ we have $e(\varphi) = 1$ for any $\mathbf{L}$-model e of T but $T \not\vdash \varphi$. The same result holds also for Łukasiewicz logic and product logic which are schematic extensions of BL (see [5]). On the other hand, the monoidal t-norm based logic (MTL), which arises from BL by omitting the divisibility axiom, enjoys the strong standard completeness theorem. Moreover, the same is valid also for involutive monoidal t-norm based logic (IMTL) which is an extension of MTL by the law of involution (i.e. the axiom schema characteristic of Łukasiewicz

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logic). Thus it is natural to ask whether ΠMTL (an extension of MTL obtained by adding the axiom schemata characteristic of product logic) satisfies this theorem as well. In this paper we are going to show that although ΠMTL belongs to the group of fuzzy logics without divisibility, it is not strongly standard complete like BL, Łukasiewicz, and product logic.

2 Preliminaries

This section recalls the basic definitions and results on MTL and its schematic extensions which we will need in the sequel. The language of MTL consists of a countable set of propositional variables, a strong conjunction &, a minimum conjunction $\wedge$, an implication $\Rightarrow$, and a truth constant $\overline{0}$. Derived connectives are defined as follows:

$$
\begin{aligned}
\varphi \vee \psi \quad & \text{is} \quad ((\varphi \Rightarrow \psi) \Rightarrow \psi) \wedge ((\psi \Rightarrow \varphi) \Rightarrow \varphi)\,, \\
\neg\varphi \quad & \text{is} \quad \varphi \Rightarrow \bar{0}\,, \\
\varphi \equiv \psi \quad & \text{is} \quad (\varphi \Rightarrow \psi) \& (\psi \Rightarrow \varphi)\,, \\
\bar{1} \quad & \text{is} \quad \neg\bar{0}\,.
\end{aligned}
$$

In [2], the authors introduced a Hilbert style calculus for MTL with an axiomatization similar to BL. They introduced new axioms for the minimum conjunction $\wedge$ and changed the divisibility axiom to a weaker form (A6). The following are the axioms of MTL:

$$
\begin{aligned}
&\text{(A1)} \quad (\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))\,, \\
&\text{(A2)} \quad \varphi \& \psi \Rightarrow \varphi\,, \\
&\text{(A3)} \quad \varphi \& \psi \Rightarrow \psi \& \varphi\,, \\
&\text{(A4)} \quad (\varphi \wedge \psi) \Rightarrow \varphi\,, \\
&\text{(A5)} \quad (\varphi \wedge \psi) \Rightarrow (\psi \wedge \varphi)\,, \\
&\text{(A6)} \quad (\varphi \& (\varphi \Rightarrow \psi)) \Rightarrow (\varphi \wedge \psi)\,, \\
&\text{(A7a)} \quad (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\varphi \& \psi \Rightarrow \chi)\,, \\
&\text{(A7b)} \quad (\varphi \& \psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))\,, \\
&\text{(A8)} \quad ((\varphi \Rightarrow \psi) \Rightarrow \chi) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \chi) \Rightarrow \chi)\,, \\
&\text{(A9)} \quad \bar{0} \Rightarrow \varphi\,.
\end{aligned}
$$

The deduction rule of MTL is modus ponens. The notion of a proof is defined in the usual way. Let T be a theory over MTL and φ be a formula. Then we write $T \vdash \varphi$ if φ is provable in the system MTL.

MTL has several important axiomatic extensions. Some of them were mentioned in the introduction. BL is an extension of MTL by the following axiom schema:

(Div) $\quad (\varphi \wedge \psi) \Rightarrow (\varphi \& (\varphi \Rightarrow \psi))\,.$

Łukasiewicz logic can be obtained by adding the law of involution to BL, i.e.

(Inv) $\quad \neg\neg\varphi \Rightarrow \varphi\,.$

Finally, product logic is an extension of BL by the following schemata:

(Π1) $\quad \varphi \wedge \neg\varphi \Rightarrow \overline{0}\,,$

(Π2) $\quad \neg\neg\chi \Rightarrow ((\varphi\&\chi \Rightarrow \psi\&\chi) \Rightarrow (\varphi \Rightarrow \psi))\,.$

It is quite natural to ask what happens if we add (Inv) respectively (Π1) and (Π2) to MTL. It was shown in [2] that the extension of MTL by (Inv), so-called IMTL, is a strictly weaker logic than Łukasiewicz logic. The analogous result was proved in [6] also for the extension of MTL by (Π1) and (Π2), so-called ΠMTL.

The semantical part of MTL is based on the notion of an MTL-algebra which is a special kind of a residuated lattice. A commutative residuated lattice $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is an algebraic structure, where $(L, *, \mathbf{1})$ is a commutative monoid, $(L, \wedge, \vee)$ is a lattice, and $(*, \rightarrow)$ form a residuated pair, i.e. $x * y \leq z$ iff $x \leq y \rightarrow z$. The operation $\rightarrow$ is called a residuum. If follows from the definition that $\rightarrow$ is antitone in the first argument and monotone in the second one. When we refer to a commutative residuated lattice, we will omit the word commutative since we will deal here only with the commutative case. The symbol a^n stands for $a * \cdots * a$ (n times). In the absence of parenthesis, $*$ is performed first, followed by $\rightarrow$, and finally $\vee$ and $\wedge$.

A residuated lattice $\mathbf{L}$ is said to be integral if $\mathbf{1}$ is the top element of L. In this case we have that $x \leq y$ iff $x \rightarrow y = \mathbf{1}$. If a residuated lattice possesses a bottom element $\mathbf{0}$, then we have $\mathbf{0} * x = \mathbf{0}$.

Definition 1 *An* MTL-*algebra is a structure* $(L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ *where the following conditions are satisfied:*

(1) $(L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ *is an integral residuated lattice,*
(2) $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ *is a bounded lattice,*

(3) $(x \to y) \vee (y \to x) = \mathbf{1}$ *for all* $x, y \in L$.

A totally ordered MTL-*algebra is called an* MTL-*chain.*

Throughout the text we will use without mentioning also the alternative signature of an MTL-algebra where the lattice operations $\wedge$ and $\vee$ are substituted by the corresponding order $\leq$.

Let $\mathcal{L}$ be an extension of MTL by given axiom schemata. The algebras of truth values for $\mathcal{L}$ (so-called $\mathcal{L}$-algebras) are defined as MTL-algebras satisfying the indentities corresponding to the given axiom schemata. For the above-mentioned extensions of MTL the corresponding algebras of truth values are called BL-algebras, MV-algebras, product algebras, IMTL-algebras, and ΠMTL-algebras respectively. Since this paper deals mainly with ΠMTL-algebras, we recall the precise definition of a ΠMTL-algebra.

Definition 2 *A* ΠMTL-*algebra* $\mathbf{L} = (L, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1})$ *is an* MTL-*algebra satisfying the following identities:*

(1) $\neg\neg z \to ((x * z \to y * z) \to (x \to y)) = \mathbf{1}\,,$
(2) $x \wedge \neg x = \mathbf{0}\,,$

where $\neg x = x \to \mathbf{0}$. *A totally ordered* ΠMTL-*algebra is called a* ΠMTL-*chain.*

An MTL-chain (ΠMTL-chain resp.) whose lattice reduct is the real interval $[0, 1]$ with the usual order is referred to as a standard MTL-chain (ΠMTL-chain resp.).

The notion of an evaluation of formulas is defined in the usual way. Given a theory T over MTL and an MTL-algebra $\mathbf{L}$, we say that an evaluation e is an $\mathbf{L}$-model of T if $e(\psi) = \mathbf{1}$ for all $\psi \in T$. Let φ be a formula. Then we write $T \models_{\mathbf{L}} \varphi$ if $e(\varphi) = \mathbf{1}$ for each $\mathbf{L}$-model e of T. In [2] Esteva and Godo proved that any axiomatic extension $\mathcal{L}$ of MTL is complete w.r.t. the class of all $\mathcal{L}$-chains.

Theorem 3 (Strong completeness) *Let* $\mathcal{L}$ *be an axiomatic extension of* MTL, *T be a theory over $\mathcal{L}$, and φ be a formula. Then $T \vdash \varphi$ iff $T \models_{\mathbf{L}} \varphi$ for each $\mathcal{L}$-chain $\mathbf{L}$.*

The latter theorem shows that any schematic extension of MTL enjoys strong completeness w.r.t. the general algebraic semantics. Nevertheless, the intended set of truth values for a fuzzy logic is the real unit interval $[0, 1]$. Thus fuzzy logicians usually ask whether a given fuzzy logic is complete w.r.t. the class of algebras whose lattice reduct is $[0, 1]$ (so-called standard completeness). It took quite long time until this kind of completeness was proved for the basic schematic extensions of MTL. The next theorem summarizes these results for the logics mentioned above (for details see [10,3,7,8,1,5]).

Theorem 4 (Standard completeness) *Let $\mathcal{L}$ be one of the following logics:* MTL*,* IMTL*,* ΠMTL*,* BL*, Łukasiewicz logic, product logic. Further, let φ be a formula over $\mathcal{L}$. Then $\vdash \varphi$ iff $\models_{\mathbf{L}} \varphi$ for each standard $\mathcal{L}$-chain $\mathbf{L}$.*

Notice that the previous theorem gives us the standard completeness only for empty theory. It is quite natural to ask whether this result can be generalized also for arbitrary theories. The answer is only partially positive. The following theorem presents results from [10,3].

Theorem 5 (Strong standard completeness) *Let $\mathcal{L}$ be either* MTL *or* IMTL*, T be a theory over $\mathcal{L}$, and φ be a formula. Then $T \vdash \varphi$ iff $T \models_{\mathbf{L}} \varphi$ for each standard $\mathcal{L}$-chain $\mathbf{L}$.*

In the case of BL, Łukasiewicz logic, and product logic we have to add a finiteness assumption for the theory unlike MTL and IMTL. The necessity of the finiteness assumption for Łukasiewicz and product logic was proved in [5]. For BL the result is well known but seems to be proved nowhere. It can be proved by reductio ad absurdum by showing that the strong standard completeness of BL implies the strong standard completeness for Łukasiewicz logic.

Theorem 6 *Let $\mathcal{L}$ be one of the following logics:* BL*, Łukasiewicz logic, product logic. Further, let T be a finite theory over $\mathcal{L}$, and φ be a formula. Then $T \vdash \varphi$ iff $T \models_{\mathbf{L}} \varphi$ for each standard $\mathcal{L}$-chain $\mathbf{L}$.*

We proved in [7] that the latter theorem holds also for ΠMTL but it has not been known so far whether it can be generalized for infinite theories. In this paper we are going to show that the finiteness assumption cannot be dropped in the following theorem.

Theorem 7 *Let T be a finite theory over* ΠMTL *and φ be a formula. Then $T \vdash \varphi$ iff $T \models_{\mathbf{L}} \varphi$ for each standard* ΠMTL*-chain $\mathbf{L}$.*

3 Filters and congruences

Before we prove the main result, we will recall some useful facts about MTL-algebras and ΠMTL-algebras. It is well-known that the congruence lattice $\mathrm{Con}(\mathbf{L})$ of an MTL-algebra $\mathbf{L}$ is isomorphic to the collection of so-called filters.

Definition 8 *Let $\mathbf{L} = (L, *, \rightarrow, \leq, \mathbf{0}, \mathbf{1})$ be an* MTL*-algebra. A filter F in $\mathbf{L}$ is a subset of L satisfying:*

(1) $\mathbf{1} \in F$,
(2) *if* $x, y \in F$, *then* $x * y \in F$,
(3) *if* $x \in F$ *and* $x \leq y$, *then* $y \in F$.

Let us denote the collection of all filters of an MTL-algebra $\mathbf{L}$ by $\mathcal{F}(\mathbf{L})$. Let $F \in \mathcal{F}(\mathbf{L})$ and $\theta \in \mathrm{Con}(\mathbf{L})$. The isomorphism between $\mathrm{Con}(\mathbf{L})$ and $\mathcal{F}(\mathbf{L})$ is described as follows.

$$F \mapsto \theta_F = \{\langle a, b\rangle \mid a \to b \in F \text{ and } b \to a \in F\} \text{ and } \theta \mapsto [\mathbf{1}]_\theta .$$

We will denote the equivalence class containing an element $x \in L$ with respect to a filter F by $[x]_F = \{a \in L \mid a\, \theta_F\, x\} = \{a \in L \mid a \to x \in F \text{ and } x \to a \in F\}$. Clearly, $[\mathbf{1}]_F = F$. It is not difficult to see that the equivalence classes are convex. Indeed, let $x < y < z$ such that $z \in [x]_F$. Then $z \to y \geq z \to x \in F$. Thus $z \to y \in F$. Consequently $y \in [x]_F$ since $y \to z = \mathbf{1}$.

A useful notion for MTL-chains is also the notion of an Archimedean class (see [4]).

Definition 9 *Let* $\mathbf{L}$ *be an* MTL-*chain,* $a, b \in L$, *and* $\sim$ *be an equivalence on* L *defined as follows:*

$$a \sim b \textit{ iff there exists } n \in \mathbb{N} \textit{ such that } a^n \leq b \leq a \textit{ or } b^n \leq a \leq b.$$

Then for any $a \in L$ *the equivalence class* $[a]_\sim$ *is called an Archimedean class.*

In [6] Hájek proved that ΠMTL-chains are exactly those MTL-chains which are cancellative.

Lemma 10 *An* MTL-*chain* $\mathbf{L}$ *is a* ΠMTL-*chain if and only if for any* $x, y, z \in L$, $z \neq \mathbf{0}$, *we have* $x * z = y * z$ *implies* $x = y$.

Let $\mathbf{L}$ be a ΠMTL-chain. Observe that by Lemma 10 we obtain for $a, b, c \in L$, $c \neq \mathbf{0}$, that $a < b$ implies $a * c < b * c$, in particular $a^2 < a$ for $0 < a < 1$ and $a * b < a$ for $b < \mathbf{1}$. Moreover, we get $a * c \to b * c = a \to b$ for $c \neq \mathbf{0}$. Indeed, as $a * (a \to b) \leq b$ in any residuated lattice, we obtain $a * c * (a \to b) \leq b * c$. Hence $a \to b \leq a * c \to b * c$. Conversely, since $a * c * (a * c \to b * c) \leq b * c$, we get $a * (a * c \to b * c) \leq b$ by Lemma 10. Thus $a * c \to b * c \leq a \to b$. Finally, notice also that $[\mathbf{0}]_F = \{\mathbf{0}\}$ for all $F \in \mathcal{F}(\mathbf{L}) - L$ since $a \to \mathbf{0} = \mathbf{0}$ for any $a > \mathbf{0}$.

The next easy result characterizes principal filters, i.e. filters generated by a single element. A principal filter F generated by b is denoted by $F(b)$.

Lemma 11 *Let* $\mathbf{L}$ *be an* MTL-*algebra and* $b \in L$. *Then the principal filter* *generated by b is of the form:*

$$F(b) = \{z \in L \mid (\exists n \in \mathbb{N})(b^n \leq z)\}\,.$$

4 Main result

In this section we are going to prove that ΠMTL is not strong standard complete, i.e. we have to show that there is a theory T and a formula φ such that $T \not\vdash \varphi$ but $T \models_{\mathbf{L}} \varphi$ for each standard ΠMTL-chain $\mathbf{L}$. From Theorem 7 it is clear that the theory T must be infinite.

We start with several auxiliary propositions. The first of them shows that in any complete ΠMTL-chain all the equivalence classes w.r.t. any filter possess maxima. In particular, it applies to each standard ΠMTL-chain since $[0, 1]$ is a complete lattice.

Lemma 12 *Let* $\mathbf{L} = (L, *, \rightarrow, \leq, \mathbf{0}, \mathbf{1})$ *be a complete* ΠMTL*-chain,* F *be a filter, and* $y \in L$*. Then* $[y]_F$ *has a maximum.*

PROOF. Assume that $F \neq \{\mathbf{1}\}$ otherwise $[y]_F = \{y\}$. Since L is a complete lattice, the supremum of $[y]_F$ exists. Let $z = \sup [y]_F$. Suppose that $z \notin [y]_F$. Let $p \in F - \{\mathbf{1}\}$ (there must be such p because F is not trivial). Then by cancellativity $p * z < z$. Moreover $p * z \in [z]_F$ since $z \rightarrow p * z = p \in F$. As the equivalence classes are disjoint and convex, $p * z$ must be an upper bound of $[y]_F$. Thus z cannot be the supremum of $[y]_F$ (a contradiction). □

The second lemma shows that if an equivalence class w.r.t. a principal filter contains a maximum, then this maximum can be somehow described.

Lemma 13 *Let* $\mathbf{L} = (L, *, \rightarrow, \leq, \mathbf{0}, \mathbf{1})$ *be an* MTL*-chain and* $x, y \in L$*. If* $\max [y]_{F(x)}$ *exists then there is* $m \in \mathbb{N}$ *such that* $x^m \rightarrow y = \max [y]_{F(x)}$*.*

PROOF. Let us denote the maximum of $[y]_{F(x)}$ by z. Since $z \rightarrow y \in F(x)$, there is $m \in \mathbb{N}$ such that $x^m \leq z \rightarrow y$ by Lemma 11. Since $*$ is commutative we get $z \leq x^m \rightarrow y$ by residuation.

Further by residuation we have $x^m \leq (x^m \rightarrow y) \rightarrow y$. Hence we get that $(x^m \rightarrow y) \rightarrow y \in F(x)$ because $x^m \in F(x)$. Since $x^m \rightarrow y \geq y$, we have $y \rightarrow (x^m \rightarrow y) = \mathbf{1} \in F(x)$. Thus $(x^m \rightarrow y) \in [y]_{F(x)}$. Consequently $z = x^m \rightarrow y$ because z is the maximum of $[y]_F$. □

Finally, the last lemma characterizes the maxima of equivalence classes by means of the residuum.

Lemma 14 *Let* $\mathbf{L} = (L, *, \to, \leq, \mathbf{0}, \mathbf{1})$ *be an* MTL*-chain. Then we have for any* $x, y \in L$*:*

(1) if $x \to y = y$ *then* $y = \max\, [y]_{F(x)}$*,*
(2) if F *is a filter in* $\mathbf{L}$ *and* $y = \max\, [y]_F$ *then* $x \to y = y$ *for all* $x \in F$*.*

PROOF.

(1) Let $F(x)$ be the principal filter generated by the element x. Assume that $x \to y = y$. Let $z \in [y]_{F(x)}$. We will show that $z \leq y$, i.e. $y = \max\, [y]_{F(x)}$. Since $z \to y \in F(x)$, there exists $n \in \mathbb{N}$ such that $x^n \leq z \to y$ by Lemma 11. By residuation we get $z \leq x^n \to y$. Since we assume that $x \to y = y$, we get after a final number of steps $x^n \to y = x^{n-1} \to (x \to y) = x^{n-1} \to y = \cdots = y$. Consequently $z \leq y$. Hence $y = \max\, [y]_{F(x)}$.
(2) Let $x \in F$. Since $x \leq (x \to y) \to y$ and $y \to (x \to y) = \mathbf{1}$, we have $(x \to y) \in [y]_F$. As y is the maximum of $[y]_F$ and $x \to y \geq y$, we get $x \to y = y$. $\square$

Now we are ready to define the desired infinite theory and the formula mentioned at the beginning of this section. Let p, q, r be propositional variables and

$$T' = \{\neg\neg r, p \Rightarrow q, \neg p \Rightarrow q\} \cup \{(p^n \Rightarrow r) \Rightarrow q \mid n \in \mathbb{N}\}\,.$$

Further, let M be the set of all formulas in the restricted language whose set of propositional variables equals $\{p, r\}$ and

$$T'' = \{\varphi \& (\varphi \Rightarrow \psi) \equiv \varphi \wedge \psi \mid \varphi, \psi \in M\}\,.$$

This means that T'' consists of all instances of divisibility axiom (Div) for formulas built from propositional variables p and r. Finally, let $T = T' \cup T''$.

In the rest of this section we will prove the following claim from which follows that T is the theory and q is the formula showing that ΠMTL is not strong standard complete.

Claim 15 *For each standard* ΠMTL*-chain* $\mathbf{L}$ *we have* $T \models_{\mathbf{L}} q$*, i.e. for each each standard* ΠMTL*-chain* $\mathbf{L}$ *and each* $\mathbf{L}$*-model* e *of* T *one has* $e(q) = 1$*. On the other hand,* $T \not\vdash q$*.*

We will prove the first part of the claim. Let $\mathbf{L}$ be any standard ΠMTL-chain and e any $\mathbf{L}$-model of T. Clearly, if $e(p) = 1$ or $e(p) = 0$ then $p \Rightarrow q$ or $\neg p \Rightarrow q$ ensures $e(q) = 1$. Thus suppose that $e(p) \neq 0, 1$. Further, $e(r) \neq 0$ otherwise we would have $e(\neg\neg r) = 0$ (i.e. e would not be an $\mathbf{L}$-model of T). Finally, if $p^n \leq r$ for some $n \in \mathbb{N}$ then $e(p^n \Rightarrow r) = 1$ and $(p^n \Rightarrow r) \Rightarrow q$ ensures that $e(q) = 1$. Hence it is sufficient to prove that there is always $n \in \mathbb{N}$ such that $p^n \leq r$. In other words, we want to show that $e(p)$ and $e(r)$ belong to the same Archimedean class provided that $e(p) > e(r)$.

Suppose not, i.e. $e(p^n) > e(r)$ for all $n \in \mathbb{N}$. Let $x = e(p)$ and $y = e(r)$. By Lemma 12 a maximum of $[y]_{F(x)}$ exists and by Lemma 13 there is $m \in \mathbb{N}$ such that $x^m \to y = \max [y]_{F(x)}$. Hence we have $x \to (x^m \to y) = x^m \to y$ by Lemma 14(2).

We have $x * (x^m \to y) = x * (x \to (x^m \to y)) = \min\{x, x^m \to y\}$. The last equality follows from the divisibility axiom (Div) for formulas in p and r. Now, since $x \to (x^m \to y) = x^m \to y < 1$, we must have $\min\{x, x^m \to y\} = x^m \to y$. On the other hand, $x < 1$ and $x^m \to y > 0$ imply $x * (x^m \to y) < (x^m \to y) = \min\{x, x^m \to y\}$, and a contradiction is reached. Thus $e(q) = 1$ for each $\mathbf{L}$-model e of T and each standard ΠMTL-chain $\mathbf{L}$.

Finally we will show that $T \not\vdash q$. Since each proof is finite, it can use only finitely many formulas from T. It follows from Theorem 3 that it is sufficient to prove that for any finite sub-theory $T_{fin} \subseteq T$ there is an $\mathbf{L}$-model e of T_{fin} such that $e(q) < 1$ for some ΠMTL-algebra $\mathbf{L}$.

Let $\mathbf{L}$ be the standard product algebra $[0,1]_\Pi$ and $m \in \mathbb{N}$ be the maximal natural number such that $(p^m \Rightarrow r) \Rightarrow q \in T_{fin}$. We evaluate the propositional variables p, r, q as follows:

$$e(p) = e(q) = \frac{1}{2}, \qquad e(r) = \frac{1}{2^{m+1}}.$$

Then we have $e(\neg\neg r) = e(p \Rightarrow q) = e(\neg p \Rightarrow q) = 1$. Further, for all $n \leq m$ we have

$$e(p^n \Rightarrow r) = \frac{1}{2^n} \to \frac{1}{2^{m+1}} = \frac{1}{2^{m+1-n}} \leq \frac{1}{2} = e(q).$$

Thus $e((p^n \Rightarrow r) \Rightarrow q) = 1$ for all possible n which may appear in T_{fin}. This means that e is a $[0,1]_\Pi$-model of T_{fin}. Since $e(q) = \frac{1}{2} < 1$, we get $T \not\vdash q$ and the next theorem follows.

Theorem 16 ΠMTL *does not enjoy the strong standard completeness theorem.*

Even since we have found the countermodel of T_{fin} for q in $[0,1]_\Pi$, we get the following corollary.

Corollary 17 *Any schematic extension between* ΠMTL *and product logic cannot be strongly standard complete.*

The latter corollary is not empty statement since we have shown in [9] that there are infinitely many schematic extensions between ΠMTL and product logic.

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