

Solution of a system of linear equations with fuzzy numbers

Rostislav Horčík¹

*Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic*

Abstract

In this paper, the interval nature of fuzzy numbers is revealed by showing that many interesting results from classical interval analysis transfer also into the fuzzy case. The paper deals with a solution of a fuzzy interval system of linear equations, i.e. a system in which fuzzy intervals (numbers) appear instead of crisp numbers.

Key words: fuzzy number, fuzzy interval, interval analysis, fuzzy arithmetic, Fuzzy Class Theory, united solution set
1991 MSC: 03B52, 65G40

1 Introduction

Equations involving fuzzy numbers and their solutions are investigated already for quite a long time. Although fuzzy numbers are called “numbers” they are rather fuzzified versions of classical crisp *intervals*. This is quite an obvious observation which was already made by several authors (for details on the connection between fuzzy numbers and classical intervals see [13,15]). Consequently, we should expect that the theory of fuzzy numbers and their arithmetic should be a fuzzified version of interval analysis, i.e. we should look for an inspiration among results of interval analysis and try to prove them in fuzzy setting.

Email address: horcik@cs.cas.cz (Rostislav Horčík).

¹ The author was partly supported by the grant No. B100300502 of the Grant Agency of the Academy of Sciences of the Czech Republic and partly by the Institutional Research Plan AV0Z10300504.

In this paper we are going to follow this line and prove several results for systems of linear equations involving fuzzy numbers. Before we introduce the topic of this paper in more details, let us discuss what are the differences between this paper and similar papers on fuzzy numbers which appeared in the literature. The first difference consists in the fuzzy logic which is used for deriving results. Most of the papers (see e.g. [4–7,12,19,21,22]) define the fuzzy arithmetic in such a way that it preserves cuts. Equivalently, it can be said that such papers use Gödel logic in order to derive their results. Recall that Gödel logic is the logic where conjunction is interpreted by the minimum (see [10]). Unlike such papers we are going to prove more general results since we work in Monoidal T-norm Based Logic (MTL) which is the logic of left-continuous t-norms and their residua (see [9]), i.e. we do not restrict the interpretation of our conjunction to be only the minimum. However, there are also a few papers in the literature which use instead of Gödel logic systems where basic connectives are conjunction, disjunction, and negation interpreted respectively by a t-norm, t-conorm, and involution (see [8,11]).

The second difference consists in the shape and the solution of equations under consideration. Many papers dealing with “fuzzy” equations focus on an equation of the form $A * X = B$ where X is an unknown fuzzy number and $*$ is the extension of an operation on reals, defined by means of Zadeh’s extension principle (see [4,8,19,22]). Usually, $*$ is one of the following operations on fuzzy numbers: addition, subtraction, multiplication, and division. There are also papers considering more general equations, e.g. quadratic equations are considered in [5,6] or equations of the form $f(X, A_1, \dots, A_n) = C$ are investigated in [21]. The solution in all above-mentioned cases is defined as a fuzzy number which satisfies the given equation if substituted for X .

Unlike the papers mentioned above, we consider in this paper a system of linear equations whose parameters are uncertain:

$$\begin{aligned} A_{11}x_1 + \dots + A_{1n}x_n &= B_1, \\ &\vdots \\ A_{m1}x_1 + \dots + A_{mn}x_n &= B_m, \end{aligned}$$

for A_{ij} , B_i fuzzy numbers, and x_j real numbers. The solution set of such a system is defined as is common in interval analysis (see e.g. [17,20]). Precisely, it is the fuzzy subset of \mathbb{R}^n such that the membership-degree of a tuple (x_1, \dots, x_n) is determined by the truth degree of the following first-order formula when interpreted in fuzzy logic:

$$\begin{aligned} (\exists a_{11} \in A_{11}) \cdots (\exists a_{mn} \in A_{mn}) (\exists b_1 \in B_1) (\exists b_m \in B_m) \\ (a_{11}x_1 + \dots + a_{1n}x_n = b_1 \ \& \ \cdots \ \& \ a_{m1}x_1 + \dots + a_{mn}x_n = b_m). \end{aligned}$$

Such a solution set is usually called united solution set in interval analysis and was considered already in [16]. In fuzzy setting this definition appeared in [7] where however only several easy observations were made and no description of the solution set offered. We consider here also other types of solution sets where some of the quantifiers above can be replaced by universal quantifier. This is usual in interval analysis (see e.g. [20]) but it appeared also in fuzzy literature. In [11] the authors used this approach in order to define a fuzzy linear program. A usage of universal quantifier was considered also in [8] containing an interesting claim indicating that the solution of the above-mentioned equation $A \cdot X = B$, where \cdot is the multiplication of fuzzy numbers, can be described as $\{x \mid (\forall a \in A)(\exists b \in B)(ax = b)\}$ which can be seen as a special case of our definition of the solution set (see Section 4).

At the end of the introduction we will present a motivational example explaining informally what is the topic of this paper. Consider a plane containing n points, say $(a_1, b_1), \dots, (a_n, b_n) \in \mathbb{R}^2$. We can ask whether there is a line p which goes through all the points. This question leads to a system of linear equations

$$\begin{aligned} a_1 k + q &= b_1, \\ &\vdots \\ a_n k + q &= b_n, \end{aligned}$$

where k is the slope of p and q is its offset (i.e. p can be described by equation $y = kx + q$). Thus the line exists iff the above system of linear equations has a solution.

Now we will modify slightly the task. We replace the coordinates a_i, b_i of the points by crisp intervals $[a_i^\uparrow, a_i^\downarrow], [b_i^\uparrow, b_i^\downarrow]$ respectively. Then an analogous question could be whether there is a line p which “goes through” all the cartesian products $[a_i^\uparrow, a_i^\downarrow] \times [b_i^\uparrow, b_i^\downarrow]$. Formally, this means that we ask if p has a non-empty intersection with each cartesian product $[a_i^\uparrow, a_i^\downarrow] \times [b_i^\uparrow, b_i^\downarrow]$. This question leads to a typical task from classical interval analysis (for details see [20]). The answer is affirmative iff the following interval system of linear equations has a solution:

$$\begin{aligned} [a_1^\uparrow, a_1^\downarrow]k + q &= [b_1^\uparrow, b_1^\downarrow], \\ &\vdots \\ [a_n^\uparrow, a_n^\downarrow]k + q &= [b_n^\uparrow, b_n^\downarrow]. \end{aligned}$$

A solution of such an interval system is usually defined as follows: we say that a pair (k, q) describing the line p is a solution iff the following classical

first-order formula holds:

$$\bigwedge_{i=1}^n \left(\exists (\alpha_i, \beta_i) \in [a_i^\uparrow, a_i^\downarrow] \times [b_i^\uparrow, b_i^\downarrow] \right) (\alpha_i k + q = \beta_i),$$

where \wedge is the classical conjunction. Observe that this formula is true if in each box $[a_i^\uparrow, a_i^\downarrow] \times [b_i^\uparrow, b_i^\downarrow]$ there exists a point which lies on p . In order to solve this problem, classical interval analysis gives us methods how to find the set of all solutions.

The purpose of this paper is to discuss what happens if we change the crisp intervals to fuzzy intervals or fuzzy numbers. Let A_1, \dots, A_n and B_1, \dots, B_n be fuzzy intervals (numbers). Analogously to the previous task we would like to find a line p which goes through all fuzzy points $A_i \times B_i$. Clearly, as the sets $A_i \times B_i$ are not crisp, some of the lines satisfy this condition better than the others. The corresponding system of equations which we have to solve is the following one:

$$\begin{aligned} A_1 k + q &= B_1, \\ &\vdots \\ A_n k + q &= B_n. \end{aligned}$$

A solution to this system of equations can be defined in the same way as in the case of classical interval analysis, only interpreted in fuzzy logic. We say that a pair (k, q) describing a line p is a solution to such an extent to which the following first-order formula holds:

$$\bigotimes_{i=1}^n (\exists (\alpha_i, \beta_i) \in A_i \times B_i) (\alpha_i k + q = \beta_i),$$

where \otimes is the strong conjunction in our fuzzy logic and \exists is interpreted by supremum as usual in first-order fuzzy logic. Then the corresponding solution set is a fuzzy set of pairs (k, q) where a pair (k, q) belongs to this set to a degree to which there are points in each fuzzy point $A_i \times B_i$ lying on the corresponding line.

Such a definition has also a reasonable interpretation. The truth degree to which a crisp point (x, y) belongs to a fuzzy point $A_i \times B_i$ can be understood as a penalty which we have to pay if the line in demand goes through this point. The greatest truth degree 1 represents no penalty and the lowest truth degree 0 the unacceptable penalty (i.e. a line going through this point cannot be a solution by no means). Then the truth degree to which a pair (k, q) describing a line p belongs to the solution set can be interpreted as follows: in each fuzzy point $A_i \times B_i$ we find the “best” point lying on p (i.e. the point with the greatest truth degree), the truth degree of this point tells us

how “good” this point is, and then we compute the conjunction of the truth degrees of all these points (i.e. we have to sum all the penalties we receive in each fuzzy point $A_i \times B_i$). The way the penalties are summed together depends on the chosen fuzzy logic. For instance, in the standard semantics of Łukasiewicz logic, where the conjunction is interpreted by the Łukasiewicz t-norm, the penalties are summed by the usual addition and truncated at a maximum penalty.

Let A_{11}, \dots, A_{mn} and B_1, \dots, B_n be fuzzy intervals (numbers). In this paper we are going to discuss how to solve the following systems of linear equations:

$$\begin{aligned} A_{11}x_1 + \dots + A_{1n}x_n &= B_1, \\ &\vdots \\ A_{m1}x_1 + \dots + A_{mn}x_n &= B_m. \end{aligned}$$

Then the above mentioned system of linear equation from the motivational example is in fact a special case of this general one. We can also understand such systems of equations as systems where parameters are uncertain, only known to belong to some given fuzzy intervals (numbers). This interpretation is further explained in Section 4. We are in fact going to generalize some results from [20] to the fuzzy case. In particular, we will show that the fundamental theorem [20, Theorem 3.4] is valid also in the fuzzy case. Then we derive a general description of the solution set for a system of linear equations with fuzzy intervals (numbers). Finally, we use this description and compute the exact shape of the solution set in the case when all the fuzzy intervals appearing in the system are trapezoidal.

The paper is organized as follows: we recall the definition of Fuzzy Class Theory which will be our main formal tool for dealing with fuzzy intervals (numbers) and fuzzy arithmetic in Section 2. Section 3 contains some technical results about fuzzy intervals. In Section 4, we define formally the most general form of the solution set considered in this paper. Then we will prove the fundamental theorem characterizing the solutions by means of fuzzy arithmetic. In Section 6, we describe the solution set when all parameters are quantified by existential quantifiers (so-called united solution set). Finally, we present how to use our results and solve a system of linear equation with fuzzy intervals of trapezoidal shape in the logic of Łukasiewicz t-norm.

2 Preliminaries

2.1 Fuzzy class theory

All the results in this paper will be derived formally in fuzzy logic according to the methodological manifesto [2]. Most of our results in this paper can be proved in any fuzzy logic that is at least as strong and expressive as $\text{MTL}\Delta$. Only at the end of the paper we restrict ourselves to a particular fuzzy logic. Let \mathcal{F} be any fuzzy propositional logic expanding $\text{MTL}\Delta$. Thus \mathcal{F} can be for instance Hájek's BL, Łukasiewicz logic, Gödel logic, or product logic (of course all of them extended by Baaz's delta operator Δ). For details on MTL see [9]. Details on fuzzy logics stronger than BL can be found in [10].

For dealing with fuzzy intervals we will use Fuzzy Class Theory (FCT) built over the logic \mathcal{F} . Originally this theory introduced in [1] was built over the logic LII. However, the definitions and basic results from [1] work in any logic extending $\text{MTL}\Delta$ (see [3]). For convenience, we reproduce basic definitions of Fuzzy Class Theory. Recall that from the point of view of formal logic, it can be characterized as Henkin-style higher-order fuzzy logic.

Definition 1 (Henkin-style second-order fuzzy logic) *Let \mathcal{F} be a logic which extends $\text{MTL}\Delta$. The Henkin-style second-order fuzzy logic over \mathcal{F} is a theory in multi-sorted first-order logic $\mathcal{F}\forall$ with sorts for atomic objects (lowercase variables) and classes (uppercase variables). Both of the sorts subsume subsorts for n -tuples, for all $n \geq 1$. Tuples are governed by the usual axioms known from classical mathematics (e.g., that tuples equal iff their respective constituents equal).*

Besides the logical predicate of identity, the only primitive predicate is the membership predicate \in between objects and classes. The axioms for \in are the following:

- (1) *The comprehension axioms $(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi)$, φ not containing X , which enable the (eliminable) introduction of comprehension terms $\{x \mid \varphi\}$ with the axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (where φ may be allowed to contain other comprehension terms).*
- (2) *The extensionality axiom $(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$.*

Convention 2 *For better readability, let us make the following conventions:*

- *The formulae $(\forall x)(x \in X \rightarrow \varphi)$ and $(\exists x)(x \in X \& \varphi)$ and the comprehension terms $\{x \mid x \in A \& \varphi\}$ are abbreviated $(\forall x \in X)\varphi$, $(\exists x \in X)\varphi$, and $\{x \in A \mid \varphi\}$ respectively (similar notation can be used for defined binary predicates).*

- The formulae $\varphi \& \dots \& \varphi$ (n times) are abbreviated φ^n .
- Let c be an atomic object. Then $\{c\}$ denotes the crisp class containing only c to degree 1, i.e. $\{c\} = \{x \mid x = c\}$.
- A chain of implications $\varphi_1 \rightarrow \varphi_2, \varphi_2 \rightarrow \varphi_3, \dots, \varphi_{n-1} \rightarrow \varphi_n$, for $n \geq 2$, will for short be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \dots \longrightarrow \varphi_n$; analogously for the equivalence connective.

Most of the formal proofs in FCT which appear in this paper use the well-known facts how the quantifiers distribute over the connectives of \mathcal{FV} . Since we will use these facts without mentioning, we recall them all now.

Lemma 3 *Assume that ν does not contain x freely. The following are theorems of \mathcal{FV} :*

- (T \forall 1) $(\forall x)(\nu \rightarrow \varphi) \leftrightarrow (\nu \rightarrow (\forall x)\varphi)$
- (T \forall 2) $(\forall x)(\varphi \rightarrow \nu) \leftrightarrow ((\exists x)\varphi \rightarrow \nu)$
- (T \forall 3) $(\exists x)(\nu \rightarrow \varphi) \rightarrow (\nu \rightarrow (\exists x)\varphi)$
- (T \forall 4) $(\exists x)(\varphi \rightarrow \nu) \rightarrow ((\forall x)\varphi \rightarrow \nu)$
- (T \forall 5) $(\forall x)\varphi(x) \leftrightarrow (\forall y)\varphi(y)$
- (T \forall 6) $(\exists x)\varphi(x) \leftrightarrow (\exists y)\varphi(y)$
- (T \forall 7) $(\exists x)(\varphi \& \nu) \leftrightarrow ((\exists x)\varphi \& \nu)$
- (T \forall 8) $(\exists x)(\nu \wedge \varphi) \leftrightarrow (\nu \wedge (\exists x)\varphi)$
- (T \forall 9) $(\exists x)(\nu \vee \varphi) \leftrightarrow (\nu \vee (\exists x)\varphi)$
- (T \forall 10) $(\forall x)(\nu \wedge \varphi) \leftrightarrow (\nu \wedge (\forall x)\varphi)$

In our proofs we will also prove by cases. The fact that this is correct follows from the following lemma.

Lemma 4 *Let T be a theory and $T \vdash \varphi \rightarrow \chi$, $T \vdash \psi \rightarrow \chi$ in \mathcal{FV} . Then $T \vdash \varphi \vee \psi \rightarrow \chi$.*

Another useful lemma states that if we know that x is equal to a term t then we can substitute it for x in a formula $\varphi(x)$.

Lemma 5 *For an arbitrary term t substitutable for x in $\varphi(x)$ it is provable that*

$$(\forall x)(x = t \rightarrow \varphi(x)) \leftrightarrow \varphi(t), \quad (1)$$

$$(\exists x)(x = t \& \varphi(x)) \leftrightarrow \varphi(t). \quad (2)$$

PROOF. (1) Left-to-right: by specification of x to t . Right to left: from the identity axiom $\varphi(t) \rightarrow (x = t \rightarrow \varphi(x))$ by generalization on x and shifting the quantifier.

(2) Left-to-right: from the identity axiom $x = t \ \& \ \varphi(x) \rightarrow \varphi(t)$ by generalization on x and shifting the quantifier to the antecedent. Right to left: $\varphi(t)$ implies $t = t \ \& \ \varphi(t)$, which implies $(\exists x)(x = t \ \& \ \varphi(x))$.

We also recall several basic operations and relations involving fuzzy classes which we will need in the sequel.

Definition 6 (Fuzzy class operations) *The following elementary fuzzy set operations can be defined:*

\emptyset	$\equiv_{\text{df}} \{x \mid 0\}$	<i>empty class</i>
\mathbb{V}	$\equiv_{\text{df}} \{x \mid 1\}$	<i>universal class</i>
$\text{Ker}(X)$	$\equiv_{\text{df}} \{x \mid \Delta(x \in X)\}$	<i>kernel</i>
$X \cap Y$	$\equiv_{\text{df}} \{x \mid x \in X \ \& \ x \in Y\}$	<i>intersection</i>
$X \sqcup Y$	$\equiv_{\text{df}} \{x \mid x \in X \ \vee \ x \in Y\}$	<i>max-union</i>

Definition 7 (Fuzzy class relations) *Further we define the following elementary relations between fuzzy sets:*

$\text{Hgt}(X)$	$\equiv_{\text{df}} (\exists x)(x \in X)$	<i>height</i>
$\text{Norm}(X)$	$\equiv_{\text{df}} (\exists x)\Delta(x \in X)$	<i>normality</i>
$\text{Crisp}(X)$	$\equiv_{\text{df}} (\forall x)\Delta(x \in X \ \vee \ x \notin X)$	<i>crispness</i>
$X \subseteq Y$	$\equiv_{\text{df}} (\forall x)(x \in X \rightarrow x \in Y)$	<i>inclusion</i>
$X \approx Y$	$\equiv_{\text{df}} (\forall x)(x \in X \leftrightarrow x \in Y)$	<i>weak bi-inclusion</i>
$X \parallel Y$	$\equiv_{\text{df}} (\exists x)(x \in X \ \& \ x \in Y)$	<i>compatibility</i>

Lemma 8 *The following formulas are provable in \mathcal{FV} :*

- (1) $A \subseteq B \rightarrow B \approx A \sqcup B$.
- (2) $\Delta(A \subseteq B) \rightarrow B = A \sqcup B$.
- (3) $\text{Crisp}(A) \rightarrow A \cap A = A$.
- (4) $\bigcap_{i=1}^m \left(\bigcup_{j=1}^n A_{ij} \right) = \bigcup_{j_1=1}^n \cdots \bigcup_{j_m=1}^n \left(\bigcap_{i=1}^m A_{ij_i} \right)$.

PROOF. In the whole proof we use the techniques developed in [1, Theorems 33 and 35] without mentioning. The following propositional formula is provable

in \mathcal{F} : $(\varphi \rightarrow \psi) \rightarrow (\varphi \vee \psi \leftrightarrow \psi)$. Thus the first statement holds. Further, by necessitation we get $\Delta(\varphi \rightarrow \psi) \rightarrow \Delta(\varphi \vee \psi \leftrightarrow \psi)$ which proves the second statement. The third formula follows from the valid propositional formula $\Delta(\varphi \vee \neg\varphi) \rightarrow \Delta(\varphi \leftrightarrow \varphi^2)$.

For the last statement we have $\&_{i=1}^m \bigvee_{j=1}^n \varphi_{ij} \leftrightarrow \&_{i=1}^m \bigvee_{j_i=1}^n \varphi_{ij_i}$. Since $\&$ distributes over \vee (i.e., $\varphi \& (\psi \vee \chi) \leftrightarrow (\varphi \& \psi) \vee (\varphi \& \chi)$), we obtain $\&_{i=1}^m \bigvee_{j_i=1}^n \varphi_{ij_i} \leftrightarrow \bigvee_{j_1=1}^n \cdots \bigvee_{j_m=1}^n \&_{i=1}^m \varphi_{ij_i}$.

2.2 Fuzzy (interval) arithmetic

Our intended universal class V of all objects is the set of real numbers \mathbb{R} endowed with the usual structure of an ordered field. However, almost all our results hold over any ordered field. The field operations and the order between objects will be denoted in the usual way, i.e. $x + y$, $x - y$, $x \leq y$, etc. Formally, one can consider that we are working within FCT extended by axioms and language of the theory of real closed fields. The way how to do this precisely is described in [1, Section 6].

Definition 9 *The following arithmetical operations and relations can be defined by Zadeh's extension principle for any fuzzy classes A , B , and a real number k :*

$$\begin{aligned}
A + B &=_{\text{df}} \{z \mid (\exists x \in A)(\exists y \in B)(z = x + y)\} && \text{addition} \\
A - B &=_{\text{df}} \{z \mid (\exists x \in A)(\exists y \in B)(z = x - y)\} && \text{subtraction} \\
kA &=_{\text{df}} Ak =_{\text{df}} \{z \mid (\exists x \in A)(z = kx)\} && \text{scalar multiplication} \\
A \leq B &\equiv_{\text{df}} (\exists x \in A)(\exists y \in B)(x \leq y) && \text{order}
\end{aligned}$$

Lemma 10 *The following are well-known facts from fuzzy arithmetic:*

- (1) $A - B = A + (-1)B$.
- (2) $A + \{0\} = A$.
- (3) $0A = \{0\}$.

Convention 11 *Tuples of elements of the universe and tuples of fuzzy classes are denoted by $\vec{x}, \vec{y}, \vec{z}, \dots$ and $\vec{A}, \vec{B}, \vec{C}, \dots$ respectively. Matrices of elements are denoted by $\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots$ and matrices of fuzzy classes by boldface capital letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$*

Let $\mathbf{A} = (A_{ij})$ be an $m \times n$ matrix of fuzzy classes. Then $(\forall \mathbf{A} \in \mathbf{A})\varphi$ stands for $(\forall a_{11} \in A_{11}) \cdots (\forall a_{mn} \in A_{mn})\varphi$. Further, the formula $(\exists \mathbf{A} \in \mathbf{A})\varphi$ stands

for $(\exists a_{11} \in A_{11}) \cdots (\exists a_{mn} \in A_{mn})\varphi$. We use the analogous conventions also for tuples of fuzzy classes.

Finally, let us introduce basic definitions on tuples of fuzzy classes. Let $\vec{A} = \langle A_1, \dots, A_n \rangle$ and $\vec{B} = \langle B_1, \dots, B_n \rangle$ be tuples of fuzzy classes and $\vec{z} = \langle z_1, \dots, z_n \rangle$ be a tuple of real numbers. Then we define

$$\vec{z} \in \vec{A} \equiv_{\text{df}} \&_{i=1}^n z_i \in A_i$$

$$\vec{A} \subseteq \vec{B} \equiv_{\text{df}} (\forall \vec{z})(\vec{z} \in \vec{A} \rightarrow \vec{z} \in \vec{B}),$$

where $(\forall \vec{z})\varphi \equiv_{\text{df}} (\forall z_1) \cdots (\forall z_n)\varphi$. If we expand the definitions, we obtain:

$$\vec{A} \subseteq \vec{B} \longleftrightarrow (\forall z_1) \cdots (\forall z_n) (\&_{i=1}^n z_i \in A_i \rightarrow \&_{i=1}^n z_i \in B_i).$$

Observe that the right hand side of the last formula is equivalent to

$$(\forall z_1 \in A_1) \cdots (\forall z_n \in A_n) (\&_{i=1}^n z_i \in B_i).$$

The addition and subtraction of matrices or vectors of fuzzy classes is defined componentwise by means of Definition 9. The multiplication of an $m \times n$ matrix $\mathbf{A} = (A_{ij})$ with a vector $\vec{x} = \langle x_1, \dots, x_n \rangle$ is defined as follows:

$$\mathbf{A}\vec{x} = \begin{pmatrix} A_{11}x_1 + \cdots + A_{1n}x_n \\ \vdots \\ A_{m1}x_1 + \cdots + A_{mn}x_n \end{pmatrix}.$$

3 Fuzzy intervals

In this section we are going to introduce the notion of a fuzzy interval and prove a couple of technical results on them. The proofs of the statements are usually easy but they may serve for a reader as a good illustration how Fuzzy Class Theory works.

A fuzzy class A is said to be convex to the degree to which the following formula holds:

$$\text{Convex}(A) \equiv_{\text{df}} (x \in A \& y \in A \& x \leq z \leq y) \rightarrow z \in A,$$

where $x \leq z \leq y$ stands for $x \leq z \& z \leq y$.

Definition 12 Let A be a fuzzy class. The degree of A being a fuzzy interval is given by the following formula:

$$\text{FInt}(A) \equiv_{\text{df}} \text{Norm}(A) \ \& \ \text{Convex}(A).$$

Fuzzy classes which are fully fuzzy intervals can be characterized by means of its down-class and up-class. A down-class and an up-class generated by a class A are defined as follows:

$$A^\downarrow =_{\text{df}} \{x \mid (\exists a \in A)(x \leq a)\},$$

$$A^\uparrow =_{\text{df}} \{x \mid (\exists a \in A)(x \geq a)\}.$$

Theorem 13 Let A be a normal class. Then A is fully convex iff $A = A^\downarrow \cap A^\uparrow$, i.e.,

$$\text{Norm}(A) \rightarrow (\Delta \text{Convex}(A) \leftrightarrow A = A^\downarrow \cap A^\uparrow).$$

PROOF. Let b be the element such that $\Delta(b \in A)$. Suppose that $\Delta \text{Convex}(A)$ holds. Then we have to show that

$$x \in A \longleftrightarrow x \in A^\downarrow \cap A^\uparrow.$$

Observe that

$$x \in A^\downarrow \cap A^\uparrow \longleftrightarrow (\exists a \in A)(x \leq a) \ \& \ (\exists a' \in A)(x \geq a').$$

Clearly, if $x \leq b$ then $\Delta(\exists a \in A)(x \leq a)$ otherwise $\Delta(\exists a' \in A)(x \geq a')$. Assume that $x \geq b$ (the case for $x \leq b$ is analogous). Then $x \in A^\downarrow \cap A^\uparrow \leftrightarrow x \in A^\downarrow$. Thus we have to show that $x \in A \leftrightarrow x \in A^\downarrow$. Firstly, we have

$$x \in A \longleftrightarrow x \in A \ \& \ x \leq x \longrightarrow (\exists a \in A)(x \leq a) \longleftrightarrow x \in A^\downarrow.$$

Conversely, we get from the convexity, normality of A , and our assumption $x \geq b$ that

$$(b \in A \ \& \ a \in A \ \& \ b \leq x \leq a \rightarrow x \in A) \longleftrightarrow (a \in A \ \& \ x \leq a \rightarrow x \in A).$$

Thus by generalization we obtain

$$(\forall a)(a \in A \ \& \ x \leq a \rightarrow x \in A) \longleftrightarrow ((\exists a \in A)(x \leq a) \rightarrow x \in A) \longleftrightarrow (x \in A^\downarrow \rightarrow x \in A).$$

Now, suppose that $A = A^\downarrow \cap A^\uparrow$. We have to show that A is convex. Assume that $z \geq b$. Then $z \in A \leftrightarrow z \in A^\downarrow$ and we have

$$x \in A \ \& \ y \in A \ \& \ x \leq z \leq y \longrightarrow y \in A \ \& \ z \leq y \longrightarrow (\exists a \in A)(z \leq a) \longleftrightarrow z \in A^\downarrow.$$

The case for $z \leq b$ is completely analogous. \square

Corollary 14 *Each fuzzy interval A to degree 1 is equal to the intersection of its down-class and up-class, i.e.,*

$$\Delta \text{FInt}(A) \rightarrow A = A^\downarrow \cap A^\uparrow.$$

Fuzzy intervals (to degree 1) are also closed under the arithmetical operations.

Lemma 15 *Let A, B be fuzzy classes. Then*

$$\Delta \text{FInt}(A) \& \Delta \text{FInt}(B) \rightarrow \Delta \text{FInt}(A + B).$$

PROOF. We have to show that $\text{Norm}(A+B)$ and $\Delta \text{Convex}(A+B)$ hold. The first condition is obvious. Let $\Delta(a \in A)$ and $\Delta(b \in B)$. Then $\Delta(a+b \in A+B)$. Now, assume that $a \leq a'$ and $b \leq b'$. Then we have the following implication:

$$x = a+b \& y = a'+b' \& x \leq z \leq y \rightarrow z = a_0+b_0 \& a \leq a_0 \leq a' \& b \leq b_0 \leq b', \quad (3)$$

where

$$a_0 = a + \frac{z-x}{y-x}(a'-a), \quad b_0 = b + \frac{z-x}{y-x}(b-b').$$

From the implication in Equation (3) we can derive:

$$a \in A \& b \in B \& a' \in A \& b' \in B \& x = a+b \& y = a'+b' \& x \leq z \leq y \rightarrow a \in A \& b \in B \& a' \in A \& b' \in B \& z = a_0+b_0 \& a \leq a_0 \leq a' \& b \leq b_0 \leq b'.$$

Recall that $a \in A \& a' \in A \& a \leq a_0 \leq a' \rightarrow a_0 \in A$ and $b \in B \& b' \in B \& b \leq b_0 \leq b' \rightarrow b_0 \in B$ since A, B are fully convex. Using this and transitivity we obtain:

$$a \in A \& b \in B \& a' \in A \& b' \in B \& x = a+b \& y = a'+b' \& x \leq z \leq y \rightarrow a_0 \in A \& b_0 \in B \& z = a_0 + b_0.$$

Finally

$$a_0 \in A \& b_0 \in B \& z = a_0+b_0 \longrightarrow (\exists c)(\exists d)(c \in A \& d \in B \& z = c+d) \longleftrightarrow z \in A+B.$$

Thus we get

$$a \in A \& b \in B \& a' \in A \& b' \in B \& x = a+b \& y = a'+b' \& x \leq z \leq y \rightarrow z \in A+B.$$

By generalization we get:

$$\begin{aligned}
& (\exists a \in A)(\exists b \in B)(x = a + b) \& \\
& (\exists a' \in A)(\exists b' \in B)(y = a' + b') \& x \leq z \leq y \rightarrow z \in A + B \longleftrightarrow \\
& (x \in A + B \& y \in A + B \& x \leq z \leq y \rightarrow z \in A + B) \longleftrightarrow \\
& \Delta \text{Convex}(A + B)
\end{aligned}$$

The proofs of other cases when $a > a'$ or $b > b'$ are completely analogous with the roles of a and a' , or b and b' interchanged. \square

Lemma 16 *Let A be a fuzzy class and k a real number. Then*

$$\Delta \text{FInt}(A) \rightarrow \Delta \text{FInt}(kA).$$

PROOF. The normality of kA is easy. Let us prove $\Delta \text{Convex}(kA)$. If $k = 0$ then it is obvious since $\{0\}$ is clearly fully convex. We will do it for $k < 0$ (the case for $k > 0$ is analogous). The following formula is valid:

$$x \leq z \leq y \longrightarrow \frac{y}{k} \leq \frac{z}{k} \leq \frac{x}{k}.$$

We also have $x \in kA \leftrightarrow x/k \in A$. Putting these together we obtain

$$x \in kA \& y \in kA \& x \leq z \leq y \longrightarrow \frac{x}{k} \in A \& \frac{y}{k} \in A \& \frac{y}{k} \leq \frac{z}{k} \leq \frac{x}{k}.$$

By convexity of A we get

$$\frac{x}{k} \in A \& \frac{y}{k} \in A \& \frac{y}{k} \leq \frac{z}{k} \leq \frac{x}{k} \longrightarrow \frac{z}{k} \in A.$$

Finally,

$$\frac{z}{k} \in A \longrightarrow z \in A \& z = k \frac{z}{k} \longrightarrow (\exists a \in A)(z = ka) \longleftrightarrow z \in kA.$$

Thus the proof is done by transitivity. \square

The remaining lemmata in this section will be stated without proofs since the proofs are easy and purely syntactical. A reader can find them in the appendix.

Lemma 17 *The following hold for any fuzzy classes A, B :*

$$A^\downarrow + B^\downarrow = (A + B)^\downarrow, \quad A^\uparrow + B^\uparrow = (A + B)^\uparrow.$$

Lemma 18 *Let A be a fuzzy class and $k \in \mathbb{R}$. If $k > 0$ then $(kA)^\downarrow = kA^\downarrow$ and $(kA)^\uparrow = kA^\uparrow$. If $k < 0$ then $(kA)^\downarrow = kA^\uparrow$ and $(kA)^\uparrow = kA^\downarrow$.*

Note that if $k = 0$ then $kA^\downarrow = \{0\}$ and $(kA)^\downarrow = \{0\}^\downarrow = (-\infty, 0]$ (analogously for up-classes).

Lemma 19 *Let A, B be fuzzy classes. Then the following is provable in FCT:*

- (1) $A \leq B \leftrightarrow A^\uparrow \parallel B^\downarrow$,
- (2) $0 \in B^\downarrow - A^\uparrow \leftrightarrow A \leq B$ and $0 \in B^\uparrow - A^\downarrow \leftrightarrow B \leq A$,
- (3) $A^\downarrow + \{0\}^\downarrow = A^\downarrow$ and $A^\uparrow + \{0\}^\uparrow = A^\uparrow$,
- (4) $\{0\}^\uparrow \parallel B^\downarrow \leftrightarrow \{0\} \parallel B^\downarrow$ and $\{0\}^\downarrow \parallel B^\uparrow \leftrightarrow \{0\} \parallel B^\uparrow$.

4 Fuzzy interval linear system and its solution set

In this section we formally define the solution set of a system of linear equations with uncertain parameters, i.e., the parameters which are known to belong to given fuzzy classes. Our intention is of course that these classes will be fuzzy intervals. However, some of our results hold generally for any fuzzy classes. Thus we define the solution set for arbitrary fuzzy classes.

Definition 20 *Let $\mathbf{A} = (A_{ij})$ be an $m \times n$ matrix of fuzzy classes (intervals) and $\vec{B} = \langle B_1, \dots, B_n \rangle$ be an n -tuple of fuzzy classes (intervals). Then the system $\mathbf{A}\vec{x} = \vec{B}$ is called fuzzy (interval) linear system.*

Thus a system of linear equations with fuzzy classes is called a fuzzy linear system whereas the system with fuzzy intervals is called a fuzzy interval linear system.

The most common approach (in classical interval analysis) how to define a solution set $\Xi_{uni}(\mathbf{A}, \vec{B})$ of a fuzzy linear system $\mathbf{A}\vec{x} = \vec{B}$ is in fact the one which we described in the introduction, i.e.

$$\Xi_{uni}(\mathbf{A}, \vec{B}) =_{\text{df}} \{\vec{x} \mid (\exists a_{11} \in A_{11}) \cdots (\exists a_{mn} \in A_{mn})(\exists b_1 \in B_1) \cdots (\exists b_n \in B_n)(\mathbb{A}\vec{x} = \vec{b})\}.$$

Note that all parameters $a_{11}, \dots, a_{mn}, b_1, \dots, b_n$ are independent. The subscript “uni” refers to the fact that this solution set is usually called *united solution set*. It is usually written in a simplified form as follows:

$$\Xi_{uni}(\mathbf{A}, \vec{B}) =_{\text{df}} \{\vec{x} \mid (\exists \mathbf{A} \in \mathbf{A})(\exists \vec{b} \in \vec{B})(\mathbb{A}\vec{x} = \vec{b})\}.$$

However it turns out that usage of the universal quantifiers in the definition is also meaningful. We will shortly present the main motivation for this coming from the very nice paper [20] on classical interval analysis. Consider a system which is to be controlled. This system is described by a system of linear equations $\mathbb{A}\vec{x} = \vec{b}$. Suppose that the entries of \mathbb{A} corresponds to the inputs of the

system and \vec{b} to its outputs. Both inputs and outputs can be of two sorts. In the set of inputs we distinguish between

- *perturbations*: the inputs on which we have no influence (e.g. noise, unknown material parameter, etc.), but we know that they belong to given fuzzy classes,
- *controls*: the inputs intended for a controller. We can set them arbitrarily but we are restricted by some constraints, i.e., they can be chosen only within given fuzzy classes.

In the set of outputs we distinguish between

- *stabilized*: the outputs which should be stabilized into given fuzzy classes (e.g. temperature of a heating),
- *controlled*: the outputs to which we must be able to attain any given values from prescribed fuzzy classes (e.g. it must be possible to put a robot's arm into any place in its operational space).

The vector \vec{x} corresponds to a state of the system. Now we are interested in those states \vec{x} for which for any perturbations and for any values of controlled outputs from the prescribed fuzzy classes, there exist suitable controls such that the stabilized outputs are within the given fuzzy classes and the controlled outputs attain the desired values. Such fuzzy class of vectors \vec{x} will be for us the most general solution set of a fuzzy linear system.

Let $\mathbf{A}\vec{x} = \vec{B}$ be a fuzzy linear system. In order to express formally the most general definition of the solution set of $\mathbf{A}\vec{x} = \vec{B}$, let us index the elements in \mathbf{A} by one index $i \in \{1, \dots, mn\}$, i.e.,

$$\mathbf{A} = \begin{pmatrix} A_1 & \cdots & A_n \\ \vdots & \ddots & \vdots \\ A_{n(m-1)+1} & \cdots & A_{mn} \end{pmatrix}.$$

Further, let π be a permutation on the set $\{1, \dots, mn\}$ and σ a permutation on the set $\{1, \dots, n\}$. Finally, let $k \in \{1, \dots, mn\}$ be the number of elements from \mathbf{A} corresponding to the perturbations and $s \in \{1, \dots, n\}$ the number of elements from \vec{B} corresponding to the controlled outputs. Then the most general definition of the solution set considered in this paper can be written as follows:

$$\begin{aligned} \Xi(\mathbf{A}, \vec{B}) = \{ \vec{x} \mid & (\forall a_{\pi(1)} \in A_{\pi(1)}) \cdots (\forall a_{\pi(k)} \in A_{\pi(k)}) \\ & (\forall b_{\sigma(1)} \in B_{\sigma(1)}) \cdots (\forall b_{\sigma(s)} \in B_{\sigma(s)}) \\ & (\exists a_{\pi(k+1)} \in A_{\pi(k+1)}) \cdots (\exists a_{\pi(mn)} \in A_{\pi(mn)}) \\ & (\exists b_{\sigma(s+1)} \in B_{\sigma(s+1)}) \cdots (\exists b_{\sigma(n)} \in B_{\sigma(n)}) (\mathbf{A}\vec{x} = \vec{b}) \}, \quad (4) \end{aligned}$$

where $\mathbb{A} = \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n(m-1)+1} & \cdots & a_{mn} \end{pmatrix}$ and $\vec{b} = \langle b_1, \dots, b_n \rangle$.

Observe that the definition of the solution set could be further generalized since the universal quantifier does not generally commute with the existential one. Thus in a more general definition we could even consider different orders of quantifiers appearing in the comprehension term. However we restrict here ourselves to the case when all universal quantifiers are followed by the existential ones.

Example 21 Let $\mathbf{A} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ and $\vec{B} = \langle B_1, B_2 \rangle$. Then one of the possible solution sets could be the following fuzzy class:

$$\Xi(\mathbf{A}, \vec{B}) = \{\vec{x} \mid (\forall a_2 \in A_2)(\forall a_3 \in A_3)(\forall b_2 \in B_2)(\exists a_1 \in A_1)(\exists a_4 \in A_4)(\exists b_1 \in B_1)(\mathbb{A}\vec{x} = \vec{b})\},$$

where $\mathbb{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $\vec{b} = \langle b_1, b_2 \rangle$.

In order to simplify the formula describing the solution set, we split the matrix \mathbf{A} and the tuple \vec{B} into two disjoint parts according to the quantifiers. We define $\mathbf{A}^\forall = (A_{ij}^\forall)$, $\mathbf{A}^\exists = (A_{ij}^\exists)$, $\vec{B}^\forall = (B_i^\forall)$, and $\vec{B}^\exists = (B_i^\exists)$, where

$$A_{ij}^\forall = \begin{cases} A_{ij} & \text{if } A_{ij} \text{ should be quantified by } \forall, \\ \{0\} & \text{otherwise,} \end{cases}$$

$$A_{ij}^\exists = \begin{cases} A_{ij} & \text{if } A_{ij} \text{ should be quantified by } \exists, \\ \{0\} & \text{otherwise,} \end{cases}$$

$$B_i^\forall = \begin{cases} B_i & \text{if } B_i \text{ should be quantified by } \forall, \\ \{0\} & \text{otherwise,} \end{cases}$$

$$B_i^\exists = \begin{cases} B_i & \text{if } B_i \text{ should be quantified by } \exists, \\ \{0\} & \text{otherwise.} \end{cases}$$

Then we have $\mathbf{A} = \mathbf{A}^\forall + \mathbf{A}^\exists$, $\vec{B} = \vec{B}^\forall + \vec{B}^\exists$. Now we can write down the formal definition of the solution set.

Definition 22 Let $(\mathbf{A}^\forall + \mathbf{A}^\exists)\vec{x} = \vec{B}^\forall + \vec{B}^\exists$ be a fuzzy linear system. Then its solution set is the following fuzzy class:

$$\Xi(\mathbf{A}^\forall, \mathbf{A}^\exists, \vec{B}^\forall, \vec{B}^\exists) =_{\text{df}} \{\vec{x} \mid ((\forall \mathbf{U} \in \mathbf{A}^\forall)(\forall \vec{u} \in \vec{B}^\forall)(\exists \mathbf{E} \in \mathbf{A}^\exists)(\exists \vec{e} \in \vec{B}^\exists)((\mathbf{U} + \mathbf{E})\vec{x} = \vec{u} + \vec{e}))\}.$$

The fact that this definition is equivalent to that given by the formula (4) follows from Lemma 5 for $t = 0$ and the fact that $x \in \{0\} \longleftrightarrow x = 0$.

5 Fundamental theorem

In this section we are going to generalize the fundamental theorem [20, Theorem 3.4] from classical logic to fuzzy logic. The theorem characterizes the solutions by means of the arithmetic defined on fuzzy classes in Subsection 2.2.

Theorem 23 (Fundamental theorem) *Let $(\mathbf{A}^\forall + \mathbf{A}^\exists)\vec{x} = \vec{B}^\forall + \vec{B}^\exists$ be a fuzzy linear system. Then a vector \vec{x} belongs to the solution set $\Xi(\mathbf{A}^\forall, \mathbf{A}^\exists, \vec{B}^\forall, \vec{B}^\exists)$ to the same degree as the formula $\mathbf{A}^\forall\vec{x} - \vec{B}^\forall \subseteq \vec{B}^\exists - \mathbf{A}^\exists\vec{x}$ holds, i.e.,*

$$\vec{x} \in \Xi(\mathbf{A}^\forall, \mathbf{A}^\exists, \vec{B}^\forall, \vec{B}^\exists) \longleftrightarrow \mathbf{A}^\forall\vec{x} - \vec{B}^\forall \subseteq \vec{B}^\exists - \mathbf{A}^\exists\vec{x}.$$

The operations in $\mathbf{A}^\forall\vec{x} - \vec{B}^\forall$ and $\vec{B}^\exists - \mathbf{A}^\exists\vec{x}$ are the arithmetical operations defined on matrices and tuples of fuzzy classes in Subsection 2.2.

PROOF. We start with the definition of the solution set $\Xi(\mathbf{A}^\forall, \mathbf{A}^\exists, \vec{B}^\forall, \vec{B}^\exists)$ and we get the following chain of equivalences:

$$\begin{aligned} \vec{x} \in \Xi(\mathbf{A}^\forall, \mathbf{A}^\exists, \vec{B}^\forall, \vec{B}^\exists) &\longleftrightarrow \\ (\forall \mathbb{U} \in \mathbf{A}^\forall)(\forall \vec{u} \in \vec{B}^\forall)(\exists \mathbb{E} \in \mathbf{A}^\exists)(\exists \vec{e} \in \vec{B}^\exists)((\mathbb{U} + \mathbb{E})\vec{x} = \vec{u} + \vec{e}) &\longleftrightarrow \\ (\forall \mathbb{U} \in \mathbf{A}^\forall)(\forall \vec{u} \in \vec{B}^\forall)(\exists \mathbb{E} \in \mathbf{A}^\exists)(\exists \vec{e} \in \vec{B}^\exists)(\mathbb{U}\vec{x} - \vec{u} = \vec{e} - \mathbb{E}\vec{x}) &\longleftrightarrow \\ (\forall \mathbb{U} \in \mathbf{A}^\forall)(\forall \vec{u} \in \vec{B}^\forall)(\exists \mathbb{E} \in \mathbf{A}^\exists)(\exists \vec{e} \in \vec{B}^\exists)(\&_{i=1}^n \mathbb{U}_i\vec{x} - u_i = e_i - \mathbb{E}_i\vec{x}), \end{aligned}$$

where \mathbb{U}_i is the i -th row of \mathbb{U} and similarly for \mathbb{E}_i . The i -th component of \vec{u} (resp. \vec{e}) is denoted by u_i (resp. e_i). Finally $\mathbb{U}_i\vec{x}$ is just multiplication of one-row matrix by the vector \vec{x} .

Since the existential quantifier distributes over $\&$, we get

$$\begin{aligned} (\forall \mathbb{U} \in \mathbf{A}^\forall)(\forall \vec{u} \in \vec{B}^\forall) \left(\&_{i=1}^n (\exists \mathbb{E}_i \in \mathbf{A}_i^\exists)(\exists e_i \in B_i^\exists)(\mathbb{U}_i\vec{x} - u_i = e_i - \mathbb{E}_i\vec{x}) \right) &\longleftrightarrow \\ (\forall \mathbb{U} \in \mathbf{A}^\forall)(\forall \vec{u} \in \vec{B}^\forall) \left(\&_{i=1}^n \mathbb{U}_i\vec{x} - u_i \in B_i^\exists - \mathbf{A}_i^\exists\vec{x} \right) &\longleftrightarrow \\ (\forall \mathbb{U} \in \mathbf{A}^\forall)(\forall \vec{u} \in \vec{B}^\forall)(\mathbb{U}\vec{x} - \vec{u} \in \vec{B}^\exists - \mathbf{A}^\exists\vec{x}), \end{aligned} \quad (5)$$

where \mathbf{A}_i^\exists is the i -th row of \mathbf{A} and B_i^\exists is the i -th component of \vec{B}^\exists .

On the other hand, we have that $\mathbf{A}^\forall\vec{x} - \vec{B}^\forall \subseteq \vec{B}^\exists - \mathbf{A}^\exists\vec{x}$ is equivalent to

$$(\forall z_1 \in \mathbf{A}_1^\forall\vec{x} - B_1^\forall) \cdots (\forall z_n \in \mathbf{A}_n^\forall\vec{x} - B_n^\forall)(\&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists\vec{x}),$$

Consider the following part of the previous formula

$$\begin{aligned} & (\forall z_n \in \mathbf{A}_n^\forall \vec{x} - B_n^\forall) (\&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \longleftrightarrow \\ & (\forall z_n) (z_n \in \mathbf{A}_n^\forall \vec{x} - B_n^\forall \rightarrow (\&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x})) \longleftrightarrow \\ & (\forall z_n) \left((\exists \mathbb{U}_n \in \mathbf{A}_n^\forall) (\exists u_n \in B_n^\forall) (z_n = \mathbb{U}_n \vec{x} - u_n) \rightarrow (\&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \right). \end{aligned}$$

We can shift the existential quantifiers outside the bracket, i.e. we get

$$\begin{aligned} & (\forall z_n) (\forall \mathbb{U}_n) (\forall u_n) \left(\mathbb{U}_n \in \mathbf{A}_n^\forall \& u_n \in B_n^\forall \& z_n = \mathbb{U}_n \vec{x} - u_n \rightarrow (\&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \right) \longleftrightarrow \\ & (\forall \mathbb{U}_n) (\forall u_n) \left(\mathbb{U}_n \in \mathbf{A}_n^\forall \& u_n \in B_n^\forall \rightarrow (\forall z_n) \left(z_n = \mathbb{U}_n \vec{x} - u_n \rightarrow (\&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \right) \right). \end{aligned}$$

By Lemma 5 for $t = \mathbb{U}_n \vec{x} - u_n$, we get

$$\begin{aligned} & (\forall z_n) \left(z_n = \mathbb{U}_n \vec{x} - u_n \rightarrow (\&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \right) \longleftrightarrow \\ & \mathbb{U}_n \vec{x} - u_n \in B_n^\exists - \mathbf{A}_n^\exists \vec{x} \& (\&_{i=1}^{n-1} z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}). \end{aligned}$$

Thus we obtain

$$(\forall \mathbb{U}_n \in \mathbf{A}_n^\forall) (\forall u_n \in B_n^\forall) \left(\mathbb{U}_n \vec{x} - u_n \in B_n^\exists - \mathbf{A}_n^\exists \vec{x} \& (\&_{i=1}^{n-1} z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \right).$$

Then we can use the same procedure to eliminate $(\forall z_i \in \mathbf{A}_i^\forall \vec{x} - \vec{B}_i^\forall)$ for each $i < n$ and finally we end up with

$$(\forall \mathbb{U}_1 \in \mathbf{A}_1^\forall) \cdots (\forall \mathbb{U}_n \in \mathbf{A}_n^\forall) (\forall u_1 \in B_1^\forall) \cdots (\forall u_n \in B_n^\forall) (\&_{i=1}^n \mathbb{U}_i \vec{x} - u_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}).$$

The last formula is equivalent to

$$(\forall \mathbb{U} \in \mathbf{A}^\forall) (\forall \vec{u} \in \vec{B}^\forall) (\mathbb{U} \vec{x} - \vec{u} \in \vec{B}^\exists - \mathbf{A}^\exists \vec{x}),$$

which is the same formula as the last formula in Equation (5). \square

6 United solution set

The fundamental theorem serves as a good starting point for computing the solution set. Although it works for arbitrary fuzzy classes, we will restrict ourselves to fuzzy intervals in the rest of the paper. This restriction is necessary if we want to obtain results like in the classical interval analysis.

The second restriction we make in this section concerns the quantifiers in the definition of solution set. More precisely, we are going to describe the solution set for a fuzzy interval linear system in the case when all the quantifiers appearing in the system are existential. This restriction allows us to separate the particular equations in the computation of the solution set. This means that

we can find the solutions for each equation separately and then the resulting solution set is just their intersection. This is not possible in general and it is necessary to solve the whole system simultaneously (see also Section 9). We leave this issue to forthcoming papers.

Assume that all quantifiers in Definition 22 of the solution set are existential. Then \mathbf{A}^\forall is the matrix of crisp fuzzy classes $\{0\}$ and the same holds for \vec{B}^\forall . Thus we will denote the solution set in this case by $\Xi(\mathbf{A}^\exists, \vec{B}^\exists)$ and call it the united solution set as in classical interval analysis. From the fundamental theorem we get

$$x \in \Xi(\mathbf{A}^\exists, \vec{B}^\exists) \longleftrightarrow \begin{pmatrix} \{0\} \\ \vdots \\ \{0\} \end{pmatrix} \subseteq \vec{B}^\exists - \mathbf{A}^\exists \vec{x},$$

The last formula is equivalent to

$$\begin{aligned} (\forall z_1) \cdots (\forall z_n) (\&_{i=1}^n z_i \in \{0\} \rightarrow \&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \longleftrightarrow \\ (\forall z_1) \cdots (\forall z_n) (\&_{i=1}^n z_i = 0 \rightarrow \&_{i=1}^n z_i \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}) \longleftrightarrow \\ \&_{i=1}^n 0 \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}. \end{aligned}$$

Thus when we compute the solution set in this case it is sufficient to find the fuzzy class of \vec{x} satisfying $0 \in B_i^\exists - \mathbf{A}_i^\exists \vec{x}$ for each i and then take their intersection. Note that generally this does not work when some of the entries of \mathbf{A} or \vec{B} are quantified by the universal quantifier as we already mentioned at the beginning of this section.

Since $B_i^\exists - \mathbf{A}_i^\exists \vec{x}$ is a fuzzy interval (to degree 1) by Lemmata 15 and 16, we get $B_i^\exists - \mathbf{A}_i^\exists \vec{x} = (B_i^\exists - \mathbf{A}_i^\exists \vec{x})^\downarrow \cap (B_i^\exists - \mathbf{A}_i^\exists \vec{x})^\uparrow$ due to Corollary 14. Thus thanks to Lemma 19(2) and Lemma 19(1), we obtain

$$\begin{aligned} 0 \in B_i^\exists - \mathbf{A}_i^\exists \vec{x} \longleftrightarrow 0 \in (B_i^\exists - \mathbf{A}_i^\exists \vec{x})^\downarrow \cap (B_i^\exists - \mathbf{A}_i^\exists \vec{x})^\uparrow \longleftrightarrow \\ 0 \in (B_i^\exists)^\downarrow - (\mathbf{A}_i^\exists \vec{x})^\uparrow \ \& \ 0 \in (B_i^\exists)^\uparrow - (\mathbf{A}_i^\exists \vec{x})^\downarrow \longleftrightarrow \\ \mathbf{A}_i^\exists \vec{x} \leq B_i^\exists \ \& \ B_i^\exists \leq \mathbf{A}_i^\exists \vec{x} \longleftrightarrow \\ (\mathbf{A}_i^\exists \vec{x})^\uparrow \parallel (B_i^\exists)^\downarrow \ \& \ (B_i^\exists)^\uparrow \parallel (\mathbf{A}_i^\exists \vec{x})^\downarrow. \end{aligned}$$

Let us introduce the following classes:

$$H_i^d = \{\vec{x} \mid (\mathbf{A}_i^\exists \vec{x})^\uparrow \parallel (B_i^\exists)^\downarrow\}, \quad H_i^u = \{\vec{x} \mid (\mathbf{A}_i^\exists \vec{x})^\downarrow \parallel (B_i^\exists)^\uparrow\}.$$

Then the solution set can be expressed as follows:

$$\Xi(\mathbf{A}^\exists, \vec{B}^\exists) = \bigcap_{i=1}^m (H_i^d \cap H_i^u). \quad (6)$$

Let $K = \{\uparrow, \downarrow\}^n$ be the set of all sequences of symbols \uparrow, \downarrow whose length is n . The j -th component of $k \in K$ will be denoted by k_j . Further, we define $\varepsilon_{jk} = 1$ if $k_j = \uparrow$ and -1 otherwise. Let $Q_k, k \in K$, be the family of all orthants of \mathbb{R}^n , i.e., we have for each Q_k :

$$Q_k = \{\vec{x} \in \mathbb{R}^n \mid \varepsilon_{1k}x_1 \geq 0 \ \& \ \cdots \ \& \ \varepsilon_{nk}x_n \geq 0\},$$

where x_j stands for the j -th component of \vec{x} . Each Q_k is obviously crisp. Observe also that for $\vec{x} \in Q_k$ we have $x_j \geq 0$ if $k_j = \uparrow$ and $x \leq 0$ if $k_j = \downarrow$.

Lemma 24 *Let $k \in K$. Then*

$$\vec{x} \in Q_k \rightarrow \left((\mathbf{A}_i^\exists \vec{x})^\uparrow \parallel (B_i^\exists)^\downarrow \leftrightarrow \left(\sum_{j=1}^n A_{ij}^{k_j} x_j \right) \parallel (B_i^\exists)^\downarrow \right),$$

$$\vec{x} \in Q_k \rightarrow \left((\mathbf{A}_i^\exists \vec{x})^\downarrow \parallel (B_i^\exists)^\uparrow \leftrightarrow \left(\sum_{j=1}^n A_{ij}^{-k_j} x_j \right) \parallel (B_i^\exists)^\uparrow \right),$$

where

$$-k_j = \begin{cases} \downarrow & \text{if } k_j = \uparrow, \\ \uparrow & \text{if } k_j = \downarrow. \end{cases}$$

PROOF. If $\vec{x} = \vec{0}$ then $\mathbf{A}_i^\exists \vec{x} = \sum_{j=1}^n A_{ij}^{k_j} x_j = \sum_{j=1}^n A_{ij}^{-k_j} x_j = \{0\}$ and the both statements follow from Lemma 19(4). Let $\vec{x} \in Q_k$ and $\vec{x} \neq \vec{0}$. Consider the following formula:

$$(\mathbf{A}_i^\exists \vec{x})^\uparrow = \left(\sum_{j=1}^n A_{ij} x_j \right)^\uparrow.$$

We would like to distribute \uparrow over the sum. It follows from Lemma 18 that we can do it only in the case when $x_j \neq 0$. In fact it is sufficient that at least for one j we have $x_j \neq 0$ (i.e., $\vec{x} \neq \vec{0}$). Then

$$(\mathbf{A}_i^\exists \vec{x})^\uparrow = \left(\sum_{x_j \neq 0} A_{ij} x_j \right)^\uparrow = \sum_{x_j \neq 0} A_{ij}^{k_j} x_j + \{0\} = \sum_{x_j \neq 0} A_{ij}^{k_j} x_j + \sum_{x_j = 0} A_{ij}^{k_j} x_j = \sum_{j=1}^n A_{ij}^{k_j} x_j.$$

Similarly, if at least one $x_j \neq 0$ then we have

$$(\mathbf{A}_i^\exists \vec{x})^\downarrow = \sum_{j=1}^n A_{ij}^{-k_j} x_j. \quad \square$$

Let us define the following fuzzy classes:

$$S_{ik}^d = \{\vec{x} \mid (\sum_{j=1}^n A_{ij}^{k_j} x_j) \parallel (B_i^{\exists})^\downarrow\}, \quad S_{ik}^u = \{\vec{x} \mid (\sum_{j=1}^n A_{ij}^{-k_j} x_j) \parallel (B_i^{\exists})^\uparrow\}.$$

Due to the latter lemma, the classes H_i^d and H_i^u in any orthant Q_k can be described by means of S_{ik}^d and S_{ik}^u respectively. Consequently, we obtain the following lemma.

Lemma 25

$$H_i^d = \bigsqcup_{k \in K} (Q_k \cap S_{ik}^d), \quad H_i^u = \bigsqcup_{k \in K} (Q_k \cap S_{ik}^u).$$

PROOF. By Lemma 24 we have $\vec{x} \in Q_k \rightarrow (\vec{x} \in H_i^d \leftrightarrow \vec{x} \in S_{ik}^d)$. Thus $\vec{x} \in Q_k \& \vec{x} \in S_{ik}^d \rightarrow \vec{x} \in H_i^d$ for each k . Consequently, $\bigvee_{k \in K} (\vec{x} \in Q_k \& \vec{x} \in S_{ik}^d) \rightarrow \vec{x} \in H_i^d$. Hence

$$\vec{x} \in \bigsqcup_{k \in K} (Q_k \cap S_{ik}^d) \longleftrightarrow \bigvee_{k \in K} (\vec{x} \in Q_k \& \vec{x} \in S_{ik}^d) \longrightarrow \vec{x} \in H_i^d.$$

On the other hand, we have $\vec{x} \in Q_k \& \vec{x} \in H_i^d \rightarrow \vec{x} \in S_{ik}^d$ by Lemma 24. Thus $(\vec{x} \in Q_k)^2 \& \vec{x} \in H_i^d \rightarrow \vec{x} \in Q_k \& \vec{x} \in S_{ik}^d$. Since $\text{Crisp}(Q_k)$ holds for each k , we get $(\vec{x} \in Q_k)^2 \leftrightarrow \vec{x} \in Q_k$. Consequently,

$$\begin{aligned} \vec{x} \in H_i^d &\longleftrightarrow \vec{x} \in H_i^d \& \bigvee_{k \in K} (\vec{x} \in Q_k) \longleftrightarrow \\ &\bigvee_{k \in K} (\vec{x} \in Q_k \& \vec{x} \in H_i^d) \longrightarrow \bigvee_{k \in K} (\vec{x} \in Q_k \& \vec{x} \in S_{ik}^d) \longleftrightarrow \vec{x} \in \bigsqcup_{k \in K} (Q_k \cap S_{ik}^d). \end{aligned}$$

The proof of the second statement is analogous. \square

From the latter lemma we obtain the final description of the united solution set.

Theorem 26

$$\Xi(\mathbf{A}^{\exists}, \mathbf{B}^{\exists}) = \bigsqcup_{k \in K} \left(\bigcap_{i=1}^m (Q_k \cap W_{ik}) \right),$$

where $W_{ik} = S_{ik}^d \cap S_{ik}^u$ and

$$S_{ik}^d = \{\vec{x} \mid (\sum_{j=1}^n A_{ij}^{k_j} x_j) \parallel (B_i^{\exists})^\downarrow\}, \quad S_{ik}^u = \{\vec{x} \mid (\sum_{j=1}^n A_{ij}^{-k_j} x_j) \parallel (B_i^{\exists})^\uparrow\}.$$

PROOF. By Lemma 25 it follows from Equation (6)

$$\begin{aligned} \Xi(\mathbf{A}^\exists, \vec{B}^\exists) &= \bigcap_{i=1}^m \left(\bigsqcup_{k \in K} (Q_k \cap S_{ik}^d) \cap \bigsqcup_{l \in K} (Q_l \cap S_{il}^u) \right) = \\ &= \bigcap_{i=1}^m \left(\bigsqcup_{k \in K} \bigsqcup_{l \in K} (Q_k \cap S_{ik}^d \cap Q_l \cap S_{il}^u) \right). \end{aligned}$$

Observe that $\vec{x} \in Q_k \cap Q_l$ iff $x_j = 0$ for all $1 \leq j \leq n$ such that $k_j \neq l_j$. Thus we get $S_{ik}^d \cap Q_k \cap Q_l = S_{il}^d \cap Q_k \cap Q_l$ and $S_{ik}^u \cap Q_k \cap Q_l = S_{il}^u \cap Q_k \cap Q_l$ because $x_j = 0$ when $k_j \neq l_j$ and $-k_j \neq -l_j$. Consequently,

$$Q_k \cap S_{ik}^d \cap Q_l \cap S_{il}^u = Q_k \cap S_{ik}^d \cap Q_l \cap S_{ik}^u \subseteq Q_k \cap S_{ik}^d \cap S_{ik}^u = Q_k \cap S_{ik}^d \cap Q_k \cap S_{ik}^u.$$

The last equality follows from the fact that each Q_k is crisp. Hence the only interesting combinations of k, l are cases when $k = l$. Thus

$$\bigsqcup_{k \in K} \bigsqcup_{l \in K} (Q_k \cap S_{ik}^d \cap Q_l \cap S_{il}^u) = \bigsqcup_{k \in K} (Q_k \cap S_{ik}^d \cap S_{ik}^u).$$

Now we have

$$\begin{aligned} \Xi(\mathbf{A}^\exists, \vec{B}^\exists) &= \bigcap_{i=1}^m \left(\bigsqcup_{k \in K} (Q_k \cap S_{ik}^d \cap S_{ik}^u) \right) = \bigcap_{i=1}^m \left(\bigsqcup_{k_i \in K} (Q_{k_i} \cap S_{ik_i}^d \cap S_{ik_i}^u) \right) = \\ &= \bigsqcup_{k_1 \in K} \cdots \bigsqcup_{k_m \in K} \left(\bigcap_{i=1}^m (Q_{k_i} \cap S_{ik_i}^d \cap S_{ik_i}^u) \right). \end{aligned}$$

Whenever $k_i \neq k_j$ then using the same argument as above we get

$$\begin{aligned} Q_{k_i} \cap Q_{k_j} \cap S_{ik_i}^d \cap S_{ik_i}^u \cap S_{jk_j}^d \cap S_{jk_j}^u &= \\ Q_{k_i} \cap Q_{k_j} \cap S_{ik_j}^d \cap S_{ik_j}^u \cap S_{jk_j}^d \cap S_{jk_j}^u &\subseteq Q_{k_j} \cap S_{ik_j}^d \cap S_{ik_j}^u \cap S_{jk_j}^d \cap S_{jk_j}^u = \\ &= Q_{k_j} \cap Q_{k_j} \cap S_{ik_j}^d \cap S_{ik_j}^u \cap S_{jk_j}^d \cap S_{jk_j}^u. \end{aligned}$$

Hence all intersections for at least one pair of different indices k_i are subsets of an intersection when all indices k_i are the same, i.e. $k_1 = \cdots = k_m$. Thus it is sufficient to make the union over only one index k .

$$\Xi(\mathbf{A}^\exists, \vec{B}^\exists) = \bigsqcup_{k_1 \in K} \cdots \bigsqcup_{k_m \in K} \left(\bigcap_{i=1}^m (Q_{k_i} \cap S_{ik_i}^d \cap S_{ik_i}^u) \right) = \bigsqcup_{k \in K} \left(\bigcap_{i=1}^m (Q_k \cap W_{ik}) \right).$$

□

Thus if we want to find the united solution of a fuzzy interval system, it is sufficient to compute the solution independently in each orthant, i.e., to compute $\bigcap_{i=1}^m (Q_k \cap W_{ik})$, and then just make the union of all results. In order

to compute the solution in an orthant, we have to be able to express any linear combination of up-classes (resp. down-classes) in the evaluation of S_{ik}^d and S_{ik}^u , and to compute the degree of compatibility of any pair of an up-class and a down-class in the evaluation of W_{ik} .

Remark 27 *Theorem 26 is also related to [20, Theorem 3.6]. Let $(\mathbf{A}^\forall + \mathbf{A}^\exists)\vec{x} = \vec{B}^\forall + \vec{B}^\exists$ be a classical interval linear system. Then [20, Theorem 3.6] states that the solution set $\Xi(\mathbf{A}^\forall, \mathbf{A}^\exists, \vec{B}^\forall, \vec{B}^\exists)$ in each orthant forms a convex polyhedron defined by a usual system of linear inequalities $\mathbb{C}\vec{x} \leq \vec{d}$ where entries in \mathbb{C} lie among the bounds of intervals appearing in $\mathbf{A}^\forall + \mathbf{A}^\exists$, and entries of \vec{d} among the bounds of intervals appearing in $\vec{B}^\forall + \vec{B}^\exists$. This results when \mathbf{A}^\forall and \vec{B}^\forall consist of the crisp singletons $\{0\}$, i.e. all quantifiers in Definition 22 of the solution set are existential, was already proved in [18].*

A generalization of this special case is presented in fact in Theorem 26. It is clear that unlike the crisp intervals, a fuzzy interval need not have a crisp lower-bound and upper-bound. Nevertheless, we can replace them respectively by its down-class and up-class. Then note that Theorem 26 is in fact of the same shape as [20, Theorem 3.6]. It tells us that a solution set in a given orthant Q_k is determined by a system of linear inequalities since $\|$ can be replaced by \leq by Lemma 19. The only difference is that we have to use the down-classes and up-classes instead of the bounds of crisp intervals.

7 United solution set for trapezoidal fuzzy intervals in Łukasiewicz logic

In this section, we want to demonstrate how to use the results from the previous section for computing concrete solution sets. For this purpose we have to restrict ourselves to one concrete logic (we choose Łukasiewicz logic) and one special type of fuzzy intervals (namely those which are known in the literature under the name trapezoidal fuzzy numbers). Of course it is possible to do it for other logics like Gödel² or product logic but the concrete solution depends on the chosen logic so that we can choose only one logic in order to keep the paper in a reasonable size. We also assume that all crisp intervals are closed.

We will work with the standard semantics of Łukasiewicz logic, i.e., with the standard MV-algebra on $[0, 1]$. Thus all the predicates can be viewed as $[0, 1]$ -valued functions on reals, $\&$ is interpreted as Łukasiewicz t-norm, \rightarrow as

² In Gödel logic we have even a stronger relation to classical interval analysis. Since the fuzzy interval arithmetic in Gödel logic behaves on each α -cut like classical interval arithmetic, we can compute in this case any α -cut of the solution set of a fuzzy interval linear system by the same methods as in the classical case.

the corresponding residuum, \wedge, \vee as min and max respectively.

We firstly introduce some notation for dealing with the chosen fuzzy intervals. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. We define its truncation $[f]_0^1 = (f \vee 0) \wedge 1$.

Lemma 28 *Let f, g be functions from \mathbb{R} to \mathbb{R} . Then $[f]_0^1 \vee [g]_0^1 = [f \vee g]_0^1$.*

PROOF. By distributivity of \vee and \wedge we get

$$[f]_0^1 \vee [g]_0^1 = ((f \vee 0) \wedge 1) \vee ((g \vee 0) \wedge 1) = ((f \vee 0 \vee g \vee 0) \wedge 1) = ((f \vee g) \vee 0) \wedge 1 = [f \vee g]_0^1.$$

□

The trapezoidal fuzzy intervals form a certain subset of piecewise linear $[0, 1]$ -valued functions. We firstly define their up-classes and down-classes. Let $d > 0$. Then

$$/a, d/(t) = \left[\left(\frac{t-a}{d} + 1 \right) \right]_0^1 = \begin{cases} 1 & \text{if } t \geq a, \\ \frac{t-a}{d} + 1 & \text{if } a-d \leq t \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

$$\backslash a, d \backslash (t) = \left[\left(\frac{a-t}{d} + 1 \right) \right]_0^1 = \begin{cases} 1 & \text{if } t \leq a, \\ \frac{a-t}{d} + 1 & \text{if } a \leq t \leq a+d, \\ 0 & \text{otherwise.} \end{cases}$$

For $d = 0$ we define

$$/a, 0/(t) = \begin{cases} 1 & \text{if } t \geq a, \\ 0 & \text{otherwise,} \end{cases} \quad \backslash a, 0 \backslash (t) = \begin{cases} 1 & \text{if } t \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we require in $/a, d/$ (resp. $\backslash a, d \backslash$) the coefficient d to be nonnegative. Typical examples of $/a, d/$ and $\backslash a, d \backslash$ are depicted in Figure 1.

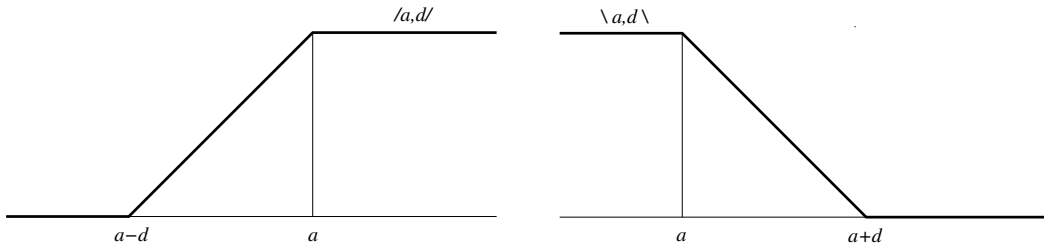


Fig. 1. Typical examples of $/a, d/$ and $\backslash a, d \backslash$.

Definition 29 *Let $a_1, a_2, d_1, d_2 \in \mathbb{R}$ such that $a_1 \leq a_2$ and $d_1, d_2 \geq 0$. Then the trapezoidal fuzzy interval $\langle /a_1, d_1/, \backslash a_2, d_2 \backslash \rangle$ is the intersection $/a_1, d_1/ \cap$*

$\backslash a_2, d_2 \backslash$ of the above defined fuzzy classes, i.e.,

$$\langle /a_1, d_1/, \backslash a_2, d_2 \backslash \rangle(t) = (/a_1, d_1/(t) + \backslash a_2, d_2 \backslash(1-t)) \vee 0 = /a_1, d_1/(t) \wedge \backslash a_2, d_2 \backslash(1-t).$$

A typical example of a trapezoidal fuzzy interval $\langle /a_1, d_1/, \backslash a_2, d_2 \backslash \rangle$ is depicted in Figure 2.

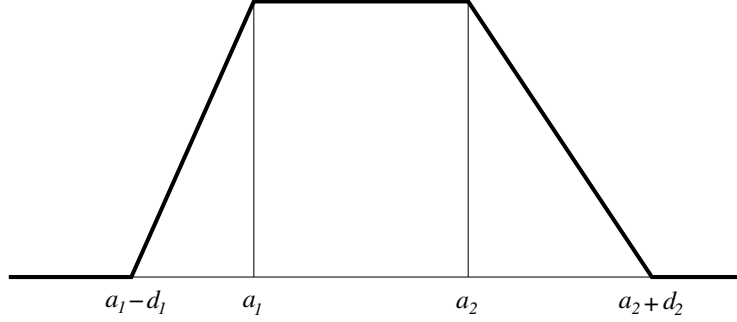


Fig. 2. A typical example of a trapezoidal fuzzy interval $\langle /a_1, d_1/, \backslash a_2, d_2 \backslash \rangle$.

Obviously we have the following lemma.

Lemma 30 *Let $\langle /a_1, d_1/, \backslash a_2, d_2 \backslash \rangle$ be a fuzzy trapezoidal interval. Then its up-class and down-class can be expressed as follows:*

$$\langle /a_1, d_1/, \backslash a_2, d_2 \backslash \rangle^\uparrow = /a_1, d_1/, \quad \langle /a_1, d_1/, \backslash a_2, d_2 \backslash \rangle^\downarrow = \backslash a_2, d_2 \backslash.$$

Further, we need to describe the arithmetical operations with trapezoidal fuzzy intervals.

Lemma 31 *The arithmetical operations with up-classes (resp. down-classes) of trapezoidal fuzzy intervals can be characterized as follows:*

$$/a, d/x = \begin{cases} /ax, dx/ & \text{if } x > 0, \\ \backslash ax, d|x| \backslash & \text{if } x < 0, \\ \{0\} & \text{otherwise.} \end{cases} \quad \backslash a, d \backslash x = \begin{cases} \backslash ax, dx \backslash & \text{if } x > 0, \\ /ax, d|x|/ & \text{if } x < 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

$$/a, d/ + /b, e/ = /a + b, d \vee e/, \quad \backslash a, d \backslash + \backslash b, e \backslash = \backslash a + b, d \vee e \backslash.$$

PROOF. Firstly, if $x = 0$ then $Ax = \{0\}$ for any fuzzy class A . Suppose that $x > 0$. Then it is not difficult to check that $(/a, d/x)(t) = /a, d/(t/x) = /ax, dx/(t)$. The case for $x < 0$ similar (only note that $dx = -d|x|$).

The second statement follows from already known facts about addition of fuzzy intervals (see e.g. [14, Theorem 6]). \square

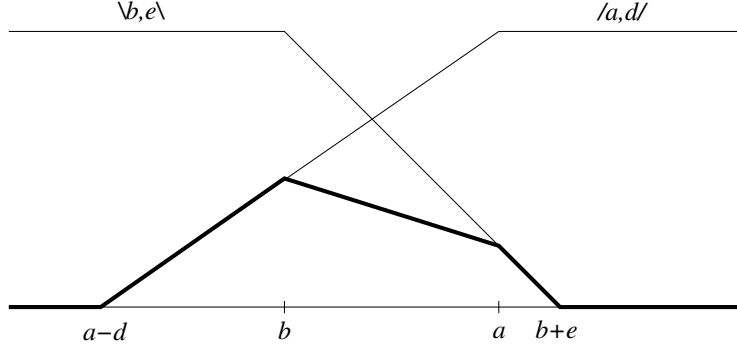


Fig. 3. The intersection of the up-class $/a, d/$ and the down-class $\backslash b, e \backslash$.

Lemma 32 *The degree of compatibility of an up-class and a down-class of trapezoidal fuzzy intervals can be computed as follows:*

$$/a, d/ \parallel \backslash b, e \backslash = /a, d/(b) \vee \backslash b, e \backslash(a).$$

Especially, we have

$$\begin{aligned} /a, 0/ \parallel \backslash b, e \backslash &= \backslash b, e \backslash(a), \\ /a, d/ \parallel \backslash b, 0 \backslash &= /a, d/(b), \\ /a, 0/ \parallel \backslash b, 0 \backslash &= \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{if } a > b. \end{cases} \end{aligned}$$

PROOF. Firstly, recall that the degree of compatibility is in fact the height of the intersection. Assume that $d, e \neq 0$. Then we have $/a, d/ \cap \backslash b, e \backslash = (/a, d/ + \backslash b, e \backslash - 1) \vee 0$. A typical situation how the intersection may look like is depicted in Figure 3. We have to prove that the function $(/a, d/ + \backslash b, e \backslash - 1) \vee 0$ attains at a or b its maximal value. Since both $/a, d/$ and $\backslash b, e \backslash$ are continuous piecewise linear, the function $/a, d/ + \backslash b, e \backslash - 1$ is also continuous and piecewise linear. Moreover, the only possible places for the maximum of $/a, d/ + \backslash b, e \backslash - 1$ are the points where two pieces are joined together, i.e., $a-d, a, b, b+e$. Clearly, $a-d$ and $b+e$ are not interesting since both of them belong to the intersection to degree 0. The cases when at least one of d, e equals 0 are trivial. \square

Now we will reformulate again the task that we want to solve. Let $\mathbf{A}\vec{x} = \vec{B}$ be a fuzzy interval linear system such that $\mathbf{A} = (A_{ij})$ is an $m \times n$ matrix of trapezoidal fuzzy intervals and \vec{B} is a vector of trapezoidal fuzzy intervals. We are going to find the united solution set $\Xi(\mathbf{A}^{\exists}, \vec{B}^{\exists})$ of the system $\mathbf{A}\vec{x} = \vec{B}$. Before we state the final theorem describing the united solution set, we introduce a further notation. Let $\vec{T} = \langle T_1, \dots, T_m \rangle$ be a tuple of fuzzy classes. Then $\cap \vec{T} = \cap_{i=1}^m T_i$. For a vector of reals $\vec{x} = \langle x_1, \dots, x_n \rangle$, we define $|\vec{x}| = \langle |x_1|, \dots, |x_n| \rangle$. Further, let $\mathbb{A} = (A_{ij})$ be an $m \times n$ matrix and $\vec{x} = \langle x_1, \dots, x_n \rangle$

a vector of reals. Then

$$\mathbb{A} \odot \vec{x} = \begin{pmatrix} A_{11}x_1 \vee \cdots \vee A_{1n}x_n \\ \vdots \\ A_{m1}x_1 \vee \cdots \vee A_{mn}x_n \end{pmatrix}.$$

Thus $\mathbb{A} \odot \vec{x}$ corresponds in fact to the usual multiplication of a matrix and a vector where all sums are replaced by the operation \vee . Finally, define the following operation for $a, b \in \mathbb{R}$:

$$a \otimes b = \begin{cases} \left[\frac{a}{b} + 1 \right]_0^1 & \text{if } b \neq 0, \\ 1 & \text{if } b = 0, a \geq 0, \\ 0 & \text{if } b = 0, a < 0, \end{cases}$$

Note that if $a, b \geq 0$ then $a \otimes b = 1$. We extend this definitions also for vectors of numbers component-wise, i.e. $\langle a_1, \dots, a_n \rangle \otimes \langle b_1, \dots, b_n \rangle = \langle a_1 \otimes b_1, \dots, a_n \otimes b_n \rangle$. Recall also the notation from the previous section. The orthants are denoted by Q_k , $k \in K = \{\uparrow, \downarrow\}^n$, k_j denotes the j -th component of k , and $-k_j$ is defined as follows:

$$-k_j = \begin{cases} \downarrow & \text{if } k_j = \uparrow, \\ \uparrow & \text{if } k_j = \downarrow. \end{cases}$$

Let \mathbb{A} be a matrix. The element in the i -th row and j -th column will be also denoted by $(\mathbb{A})_{ij}$.

Theorem 33 *Let $\mathbf{A}\vec{x} = \vec{B}$ be a fuzzy interval linear system such that $\mathbf{A} = (A_{ij})$ is an $m \times n$ matrix of trapezoidal fuzzy intervals $A_{ij} = \langle /a_{ij}^\uparrow, c_{ij}^\uparrow /, \backslash a_{ij}^\downarrow, c_{ij}^\downarrow \backslash \rangle$ and $\vec{B} = \langle B_1, \dots, B_n \rangle$ is a tuple of trapezoidal fuzzy intervals $B_i = \langle /b_i^\uparrow, e_i^\uparrow /, \backslash b_i^\downarrow, e_i^\downarrow \backslash \rangle$. Then the solution set $\Xi(\mathbf{A}^\exists, \vec{B}^\exists)$ can be described as follows:*

$$\Xi(\mathbf{A}^\exists, \vec{B}^\exists) = \bigsqcup_{k \in K} \left(Q_k \cap \bigcap \vec{T}_k \cap \bigcap \vec{R}_k \right),$$

where

$$\vec{T}_k(\vec{x}) = \left(\vec{b}_\downarrow - \mathbb{A}_k^\uparrow \vec{x} \right) \otimes \left(\vec{e}_\downarrow \vee \left(\mathbb{C}_k^\uparrow \odot |\vec{x}| \right) \right),$$

$$\vec{R}_k(\vec{x}) = \left(\mathbb{A}_k^\downarrow \vec{x} - \vec{b}_\uparrow \right) \otimes \left(\vec{e}_\uparrow \vee \left(\mathbb{C}_k^\downarrow \odot |\vec{x}| \right) \right),$$

$\vec{b}_\downarrow = \langle b_1^\downarrow, \dots, b_n^\downarrow \rangle$, $\vec{b}_\uparrow = \langle b_1^\uparrow, \dots, b_n^\uparrow \rangle$, $\vec{e}_\downarrow = \langle e_1^\downarrow, \dots, e_n^\downarrow \rangle$, $\vec{e}_\uparrow = \langle e_1^\uparrow, \dots, e_n^\uparrow \rangle$, and \mathbb{A}_k^\uparrow , \mathbb{A}_k^\downarrow , \mathbb{C}_k^\uparrow , \mathbb{C}_k^\downarrow are $m \times n$ matrices such that $(\mathbb{A}_k^\uparrow)_{ij} = a_{ij}^{k_j}$, $(\mathbb{A}_k^\downarrow)_{ij} = a_{ij}^{-k_j}$, $(\mathbb{C}_k^\uparrow)_{ij} = c_{ij}^{k_j}$, $(\mathbb{C}_k^\downarrow)_{ij} = c_{ij}^{-k_j}$.

PROOF. According to Theorem 26 we have to prove that

$$Q_k \cap \bigcap \vec{T}_k \cap \bigcap \vec{R}_k = Q_k \cap \bigcap_{i=1}^m (S_{ik}^d \cap S_{ik}^u).$$

Firstly we can rewrite the expression $\bigcap_{i=1}^m (S_{ik}^d \cap S_{ik}^u)$ as $\bigcap_{i=1}^m S_{ik}^d \cap \bigcap_{i=1}^m S_{ik}^u$. We want to show that $\bigcap \vec{T}_k = \bigcap_{i=1}^m S_{ik}^d$ and $\bigcap \vec{R}_k = \bigcap_{i=1}^m S_{ik}^u$ for $\vec{x} \in Q_k$. Let fix $\vec{x} \in Q_k$. Then it is sufficient to prove that the truth-degree of $\vec{x} \in S_{ik}^d$ (resp. $\vec{x} \in S_{ik}^u$) equals i -th component of $\vec{T}_k(\vec{x})$ (resp. $\vec{R}_k(\vec{x})$). Recall that

$$\vec{x} \in S_{ik}^d \longleftrightarrow \left(\sum_{j=1}^n A_{ij}^{k_j} x_j \right) \parallel (B_i^\exists)^\downarrow.$$

We have $(B_i^\exists)^\downarrow = \setminus b_i^\downarrow, e_i^\downarrow \setminus$ and $A_{ij}^{k_j} = /a_{ij}^{k_j}, c_{ij}^{k_j}/$ if $k_j = \uparrow$ and $\setminus a_{ij}^{k_j}, c_{ij}^{k_j} \setminus$ otherwise. Assume that all $x_j \neq 0$. Then $k_j = \uparrow$ (i.e., $x_j \geq 0$) implies $A_{ij}^{k_j} x_j = /a_{ij}^{k_j} x_j, c_{ij}^{k_j} x_j/$ and $k_j = \downarrow$ (i.e., $x_j \leq 0$) implies $A_{ij}^{k_j} x_j = /a_{ij}^{k_j} x_j, c_{ij}^{k_j} |x_j|/$. Thus we can write $A_{ij}^{k_j} x_j = /a_{ij}^{k_j} x_j, c_{ij}^{k_j} |x_j|/$ for both values of k_j . Consequently, we get by Lemma 31

$$\sum_{j=1}^n A_{ij}^{k_j} x_j = \sum_{j=1}^n /a_{ij}^{k_j} x_j, c_{ij}^{k_j} |x_j|/ = / \sum_{j=1}^n a_{ij}^{k_j} x_j, \bigvee_{j=1}^n c_{ij}^{k_j} |x_j| / = /(\mathbb{A}_k^\uparrow)_i \vec{x}, (\mathbb{C}_k^\uparrow)_i \odot |x_j| /, \quad (7)$$

where $(\mathbb{A}_k^\uparrow)_i$ is the $1 \times n$ matrix constituted from i -th row of \mathbb{A}_k^\uparrow and similarly for $(\mathbb{C}_k^\uparrow)_i$. Observe that Equation (7) holds also in the case when some (but not all) $x_j = 0$ by Lemma 19(3) (in particular we use $/a, d/ + \{0\} = /a, d/ + /0, 0/$). Thus the only case when Equation (7) does not hold is the case when all $x_j = 0$. Then $\sum_{j=1}^n A_{ij}^{k_j} x_j = \{0\}$.

According to Lemma 32 if we want to evaluate the degree of compatibility between $\sum_{j=1}^n A_{ij}^{k_j} x_j$ and $(B_i^\exists)^\downarrow$ it is sufficient to compute the following maximum:

$$/(\mathbb{A}_k^\uparrow)_i \vec{x}, (\mathbb{C}_k^\uparrow)_i \odot |x_j| / (b_i^\downarrow) \vee \setminus b_i^\downarrow, e_i^\downarrow \setminus ((\mathbb{A}_k^\uparrow)_i \vec{x}). \quad (8)$$

Note that we can use the latter formula also for the case when all $x_j = 0$ by Lemma 19(4) since then we can replace $\{0\}$ by $/0, 0/$.

We want to show that the maximum from (8) equals to the i -th component of \vec{T}_k . Recall that the i -th component of \vec{T}_k equals

$$(\vec{T}_k)_i = (b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}) \odot (e_i^\downarrow \vee ((\mathbb{C}_k^\uparrow)_i \odot |\vec{x}|)).$$

There are several cases. Firstly, suppose $(\mathbb{C}_k^\uparrow)_i \odot |x_j| > 0$ and $e_i^\downarrow > 0$. Then (8)

can be rewritten as follows:

$$\begin{aligned}
\left[\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{(\mathbb{C}_k^\uparrow)_i \odot |x_j|} + 1 \right]_0^1 \vee \left[\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{e_i^\downarrow} + 1 \right]_0^1 = \\
\left[\left(\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{(\mathbb{C}_k^\uparrow)_i \odot |x_j|} + 1 \right) \vee \left(\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{e_i^\downarrow} + 1 \right) \right]_0^1 = \\
\left[\left(\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{(\mathbb{C}_k^\uparrow)_i \odot |x_j|} \vee \frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{e_i^\downarrow} \right) + 1 \right]_0^1. \quad (9)
\end{aligned}$$

If $b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x} \geq 0$ then (9) equals 1 since $(\mathbb{C}_k^\uparrow)_i \odot |x_j| > 0$ and $e_i^\downarrow > 0$. Also $(\vec{T}_k)_i$ equals 1 in this case. Thus assume that $b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x} < 0$. Then (9) can be further rewritten as follows:

$$\begin{aligned}
\left[\left(\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{(\mathbb{C}_k^\uparrow)_i \odot |x_j|} \vee \frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{e_i^\downarrow} \right) + 1 \right]_0^1 = \\
\left[(b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}) \left(\frac{1}{(\mathbb{C}_k^\uparrow)_i \odot |x_j|} \wedge \frac{1}{e_i^\downarrow} \right) + 1 \right]_0^1 = \\
\left[\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{e_i^\downarrow \vee ((\mathbb{C}_k^\uparrow)_i \odot |x_j|)} + 1 \right]_0^1 = (\vec{T}_k)_i. \quad (10)
\end{aligned}$$

Now assume that $(\mathbb{C}_k^\uparrow)_i \odot |x_j| = 0$ and $e_i^\downarrow > 0$. Then by Lemma 32 we have that (8) equals

$$\left[\frac{b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}}{e_i^\downarrow} + 1 \right]_0^1 = (b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x}) \odot e_i^\downarrow = (\vec{T}_k)_i.$$

Similarly we can argue in the case when $(\mathbb{C}_k^\uparrow)_i \odot |x_j| > 0$ and $e_i^\downarrow = 0$. Finally, if $(\mathbb{C}_k^\uparrow)_i \odot |x_j| = 0$ and $e_i^\downarrow = 0$. Then both (8) and $(\vec{T}_k)_i$ equals 1 if $b_i^\downarrow - (\mathbb{A}_k^\uparrow)_i \vec{x} \geq 0$ and 0 otherwise.

The proof of the fact that the i -th component of $\vec{R}_k(\vec{x})$ is equivalent to the truth-degree of $\vec{x} \in S_{ik}^u$ is completely analogous. \square

8 Example

In this section we are going to illustrate Theorem 33 on a concrete example. We will in fact consider a fuzzified version of a favorite example from classical interval analysis.

Example 34 Consider the following classical interval linear system:

$$\begin{pmatrix} [2, 4] & [-2, 1] \\ [-1, 2] & [2, 4] \end{pmatrix} \vec{x} = \begin{pmatrix} [-2, 2] \\ [-2, 2] \end{pmatrix}. \quad (11)$$

The united solution set of this system is depicted in Figure 4.

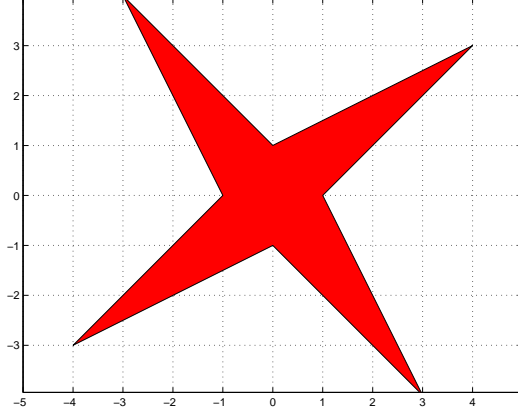


Fig. 4. The united solution set $\Xi(\mathbf{A}^\exists, \vec{B}^\exists)$ of the system from Example 35.

Now we fuzzify the latter example and then describe its united solution set.

Example 35 Consider the following fuzzy interval linear system with trapezoidal fuzzy intervals:

$$\begin{pmatrix} \langle /2, \frac{1}{2}/, \backslash 4, \frac{1}{2} \backslash \rangle & \langle /-2, \frac{1}{2}/, \backslash 1, \frac{1}{2} \backslash \rangle \\ \langle /-1, \frac{1}{2}/, \backslash 2, \frac{1}{2} \backslash \rangle & \langle /2, \frac{1}{2}/, \backslash 4, \frac{1}{2} \backslash \rangle \end{pmatrix} \vec{x} = \begin{pmatrix} \langle /-2, \frac{1}{2}/, \backslash 2, \frac{1}{2} \backslash \rangle \\ \langle /-2, \frac{1}{2}/, \backslash 2, \frac{1}{2} \backslash \rangle \end{pmatrix} \quad (12)$$

In this case $n = 2$. Thus we have four orthants:

$$\begin{aligned} Q_1 &= \{\vec{x} \mid x_1 \geq 0 \ \& \ x_2 \geq 0\}, & Q_2 &= \{\vec{x} \mid x_1 \leq 0 \ \& \ x_2 \geq 0\}, \\ Q_3 &= \{\vec{x} \mid x_1 \leq 0 \ \& \ x_2 \leq 0\}, & Q_4 &= \{\vec{x} \mid x_1 \geq 0 \ \& \ x_2 \leq 0\}. \end{aligned}$$

Due to simplicity we replaced the subscripts of orthants by numbers, i.e. $Q_1 = Q_{\{\uparrow, \uparrow\}}$, $Q_2 = Q_{\{\downarrow, \uparrow\}}$, etc. Similarly we denote other object e.g. $\mathbb{A}_2^\uparrow = \mathbb{A}_{\{\downarrow, \uparrow\}}^\uparrow$.

Since the united solution set is in fact the union over all orthants, we describe the united solution set only in Q_2 . This means that we compute \vec{T}_2 and \vec{R}_2 . We have

$$\vec{b}_\downarrow = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \vec{b}_\uparrow = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \quad \vec{e}_\downarrow = \vec{e}_\uparrow = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

$$\mathbb{A}_2^\uparrow = \begin{pmatrix} 4 & -2 \\ 2 & 2 \end{pmatrix}, \quad \mathbb{A}_2^\downarrow = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}, \quad \mathbb{C}_2^\uparrow = \mathbb{C}_2^\downarrow = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then

$$\begin{aligned} \vec{T}_2(\vec{x}) &= \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ 2 & 2 \end{pmatrix} \vec{x} \right) \oslash \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \vee \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \odot |\vec{x}| \right) \\ &= \begin{pmatrix} 2 - (4x_1 - 2x_2) \\ 2 - (2x_1 + 2x_2) \end{pmatrix} \oslash \begin{pmatrix} \frac{1}{2} \vee \frac{1}{2}|x_1| \vee \frac{1}{2}|x_2| \\ \frac{1}{2} \vee \frac{1}{2}|x_1| \vee \frac{1}{2}|x_2| \end{pmatrix}, \\ \vec{R}_2(\vec{x}) &= \left(\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \vec{x} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right) \oslash \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \vee \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \odot |\vec{x}| \right) \\ &= \begin{pmatrix} 2x_1 + x_2 + 2 \\ -x_1 + 4x_2 + 2 \end{pmatrix} \oslash \begin{pmatrix} \frac{1}{2} \vee \frac{1}{2}|x_1| \vee \frac{1}{2}|x_2| \\ \frac{1}{2} \vee \frac{1}{2}|x_1| \vee \frac{1}{2}|x_2| \end{pmatrix}. \end{aligned}$$

The united solution set $\Xi(\mathbf{A}^\exists, \vec{B}^\exists)$ is depicted in Figure 5. Notice that the kernel of $\Xi(\mathbf{A}^\exists, \vec{B}^\exists)$ is the same as the united solution set from Example 34.

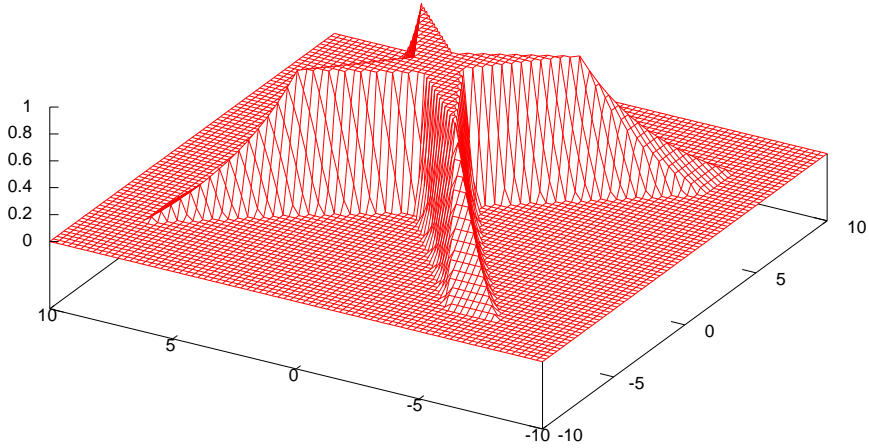


Fig. 5. The united solution set $\Xi(\mathbf{A}^\exists, \vec{B}^\exists)$ of the system from Example 35.

9 Conclusions

In this paper we have described a solution set for a fuzzy interval linear system. As a background theory we have used the Fuzzy Class Theory (FCT) developed in [1] which helps substantially with the formalization and the complex reasoning about the fuzzy interval linear systems and their solution sets. Thus FCT seems to be a good candidate also for further research in this direction.

The connection of our results with those coming from classical interval analysis is well visible in Theorem 33. It is known that the united solution set of a classical crisp interval linear system forms a convex polyhedron in each orthant (see [20, Theorem 3.6]). In fact, this is valid for any solution set as it was defined here (i.e., where all universal quantifiers are followed by the existential ones). Thus the solution set is described in each orthant by a system of linear inequalities. In Theorem 33 this system of inequalities is reflected in the first parts of the expressions for $\vec{T}_k(\vec{x})$ and $\vec{R}_k(\vec{x})$ respectively, i.e., $\vec{b}_\downarrow - \mathbb{A}_k^\uparrow \vec{x}$ and $\mathbb{A}_k^\downarrow \vec{x} - \vec{b}_\uparrow$. The second parts (i.e., $\vec{e}_\downarrow \vee (\mathbb{C}_k^\uparrow \odot |\vec{x}|)$ and $\vec{e}_\uparrow \vee (\mathbb{C}_k^\downarrow \odot |\vec{x}|)$) only represent the fuzziness of the resulting solution set. Thus they have no influence on the result when all intervals appearing in the linear system are crisp.

Let us also discuss the possible future work. The classical interval analysis is of course much more developed as a theory than its fuzzy generalization. Thus it would be quite natural to generalize also other results to fuzzy logics. For instance it would be interesting to introduce somehow Kaucher's interval arithmetic on fuzzy intervals. Kaucher's arithmetic in classical interval analysis is used for computing outer and inner estimations of the solution set since the complete computation of the solution set is very complex due to the fact that the number of orthants grows exponentially with the dimension of underlying vector space \mathbb{R}^n . However, the direct generalization to the fuzzy case is not possible since the addition in classical interval arithmetic is cancellative unlike the addition of fuzzy intervals (for details see [20]).

Another possible direction is to find a description of the general solution set as defined in Definition 22. As we mentioned at the beginning of Section 6, in this case it is not possible to separate the computation of the solution set according to the particular equations. The problem consists in the definition of the inclusion between two tuples of fuzzy classes \vec{A}, \vec{B} that appeared at the end of Subsection 2.2. The definition in fact states that $\vec{A} \subseteq \vec{B}$ is equivalent to the inclusion of the corresponding cartesian products of components of \vec{A} and \vec{B} . If we want to separate the particular equations, we have to know that the inclusion of the cartesian products is equivalent to the conjunction of the inclusions of particular components of \vec{A} and \vec{B} . But in Fuzzy Class Theory it is not difficult to find a counterexample showing that this is not true in general.

Let A be a singleton containing an element a to degree $\frac{1}{2}$ and $b \neq a$. Then in Łukasiewicz logic $\langle A, A \rangle \subseteq \langle \{b\}, \{b\} \rangle$ holds to degree 1 since $a \in A \ \& \ a \in A$ holds to degree 0 (i.e., $A \times A = \emptyset$). On the other hand $A \subseteq \{b\}$ does not hold to degree 1 since the formula $a \in A \rightarrow a \in \{b\}$ holds only to degree $\frac{1}{2}$. Thus $A \subseteq \{b\} \ \& \ A \subseteq \{b\}$ is not valid at all (the truth degree is 0). Consequently, if we want to use the fundamental theorem for computing a solution set of fuzzy linear system, we have to work with all equations simultaneously.

Appendix

Here we collect syntactical proofs of the lemmata from Section 3 which were stated without proofs. In most cases the proofs are based on three steps. Firstly, the definitions are expanded. Secondly, some quantifier shifts are applied and finally some properties of real numbers are used.

Lemma 17 *The following hold for any fuzzy classes A, B :*

$$A^\downarrow + B^\downarrow = (A + B)^\downarrow, \quad A^\uparrow + B^\uparrow = (A + B)^\uparrow.$$

PROOF. We will prove only the first part of the statement. The second one is similar.

$$\begin{aligned} z \in A^\downarrow + B^\downarrow &\longleftrightarrow (\exists a' \in A^\downarrow)(\exists b' \in B^\downarrow)(z = a' + b') \longleftrightarrow \\ &(\exists a')(\exists b') ((\exists a \in A)(a' \leq a) \ \& \ (\exists b \in B)(b' \leq b) \ \& \ z = a' + b') \longleftrightarrow \\ &(\exists a')(\exists b')(\exists a)(\exists b) (a \in A \ \& \ b \in B \ \& \ a' \leq a \ \& \ b' \leq b \ \& \ z = a' + b') \longleftrightarrow \\ &(\exists a \in A)(\exists b \in B)(\exists a')(\exists b')(a' \leq a \ \& \ b' \leq b \ \& \ z = a' + b') \longleftrightarrow \\ &(\exists a \in A)(\exists b \in B)(z \leq a + b). \end{aligned}$$

Now we will show that also $z \in (A + B)^\downarrow$ is equivalent to the formula $(\exists a \in A)(\exists b \in B)(z \leq a + b)$.

$$\begin{aligned} z \in (A + B)^\downarrow &\longleftrightarrow (\exists c \in A + B)(z \leq c) \longleftrightarrow \\ &(\exists c) ((\exists a \in A)(\exists b \in B)(c = a + b) \ \& \ z \leq c) \longleftrightarrow \\ &(\exists a \in A)(\exists b \in B)(\exists c)(c = a + b \ \& \ z \leq c) \longleftrightarrow \\ &(\exists a \in A)(\exists b \in B)(z \leq a + b). \quad \square \end{aligned}$$

Lemma 18 *Let A be a fuzzy class and $k \in \mathbb{R}$. If $k > 0$ then $(kA)^\downarrow = kA^\downarrow$ and $(kA)^\uparrow = kA^\uparrow$. If $k < 0$ then $(kA)^\downarrow = kA^\uparrow$ and $(kA)^\uparrow = kA^\downarrow$.*

PROOF. We will prove only the claim $(kA)^\downarrow = kA^\uparrow$ for $k < 0$. The proofs of other claims are analogous.

$$\begin{aligned}
x \in kA^\uparrow &\longleftrightarrow (\exists a' \in A^\uparrow)(x = ka') \longleftrightarrow (\exists a') ((\exists a \in A)(a' \geq a) \& x = ka') \longleftrightarrow \\
(\exists a \in A)(\exists a')(a' \geq a \& x = ka') &\longleftrightarrow (\exists a \in A) \left(\frac{x}{k} \geq a \right) \longleftrightarrow (\exists a \in A)(x \leq ka).
\end{aligned}$$

In the last equivalence we use the fact that $k < 0$. Now we will show that also $x \in (kA)^\downarrow$ is equivalent to the formula $(\exists a \in A)(x \leq ka)$.

$$\begin{aligned}
x \in (kA)^\downarrow &\longleftrightarrow (\exists a' \in kA)(x \leq a') \longleftrightarrow (\exists a') ((\exists a \in A)(a' = ka) \& x \leq a') \longleftrightarrow \\
(\exists a \in A)(\exists a')(a' = ka \& x \leq a') &\longleftrightarrow (\exists a \in A)(x \leq ka).
\end{aligned}$$

□

Lemma 19 *Let A, B be fuzzy classes. Then the following is provable in FCT:*

- (1) $A \leq B \leftrightarrow A^\uparrow \parallel B^\downarrow$,
- (2) $0 \in B^\downarrow - A^\uparrow \leftrightarrow A \leq B$ and $0 \in B^\uparrow - A^\downarrow \leftrightarrow B \leq A$,
- (3) $A^\downarrow + \{0\}^\downarrow = A^\downarrow$ and $A^\uparrow + \{0\}^\uparrow = A^\uparrow$,
- (4) $\{0\}^\uparrow \parallel B^\downarrow \leftrightarrow \{0\} \parallel B^\downarrow$ and $\{0\}^\downarrow \parallel B^\uparrow \leftrightarrow \{0\} \parallel B^\uparrow$.

PROOF.

(1) We have

$$\begin{aligned}
A^\uparrow \parallel B^\downarrow &\longleftrightarrow (\exists x)(x \in A^\uparrow \& x \in B^\downarrow) \longleftrightarrow \\
(\exists x)((\exists a \in A)(x \geq a) \& (\exists b \in B)(x \leq b)) &\longleftrightarrow \\
(\exists x)(\exists a)(\exists b)(a \in A \& b \in B \& x \geq a \& x \leq b) &\longleftrightarrow \\
(\exists a)(\exists b)(a \in A \& b \in B \& (\exists x)(x \geq a \& x \leq b)) &\longleftrightarrow \\
(\exists a)(\exists b)(a \in A \& b \in B \& a \leq b) &\longleftrightarrow A \leq B.
\end{aligned}$$

(2) We will prove only the first part of the lemma since the second one is similar.

$$\begin{aligned}
0 \in B^\downarrow - A^\uparrow &\longleftrightarrow (\exists x \in B^\downarrow)(\exists y \in A^\uparrow)(0 = x - y) \longleftrightarrow \\
(\exists x)(\exists y)((\exists b \in B)(x \leq b) \& (\exists a \in A)(y \geq a) \& x = y) &\longleftrightarrow \\
(\exists a)(\exists b)(b \in B \& a \in A \& (\exists x)(\exists y)(x \leq b \& y \geq a \& x = y)) &\longleftrightarrow \\
(\exists a \in A)(\exists b \in B)(a \leq b) &\longleftrightarrow A \leq B.
\end{aligned}$$

(3) We will prove only the first part of the lemma since the second one is

analogous.

$$\begin{aligned}
z \in A^\downarrow + \{0\}^\downarrow &\longleftrightarrow (\exists a' \in A^\downarrow)(\exists b' \in \{0\}^\downarrow)(z = a' + b') \longleftrightarrow \\
&(\exists a')(\exists b')((\exists a \in A)(a' \leq a) \& (\exists b \in \{0\})(b' \leq b) \& z = a' + b') \longleftrightarrow \\
&(\exists a')(\exists b')(\exists a)(a \in A \& a' \leq a \& b' \leq 0 \& z = a' + b') \longleftrightarrow \\
&(\exists a)(a \in A \& (\exists a')(\exists b')(a' \leq a \& b' \leq 0 \& z = a' + b')) \longleftrightarrow \\
&(\exists a \in A)(z \leq a) \longleftrightarrow z \in A^\downarrow.
\end{aligned}$$

(4) We will prove again only the first statement.

$$\begin{aligned}
\{0\}^\uparrow \parallel B^\downarrow &\longleftrightarrow (\exists x)(x \in \{0\}^\uparrow \& x \in B^\downarrow) \longleftrightarrow \\
&(\exists x)(x \geq 0 \& (\exists b)(b \in B \& x \leq b)) \longleftrightarrow \\
&(\exists b)(b \in B \& (\exists x)(x \geq 0 \& x \leq b)) \longleftrightarrow \\
&(\exists b \in B)(0 \leq b) \longleftrightarrow 0 \in B^\downarrow \longleftrightarrow \{0\} \parallel B^\downarrow,
\end{aligned}$$

where we use the validity of the formula $(\exists x)(x \geq 0 \& x \leq b) \leftrightarrow 0 \leq b$. \square

References

- [1] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.
- [2] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: a methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.
- [3] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.
- [4] L. Biacino and A. Lettieri. Equations with fuzzy numbers. *Information Sciences*, 47:63–76, 1989.
- [5] J. J. Buckley and Y. Qu. Solving linear and quadratic fuzzy equations. *Fuzzy Sets and Systems*, 38:43–59, 1990.
- [6] J. J. Buckley and Y. Qu. Solving fuzzy equations: A new solution concept. *Fuzzy Sets and Systems*, 39:291–301, 1991.
- [7] J. J. Buckley and Y. Qu. Solving systems of linear fuzzy equations. *Fuzzy Sets and Systems*, 43:33–43, 1991.
- [8] D. Dubois and H. Prade. Fuzzy-set-theoretic differences and inclusions and their use in the analysis of fuzzy equations. *Control and Cybernetics*, 13:129–146, 1984.

- [9] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124:271–288, 2001.
- [10] P. Hájek. *Metamathematics of fuzzy logic*, volume 4 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 1998.
- [11] M. Inuiguchi, J. Ramík, T. Tanino, and M. Vlach. Satisficing solutions and duality in interval and fuzzy linear programming. *Fuzzy Sets and Systems*, 135:151–177, 2003.
- [12] M. Inuiguchia, J. Ramík, and T. Tanino. Oblique fuzzy vectors and their use in possibilistic linear programming. *Fuzzy Sets and Systems*, 135:123–150, 2003.
- [13] W. A. Lodwick and K. D. Jamison. Special issue: interfaces between fuzzy set theory and interval analysis. *Fuzzy Sets and Systems*, 135:1–3, 2003.
- [14] R. Mesiar. Triangular-norm-based addition of fuzzy intervals. *Fuzzy Sets and Systems*, 91:231–237, 1997.
- [15] R. Moore and W. Lodwick. Interval analysis and fuzzy set theory. *Fuzzy Sets and Systems*, 135:5–9, 2003.
- [16] R. E. Moore. *Methods and Applications of Interval Analysis*. SIAM, Philadelphia, 1979.
- [17] A. Neumaier. *Interval Methods for Systems of Equations*. Cambridge University Press, Cambridge, 1990.
- [18] W. Oettli. On the solution set of a linear system with inaccurate coefficients. *SIAM Journal on Numerical Analysis*, 2:115–118, 1965.
- [19] E. Sanchez. Solution of fuzzy equations with extended operations. *Fuzzy Sets and Systems*, 12:237–248, 1984.
- [20] S. P. Shary. A new technique in systems analysis under interval uncertainty and ambiguity. *Reliable Computing*, 8:321–418, 2002.
- [21] J. Wasowski. On solution of fuzzy equations. *Control and Cybernetics*, 26:653–658, 1997.
- [22] R. Zhao and R. Govind. Solutions of algebraic equations involving generalized fuzzy numbers. *Information Sciences*, 56:199–243, 1991.