Formal systems of fuzzy logic and their fragments*

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Abstract

Formal systems of fuzzy logic (including the well-known Łukasiewicz and Gödel-Dummett infinite-valued logics) are well-established logical systems and respected members of the broad family of the so-called substructural logics closely related to the famous logic BCK. The study of fragments of logical systems is an important issue of research in any class of non-classical logics. Here we study the fragments of nine prominent fuzzy logics to all sublanguages containing implication. However, the results achieved in the paper for those nine logics are usually corollaries of theorems with much wider scope of applicability. In particular, we show how many of these fragments are really distinct and we find axiomatic systems for most of them. In fact, we construct strongly separable axiomatic systems for eight of our nine logics. We also fully answer the question for which of the studied fragments the corresponding class of algebras forms a variety. Finally, we solve the problem how to axiomatize predicate versions of logics without the lattice disjunction (an essential connective in the usual axiomatic system of fuzzy predicate logics).

Keywords: mathematical fuzzy logic, BCK-algebras, BCK, FBCK, monoidal t-norm logic.

1 Introduction

Many formal systems of fuzzy logic (or simply just *fuzzy logics*) were introduced and developed in the last decades. Also some well-established many-valued logics, like Łukasiewicz ([42, 41]) or Gödel-Dummett logic ([20, 10]), have been adopted into a general framework of fuzzy logics (as extensions of Hájek's Basic Fuzzy Logic). Since Hájek's monograph [22] appeared, the fuzzy logics have been considered *mathematical and formal* non-classical logics *sui juris*. After then the above mentioned dozens of logical systems were introduced and studied, with contributions made by prominent logicians (like A. Avron, M. Baaz, R. Cignoli, D. Gabbay, M. Laskowski, F. Montagna, D. Mundici, H. Ono, J. Paris, J.C. Shepherdson, etc.). During these years some of the logics have become central and more studied than others and this paper is aimed to contribute to the development of nine of these prominent logics (see Subsection 2.3 for their formal definitions). Of course the selection of 'prominent' logics is always questionable, to overcome this issue we tried to prove our results in as general form as possible so they are surely applicable to much wider classes of logics as well.

It turned out that these prominent fuzzy logics are natural expansions of the famous logic BCK. This logic was introduced by C.A. Meredith (see e.g. [48, 40]) as a pure implicational logic being contained in many important logics (classical, intuitionistic, positive, etc.). Precisely it is the implicational fragment of Full Lambek Logic with exchange and weakening (or Intuitionistic Affine Linear Logic). One of the results of this paper is that by adding just one axiom to the axioms of BCK we axiomatize the implicative fragment of the fuzzy logic MTL.

This paper can be read in two different ways by two different groups of readers. First, the readers familiar with and/or interested in fuzzy logics can read this paper in a top-to-bottom

*The work of the first and the second author was supported by project 1M0021620808 of the Ministry of Education, Youth and Sports of the Czech Republic. The third author was partly supported by the Program "Information Society" under project 1ET100300517 and partly by Institutional Research Plan AV0Z10300504.

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fashion—starting with the mentioned fuzzy logics, we study a class of their fragments to propositional languages lacking some connectives. Second, the readers familiar with and/or interested in the logic BCK or implicational fragments of other non-classical logics (Linear, Intuitionistic, etc.) can read this paper in a bottom-to-top fashion—starting with BCK, we study a class of its expansions to languages with more connectives.

1.1 Bottom-to-top fashion: from BCK to fuzzy logics

It is well-known that the corresponding algebraic semantics of the logic BCK is the class of BCK-algebras (introduced by Imai and Iséki in [32], see also survey paper [34]) and that we can define an order in each BCK-algebra. If this order is linear then we call such an algebra a BCK-chain. Finally, it is also known (see [16]) how to axiomatically characterize subdirect products of BCK-chains—we need to add the axiom of prelinearity:

$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi).$$

The logic BCK plus this axiom of prelinearity will be the starting point for us in this paper—we call this logic *Fuzzy* BCK logic (FBCK for short). This logic is obviously complete with respect to the BCK-chains and this is the rationale for the name "Fuzzy BCK", as the authors believe that completeness w.r.t. chains is the constitutive feature of all fuzzy logics (see Subsection 2.1 and especially papers [8, 4] where mathematical, philosophical, and methodological reasons for this belief are presented).

The expansions of BCK to richer propositional languages was thoroughly studied in the famous paper by Ono and Komori [46] (see also older papers by Idziak [30, 31]). They construct a class of logics corresponding to expansions of BCK in all propositional languages containing implication and some of the following connectives: $\&, \vee, \wedge, \overline{0}$. In Section 3 we study the analogous class of logics expanding FBCK, it turns out that many of these logics coincide with some fragments of prominent fuzzy logics (see Section 4 for more details).

The added axiom of prelinearity makes some things simpler and some more complex. The complication is caused by the fact that we cannot base our study on some forms of sequent calculi (central in [46]) as the prelinearity axioms forces us to move to hyper-sequent calculi (originated by Avron in [3], for a survey of these calculi for fuzzy logics see [43]). Also the Kripke semantics, used in [46] behaves in a rather peculiar way in these logics. Thus we decided to base our study on the Hilbert style axiomatizations and algebraic semantics. The simplification lies in the fact the we can restrict all our algebraic reasoning to linearly ordered algebras and that we can use some strong metalogical properties (e.g., Prelinearity Property).

There is a famous result by Wronski ([58]) stating that the class of BCK-algebras is not a variety. In fact the problem of characterizing which classes of BCK-algebras are varieties is central in the theory of BCK-algebras. In this paper we contribute to this topic by establishing a general theorem which answers this question for a rather wide class of logics, in particular for all the logics we study in this paper. The last two sections are devoted to the study of interdefinability of connectives and to the first-order variants of our logics. We find axiomatic systems for all these first-order logics (a rather non-trivial task as all the axiomatic systems of the usual first-order fuzzy logics rely on the lattice disjunction) and prove the completeness theorems.

1.2 Top-to-bottom fashion: from fuzzy logics to their fragments

One of the issues studied in the framework of fuzzy logics was the study of the $\overline{0}$-free fragments of these logics (see [27]). As fuzzy logics are usually presented in the language containing implication, strong conjunction, lattice conjunction and disjunction, and the truth constant for falsity, there is a natural question: what about the other fragments? As we always want to keep implication in our language (because the implication-less fragments of our fuzzy logics are essentially classical, see [1]) we are going to study fragments of nine different logic to 16 different propositional languages, i.e., 144 different logics. Obviously, not all these logics are distinct, some of them are termwise equivalent, as some of the connectives become definable. In Section 6 we fully solve this problem by showing that exactly 57 of these logics are mutually distinct.

For each of these logics we can find its corresponding class of algebras. It can be easily shown that all these classes are quasivarieties. This entails an interesting problem of characterizing which of these classes are in fact varieties. In Section 5 we fully solve this problem. As the notion of fragment is defined implicitly (as the logic proving the same formulas in the restricted language as the original logic) one of our main goals is to find an axiomatic system for each of our fragments. After we prepare (in Section 3) some crucial definitions and theorems and establish a connection between fuzzy logics and expansions of BCK, we solve most of the cases—only for six fragments we are unable to find their axiomatic systems (but these cases are strongly interrelated as a solution of one of them would lead to the solutions of others). At the end, we turn our attention to the first-order logics. In the usual approach to predicate fuzzy logics the lattice disjunction is a crucial connective needed for axiomatization of these logics. In the Section 7 we show how to overcome this obstacle in logics where this connective is not present. We prove the completeness theorem and show that our solution brings nothing new when the disjunction is present.

2 Preliminaries

2.1 Weakly implicative fuzzy logics

We start by recalling some definitions concerning general theory of logical calculi (for details see e.g. [56]). The notions of a propositional language $\mathcal{L}$ and $\mathcal{L}$-formula are defined as usual. By $\mathcal{L}$-theory we understand a set of $\mathcal{L}$-formulas. A *logic* $\mathbf{L}$ in the language $\mathcal{L}$ is a finitary structural consequence relation in the sense of Tarski on the set of $\mathcal{L}$-formulas (we omit $\mathcal{L}$ if the language is clear from the context).

We use the usual denotation $T \vdash_{\mathbf{L}} \varphi$ for the fact that a formula φ is a consequence of a theory T. Each logic $\mathbf{L}$ has an axiomatic system[1] and we define the notion of the proof in the usual way (as a sequence of formulas). Then we obtain $T \vdash_{\mathbf{L}} \varphi$ iff there is a proof of φ from the theory T. By $\mathbf{L} + \mathcal{A}$ we denote the logic resulting from $\mathbf{L}$ by adding a set of axioms $\mathcal{A}$ to its axiomatic system.

A logic $\mathbf{L}_2$ in a language $\mathcal{L}_2$ is said to be an *expansion* of $\mathbf{L}_1$ in $\mathcal{L}_1$ if for each $\mathcal{L}_1$-theory T and each $\mathcal{L}_1$-formula φ we have $T \vdash_{\mathbf{L}_1} \varphi$ implies $T \vdash_{\mathbf{L}_2} \varphi$. The expansion is *conservative* if also $T \vdash_{\mathbf{L}_2} \varphi$ implies $T \vdash_{\mathbf{L}_1} \varphi$; in this case we also say that $\mathbf{L}_1$ is the $\mathcal{L}_1$-fragment of $\mathbf{L}_2$ and denote $\mathbf{L}_1$ by $\mathbf{L}_2 \restriction \mathcal{L}_1$. Observe that the $\mathcal{L}_1$-fragment of $\mathbf{L}_2$ is uniquely determined. If the languages coincide we speak about *extension* rather than expansion; we say that an extension is *axiomatic* if there is a set of axioms $\mathcal{A}$ such that $\mathbf{L}_2 = \mathbf{L}_1 + \mathcal{A}$.

The logic $\mathbf{L}\restriction(\mathcal{L} \setminus \{c\})$ is sometimes called the c-free fragment of $\mathbf{L}$. Similarly we define the c-free reduct of an algebra[2]. An algebra $\mathbf{B}$ in the signature $\mathcal{L}$ is called the c-free reduct of $\mathbf{A}$ if $\mathbf{B}$ is the reduct of $\mathbf{A}$ on the language $\mathcal{L} \setminus \{c\}$ (analogously for subreducts).

Weakly implicative logics were introduced in [8] and they generalize the Rasiowa's implicative logics (see [49]) by removing the rule of *weakening* (W). The aim of this class of logics is to provide a general framework for logics with 'reasonable' implication and to provide a starting point for a general formal definition of the notion of fuzzy logic.

Definition 2.1 *Let* $\mathbf{L}$ *be a logic in* $\mathcal{L}$*. We say that* $\mathbf{L}$ *is a* weakly implicative *logic if:*

(Ref)	$\vdash \varphi \to \varphi$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$
(WT)	$\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$
(Cng^i_c)	$\varphi \to \psi, \psi \to \varphi \vdash c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \to c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$ *for* n*-ary connective* $c \in \mathcal{L}$ *and* $i \leq n$.

[1] Understood as a set of axioms and finitary deduction rules; by a logical axiom (rule) we always mean an axiom (rule) schema in this text.

[2] We need to assume some background on Universal Algebra, see some good reference book such as [7].

A weakly implicative logic is implicative *if further:*

(W) $\varphi \vdash \psi \to \varphi$

Now we introduce the class of weakly implicative fuzzy logics.

Definition 2.2 *A (weakly) implicative logic* **L** *is a (weakly) implicative* fuzzy *logic if the following meta-rule (called Prelinear Property—$\mathcal{PP}$) is valid in* **L***:*

$$\frac{T, \varphi \to \psi \vdash_{\mathbf{L}} \chi \qquad T, \psi \to \varphi \vdash_{\mathbf{L}} \chi}{T \vdash_{\mathbf{L}} \chi}$$

Sometimes we just say that the logic is fuzzy. As we do not want to go into details about general semantics of (weakly) implicative logics (logical matrices) we gave just a syntactic definition. The original definition in [8] involves both syntax and semantics and the definition above is just an equivalent characterization[3]. Roughly speaking: the conditions of weakly implicative logics allow us to uniformly define an order relation in each logical matrix of a given logic; and fuzzy logics are those weakly implicative logics which are complete w.r.t. matrices where the defined order is linear (we give a formalization of this idea in a particular case in the next section). The rationale behind labeling this class of logics (consequence relations) as fuzzy logics can be found in the opinion shared in the community of 'fuzzy logicians' that fuzzy logic investigates the *comparative notion of truth,* or *degrees of truth.* Then it is natural to assume that the truth-values, if they are to represent *degrees of truth,* are *comparable,* i.e., totally ordered; for more (deeper) mathematical, philosophical, and methodological reasons supporting this terminology see [8, 4].

We conclude this subsection by recalling the notion of Local Deduction Theorem (we use a rather special version sufficient for our needs in this paper) and its relation with the notion of fuzzy logic. First, we introduce one technical notion.

Definition 2.3 *Let us define $\varphi \to^0 \psi = \psi$ and $\varphi \to^{i+1} \psi = \varphi \to (\varphi \to^i \psi)$. Further for a sequence $\bar{\varphi} = \varphi_1, \ldots, \varphi_n$ we use $\bar{\varphi} \to \psi$ for $\varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots)$.*

Observe that for $\bar{\varphi}$ being a sequence consisting of a formula φ n-times we have $\bar{\varphi} \to \psi = \varphi \to^n \psi$.

Definition 2.4 (Local Deduction Theorem) *Let* **L** *be a weakly implicative logic. We say that* **L** *has the* Local Deduction Theorem *if for each theory $T \cup \{\varphi, \psi\}$ we have: $T \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ iff there is n such that $T \vdash_{\mathbf{L}} \varphi \to^n \psi$.*

Proposition 2.5 ([8]) *Let* **L** *be a weakly implicative logic with Local Deduction Theorem such that for each i and j we have: $(\varphi \to \psi) \to^i \chi, (\psi \to \varphi) \to^j \chi \vdash_{\mathbf{L}} \chi$. Then* **L** *is a weakly implicative fuzzy logic.*

2.2 BCK logic and BCK-algebras

The logic BCK was introduced by Meredith, see e.g., [48, 40]. It has one basic binary connective $\to$. Its only deduction rule is *modus ponens* and it has the following axioms:

(B) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(C) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(K) $\varphi \to (\psi \to \varphi)$

In BCK we can define the nullary connective (truth constant) $\overline{1}$ as $\varphi \to \varphi$.

Proposition 2.6 *The following formulas are provable in* BCK*:*

1. $(\bar{\varphi} \to (\bar{\psi} \to \chi)) \to (\bar{\psi} \to (\bar{\varphi} \to \chi))$

[3] In fact, the equivalence holds in finitary logic only, but it is sufficient for us in this paper.

2. $(\varphi \to \psi) \to ((\bar{\chi} \to \varphi) \to (\bar{\chi} \to \psi))$

3. $(\psi' \to \psi) \to [((\varphi \to \psi) \to \chi) \to ((\varphi \to \psi') \to \chi)]$

4. $\varphi \to \overline{1}$

5. $\varphi \to \varphi$

6. $(\overline{1} \to \varphi) \to \varphi$

Clearly BCK is an implicative logic and so are its expansions satisfying axioms (Cng^i_c) for the additional connectives c. As a special case of [8, Corollary 8] we obtain:

Theorem 2.7 *Let* **L** *be an implicative logic expanding* BCK*. Then* **L** *has Local Deduction Theorem iff* **L** *has an axiomatic system containing modus ponens as the only deduction rule.*

It is well-known that BCK is sound and complete with respect to the widely studied class of BCK-algebras (introduced in [32], see also survey paper [34]). In fact, the logic BCK is *algebraizable* in the sense of Blok and Pigozi (see [5]) and BCK-algebras are its *equivalent algebraic semantics* with the translations $\Delta(p, q) = \{p \to q, q \to p\}$ and $E(p) = \{p \approx \overline{1}\}$.

Definition 2.8 *A* BCK-*algebra is a structure* $\mathbf{L} = (L, \to, \overline{1})$ *such that:*

1. $(x \to y) \to ((y \to z) \to (x \to z)) = \overline{1}$

2. $x \to ((x \to y) \to y) = \overline{1}$

3. $x \to x = \overline{1}$

4. $x \to \overline{1} = \overline{1}$

5. if $x \to y = \overline{1}$ *and* $y \to x = \overline{1}$ *then* $x = y$

In each BCK-algebra we can introduce an order $\leq$ (with the top element $\overline{1}$) as $x \leq y$ iff $x \to y = \overline{1}$. If this order is linear we call the algebra a BCK-chain. BCK-algebras obviously form a quasivariety and it can be shown that they do not form a variety [58].

It can be easily shown that BCK is not a fuzzy logic. The logic FBCK was introduced and shown to be *the weakest* weakly implicative fuzzy logic extending BCK in [8].

Definition 2.9 *The logic* FBCK *is an extension of* BCK *by the following axioms (for each n):*

$$((\varphi \to \psi) \to^n \chi) \to (((\psi \to \varphi) \to^n \chi) \to \chi)\,.$$

The additional axioms (later we will denote them $F_{n,n}$, see Definition 3.8) were also studied in the context of BCK-algebras in [19].

2.3 The logic MTL and its axiomatic extensions

Now we are going to introduce the 'prominent' fuzzy logics which we mentioned in the introduction. The weakest of them is the so-called Monoidal T-norm Based Logic which was introduced by Esteva and Godo in [13]. It is the weakest fuzzy logic with the structural rules of exchange and weakening studied in the literature and the majority of existing fuzzy logics are its expansions. There are dozens of papers dealing with all possible aspects of this logic. In this subsection we recall its axiomatic system and algebraic semantics only. Other basic properties of this logic will be obtained as corollaries of more general theorems further in the paper. The logic MTL has a basic nullary connective $\overline{0}$ and three binary connectives $\to$, $\wedge$, and $\&$. Further connectives are defined as follows:

$$\begin{array}{rll} \varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \neg\varphi & \text{is} & \varphi \to \overline{0}, \\ \overline{1} & \text{is} & \neg\overline{0}. \end{array}$$

The logic MTL is axiomatized by modus ponens and the following axioms:

$(A1)$	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$(A2)$	$\varphi\&\psi \to \varphi$
$(A3)$	$\varphi\&\psi \to \psi\&\varphi$
$(A4a)$	$\varphi\&(\varphi \to \psi) \to \varphi \wedge \psi$
$(A4b)$	$\varphi \wedge \psi \to \varphi$
$(A4c)$	$\varphi \wedge \psi \to \psi \wedge \varphi$
$(A5a)$	$(\varphi \to (\psi \to \chi)) \to (\varphi\&\psi \to \chi)$
$(A5b)$	$(\varphi\&\psi \to \chi) \to (\varphi \to (\psi \to \chi))$
$(A6)$	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
$(A7)$	$\overline{0} \to \varphi$

The following are further important axioms

$(A4)$	$\varphi\&(\varphi \to \psi) \to \psi\&(\psi \to \varphi)$
$(IDEMP)$	$\varphi \to \varphi\&\varphi$
(DN)	$\neg\neg\varphi \to \varphi$
$(\Pi1)$	$\neg\neg\chi \to ((\chi\&\varphi \to \chi\&\psi) \to (\varphi \to \psi))$
$(\Pi2)$	$\neg\varphi \wedge \varphi \to \overline{0}$

The following logics (defined by adding axioms to MTL) are widely studied in the literature.

Name	symbol	added axioms	references
involutive MTL	IMTL	(DN)	[12, 13]
strict MTL	SMTL	$(\Pi2)$	[12, 13]
cancellative MTL	ΠMTL	$(\Pi1), (\Pi2)$	[29, 23]
basic fuzzy logic	BL	$(A4)$	[21, 22]
strict BL	SBL	$(A4), (\Pi2)$	[14, 22]
Łukasiewicz logic	Ł	$(A4), (DN)$	[42, 22]
Gödel logic	G	$(A4), (IDEMP)$	[10, 22]
product logic	Π	$(A4), (\Pi1), (\Pi2)$	[26, 22]

Definition 2.10 *An* MTL-algebra *is a structure* $\mathbf{B} = (B, \wedge, \vee, \&, \to, \overline{0}, \overline{1})$*, where*

(1) $(L, \wedge, \vee, \overline{0}, \overline{1})$ is a bounded lattice,
(2) $(L, \&, \overline{1})$ is a commutative monoid,
(3) $z \leq (x \to y)$ iff $x\&z \leq y$ for all x, y, z, (residuation)
(4) $(x \to y) \vee (y \to x) = \overline{1}$. (prelinearity)

If the lattice is linearly ordered we say that $\mathbf{B}$ *is an* MTL-*chain.*

Observe that the $\to$-reduct of an MTL-algebra is a BCK-algebra. The logic MTL is again algebraizable and MTL-algebras are its *equivalent algebraic semantics* with the same translations as in BCK.

3 The logic FBCK and its expansions

The goal of this paper is to study the fragments of MTL (and other important fuzzy logics introduced above) with respect to all their possible sublanguages. Following the arguments given in the introduction, that not all sublanguages are interesting/important/worth to study in this paper, we restrict ourselves to the following convention on propositional languages.

Convention 3.1 *We say that* $\mathcal{L}$ *is a* propositional language *if* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \&, \vee, \wedge, \overline{0}\}$.

In order to find axiomatizations of ‘our fragments’, we follow the ‘reverse’ route: we start from a weak logic FBCK (in the language containing $\to$ as the only primitive connective) and study its expansions to each propositional language. Then, in Section 4, we show that the fragments we want to axiomatize coincide with the corresponding expansions of FBCK.

3.1 Syntax

Now we introduce sets of axioms for particular connectives we use in this text (notice that we use the same names for the “implicational” versions of axioms of MTL, but different names for the additional ones).

Definition 3.2 *The following axioms are the elements of* $\mathcal{AX}(\to)$*:*

$(A1)$	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$(A2)$	$\varphi \to (\psi \to \varphi)$
$(A3)$	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
$(A6)$	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

The following axioms are the elements of $\mathcal{AX}(\&)$*:*

$(A5a)$	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
$(A5b)$	$(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$

The following axiom is the element of $\mathcal{AX}(\overline{0})$*:*

$(A7)$	$\overline{0} \to \varphi$

The following axioms are the elements of $\mathcal{AX}(\wedge)$*:*

$(A4a)$	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
$(A4b)$	$\varphi \wedge \psi \to \varphi$
$(A4c)$	$\varphi \wedge \psi \to \psi \wedge \varphi$

The following axioms are the elements of $\mathcal{AX}(\vee)$*:*

$(A4\vee a)$	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
$(A4\vee b)$	$\varphi \to \varphi \vee \psi$
$(A4\vee c)$	$\varphi \vee \psi \to \psi \vee \varphi$

Finally we define the following important axioms:

(Div)	$((\varphi \to \psi) \to (\varphi \to \chi)) \to ((\psi \to \varphi) \to (\psi \to \chi))$	*divisibility*
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	*Wajsberg axiom*
$(Cont)$	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	*contraction*
(SBL)	$(\varphi \to \neg\varphi) \to \neg\varphi$	*strictness*

Definition 3.3 *Let* $\mathcal{L}$ *be a propositional language. We define* $\mathcal{AX}(\mathcal{L}) = \bigcup_{c \in \mathcal{L}} \mathcal{AX}(c)$*. We also define the axiomatic system* $\mathcal{AX}(\mathcal{L})^-$ *as* $\mathcal{AX}(\mathcal{L})$ *without the axiom* $(A6)$*.*

Let us denote the logic axiomatized by $\mathcal{AX}(\mathcal{L})^-$ *and modus ponens by* $\mathrm{BCK}^{\mathcal{L}}$*. The logic axiomatized by* $\mathcal{AX}(\mathcal{L})$ *and modus ponens is denoted by* $\mathrm{FBCK}^{\mathcal{L}}$*.*

We will write BCK instead of $\mathrm{BCK}^{\{\to\}}$. This convention is sound because BCK is axiomatized by $\mathcal{AX}(\to)^-$ (see Subsection 2.2). After Corollary 3.11 (where we show that FBCK defined in Definition 2.9 is axiomatized by $\mathcal{AX}(\to)$) we will write FBCK instead of $\mathrm{FBCK}^{\{\to\}}$. One of the main goals of this paper is to show that for each propositional language $\mathcal{L}$, $\mathcal{AX}(\mathcal{L})$ is an axiomatic system of the $\mathcal{L}$-fragment of MTL. The proof of the following proposition is straightforward.

Proposition 3.4 *Let* $\mathcal{L}$ *be a propositional language and let* $\mathbf{L}$ *be any* axiomatic *extension of* $\mathrm{BCK}^{\mathcal{L}}$ *(e.g.,* $\mathrm{FBCK}^{\mathcal{L}}$*). Then* $\mathbf{L}$ *is an implicative logic.*

One easily shows that $\mathrm{BCK}^{\{\to,\&\}}$ proves the forms of axioms $(A2)$ and $(A3)$ from the definition of MTL. Furthermore, checking [22, Lemma 2.2.8] one sees that $\mathrm{BCK}^{\{\to,\&\}}$ proves $\varphi \to (\psi \to \varphi\&\psi)$, $(\varphi \to \psi) \to (\varphi\&\chi \to \psi\&\chi)$, and the associativity of &.

Lemma 3.5 *If* $\mathbf{L}$ *is an expansion of* $\mathrm{BCK}^{\{\to,\&\}}$, *then* $\vdash_{\mathbf{L}} (Div)$ *iff* $\vdash_{\mathbf{L}} (A4)$. *Furthermore, if* $\mathbf{L}$ *is an expansion of* $\mathrm{BCK}^{\{\to,\overline{0},\wedge\}}$, *then* $\vdash_{\mathbf{L}} (SBL)$ *iff* $\vdash_{\mathbf{L}} (\Pi 2)$. *Finally, if* $\mathbf{L}$ *is an expansion of* $\mathrm{BCK}^{\{\to,\overline{0}\}}$ *and* $\vdash_{\mathbf{L}} (Waj)$ *then* $\vdash_{\mathbf{L}} (DN)$.

Proof: The first claim: first notice that $\varphi\&(\varphi \to \psi) \to \chi$ is equivalent with $(\varphi \to \psi) \to (\varphi \to \chi)$ (using axioms $(A5a)$, $(A5b)$). Let us assume $\vdash_{\mathbf{L}} (Div)$, to prove $(A4)$ just set $\chi = \varphi\&(\varphi \to \psi)$. The other direction is a simple consequence of axiom $(A1)$.

The second claim: let us assume $\vdash_{\mathbf{L}} (\Pi 2)$ and observe $(\varphi \to \varphi \wedge \neg\varphi) \to (\varphi \to \overline{0})$ (due to axiom $(A1)$). Notice that also $(\varphi \to \neg\varphi) \to (\varphi \to \varphi \wedge \neg\varphi)$ (due to axiom $(A4a)$). Transitivity completes the proof of this direction. The converse direction: start with $\neg\varphi \to (\varphi \to \overline{0})$. Then use axiom $(A4b)$ twice (and transitivity) and obtain $(\varphi \wedge \neg\varphi) \to ((\varphi \wedge \neg\varphi) \to \overline{0})$. Axiom (SBL) for φ being $(\varphi \wedge \neg\varphi)$ completes the proof.

The proof of the third claim is almost straightforward. □

Now we can show that many logics introduced in the Subsection 2.3 can be axiomatized using our new axioms. We summarize the results in Table 1. The proof is rather simple, just rewrite the axioms for & using the residuation and use the previous lemma. Later we will see that some of the axiomatic systems mentioned in Table 1 are redundant.

Table 1: Alternative axiomatic systems for some prominent fuzzy logics

Logic	Axiomatic system
MTL	$\mathcal{AX}(\to,\&,\wedge,\overline{0})$
IMTL	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (DN)$
SMTL	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (SBL)$
ΠMTL	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (SBL), (\Pi 1)$
BL	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (Div)$
SBL	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (Div), (SBL)$
Ł	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (Div), (Waj)$
G	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (Div), (Cont)$
Π	$\mathcal{AX}(\to,\&,\wedge,\overline{0}) + (Div), (SBL), (\Pi 1)$

Remark 3.6 *There are many other well-known logics outside the realm of "fuzzy" logics which are axiomatizable using our axioms, e.g. we can easily obtain:*

1. BCK^s *is axiomatized by* $\mathcal{AX}(\to,\&)^-$ *(see [30]).*
2. $\mathrm{FL_{ew}}$ *is axiomatized by* $\mathcal{AX}(\to,\&,\wedge,\vee,\overline{0})^-$ *(see [46]).*
3. *The implicational fragment of Intuitionistic logic is axiomatized by* $\mathcal{AX}(\to)^- + (Cont)$ *(see [50]).*
4. *Intuitionistic logic is axiomatized by* $\mathcal{AX}(\to,\wedge,\vee,\overline{0})^- + (Cont)$.

From Theorem 2.7 we immediately obtain:

Theorem 3.7 (Local Deduction Theorem) *Let* $\mathcal{L}$ *be a propositional language and let* $\mathbf{L}$ *be any* axiomatic *extension of* $\mathrm{BCK}^{\mathcal{L}}$ *(e.g.,* $\mathrm{FBCK}^{\mathcal{L}}$*). Then* $\mathbf{L}$ *has the Local Deduction Theorem.*

Definition 3.8 *Let* $n, m > 0$. *We define the formulas* $F_{n,m}$ *as follows:*

$$((\varphi \to \psi) \to^n \chi) \to (((\psi \to \varphi) \to^m \chi) \to \chi)\,.$$

Lemma 3.9 *Let* **L** *be a logic expanding* BCK*; m, n, n' natural numbers, and $n' \leq n$. Further assume that $F_{n,m}$ is a theorem of* **L***. Then also $F_{n',m}$, $F_{m,n}$, and $F_{2n,m}$ are theorems of* **L***.*

Proof: The proof of the first part is a simple consequence of the axiom $(A2)$. The second part easily follows from axiom $(A3)$. We give a formal proof of the last part. Let $C = ((\psi \to \varphi) \to^m \chi) \to \chi$

(i)	$\vdash ((\varphi \to \psi) \to^n \chi) \to C$	$F_{n,m}$
(ii)	$\vdash ((\psi \to \varphi) \to^m \chi) \to ((\psi \to \varphi) \to^m \chi)$	Proposition 2.6 part 5.
(iii)	$\vdash (\psi \to \varphi) \to^m C$	(ii) and Proposition 2.6 part 1.
(iv)	$\vdash ((\varphi \to \psi) \to^n C) \to C$	(iii) and $F_{m,n}$ (for φ and ψ interchanged)
(v)	$\vdash [(\varphi \to \psi) \to^n ((\varphi \to \psi) \to^n \chi)] \to C$	$(i), (iv)$, and Proposition 2.6 part 3.
(vi)	$\vdash ((\varphi \to \psi) \to^{2n} \chi) \to C$	definition

□

Observe that $F_{1,1}$ is the axiom (A6). Thus we obtain the following two corollaries.

Corollary 3.10 *Let* **L** *be a logic expanding* $\text{FBCK}^{\{\to\}}$. *Then* $\vdash_{\mathbf{L}} F_{n,m}$ *for each n, m.*

Corollary 3.11 *The logic* FBCK *is axiomatized by* $\mathcal{AX}(\to)$.

As mentioned above we write FBCK instead of $\text{FBCK}^{\{\to\}}$ from now on. This last corollary was also independently proven in [54].

We know that all logics axiomatically extending $\text{FBCK}^{\mathcal{L}}$ are implicative, have the local deduction theorem, and for each i and j it holds: $(\varphi \to \psi) \to^i \chi, (\psi \to \varphi) \to^j \chi \vdash \chi$. Thus we can use Proposition 2.5 to conclude that all these logics are implicative fuzzy logics.

Theorem 3.12 *Let $\mathcal{L}$ be a propositional language and let* **L** *be an axiomatic extension of* $\text{FBCK}^{\mathcal{L}}$. *Then* **L** *is an implicative fuzzy logic.*

At the end of this section we prove some facts about their relationship of the additional axioms (Waj), $(Cont)$, and (Div). The proof of the following proposition is well-known in the case that we have either $\overline{0}$ or & in the language (see [27]), however our proof uses implication only.

Proposition 3.13 *The logic* $\text{FBCK} + (Waj)$ *proves the axiom* (Div).

Proof: We give a formal proof:

(i)	$\psi \to \chi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$	simple
(ii)	$\chi \to \psi, \varphi \to \chi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$	simple
(iii)	$\vdash ((\psi \to \chi) \to (\varphi \to \chi)) \to [((\varphi \to \chi) \to \chi) \to ((\psi \to \chi) \to \chi)]$	transitivity
(iv)	$\vdash ((\psi \to \chi) \to (\varphi \to \chi)) \to [((\chi \to \varphi) \to \varphi) \to ((\chi \to \psi) \to \psi)]$	(iii) and (Waj) twice
(v)	$\chi \to \psi, \chi \to \varphi \vdash ((\psi \to \chi) \to (\varphi \to \chi)) \to (\varphi \to \psi)$	(iv)
(vi)	$\chi \to \psi, \chi \to \varphi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [((\psi \to \chi) \to (\varphi \to \chi)) \to (\varphi \to \chi)]$	(v), trans.
(vii)	$\chi \to \psi, \chi \to \varphi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [((\varphi \to \chi) \to (\psi \to \chi)) \to (\psi \to \chi)]$	(vi), (Waj).
(viii)	$\vdash (\psi \to \varphi) \to ((\varphi \to \chi) \to (\psi \to \chi))$	transitivity
(ix)	$\chi \to \psi, \chi \to \varphi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$	(vii) and (viii)

(x) $\chi \to \psi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$ (ii), (ix), and $\mathcal{PP}$

(xi) $\vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$ (i), (x), and $\mathcal{PP}$

□

Remark 3.14 *Notice the use of prelinearity property in the proof. It is known that* FBCK+(Waj) *is the implicational fragment of the Lukasiewicz logic (see e.g., [38, 51]) and thus it proves* (Div)*. The proof of this fact utilize the completeness theorems and some algebraic reasoning. However, the authors of the present paper are not aware of any existing formal proof (in the presented axiomatic system) of this fact.*

A similar remark could be made about the following proposition (as it is well-known that BCK+ $(Cont)$ *is the implicational fragment of intuitionistic logic (see e.g., [50]), we know that* BCK + $(Cont)$ *proves* (Div)*).*

Proposition 3.15 *The logic* FBCK + $(Cont)$ *proves the axiom* (Div)*.*

Proof: We give a formal proof:

(i) $\varphi \to \psi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$ simple

(ii) $\vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [\psi \to (\varphi \to \chi)]$ $(A2)$ and transitivity

(iii) $\psi \to \varphi \vdash (\varphi \to \chi) \to (\psi \to \chi)$ transitivity

(iv) $\psi \to \varphi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [\psi \to (\psi \to \chi)]$ (ii), (iii), and transitivity

(v) $\psi \to \varphi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to (\psi \to \chi)$ (iv) and $(Cont)$

(vi) $\psi \to \varphi \vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$ (v) and $(A2)$

(xi) $\vdash [(\varphi \to \psi) \to (\varphi \to \chi)] \to [(\psi \to \varphi) \to (\psi \to \chi)]$ (i), (vi), and $\mathcal{PP}$

□

3.2 Semantics

Definition 3.16 *Let* $\mathcal{L}$ *be a propositional language and let* $\mathbf{L}$ *be any axiomatic extension of* $\mathrm{BCK}^{\mathcal{L}}$ *(e.g.,* $\mathrm{FBCK}^{\mathcal{L}}$*). We say that an algebra* $\mathbf{B}$ *in the language* $\mathcal{L} \cup \{\bar{1}\}$ *is an* $\mathbf{L}$*-algebra if:*

- *its* $\{\to_{\mathbf{B}}, \bar{1}_{\mathbf{B}}\}$*-reduct is a* BCK*-algebra,*
- *all axioms from* $\mathbf{L}$ *are* $\mathbf{B}$*-tautologies*[4]*.*

If the order (induced by BCK*-algebraic reduct) is linear we say that* $\mathbf{B}$ *is an* $\mathbf{L}$*-chain.*

Finally, if the domain of $\mathbf{B}$ *is the unit interval* $[0, 1]$ *and the order is the usual order of reals, we say that* $\mathbf{B}$ *is a* standard $\mathbf{L}$*-algebra.*

Recall that the interpretations of operations & and → in a standard MTL-algebra are a left-continuous t-norm and its residuum. For a comprehensive survey on t-norms see [37].

Observe that the class of **L**-algebras is always a quasivariety. In Section 5 we answer the question for which axiomatic extensions of BCK (studied in this paper) this class is a variety.

This general definition has many special cases and many algebras defined above are well-known under different names. Let us summarize the most well-known ones:

[4] $\bar{1}_{\mathbf{B}}$ is the only designated element of $\mathbf{B}$. Thus the basic semantical notions like evaluation, model, or tautology could be defined as usual.

Our symbol	usual name(s)	references
$\mathrm{BCK}^{\{\to\}}$	BCK-algebra	[32, 34]
$\mathrm{BCK}^{\{\to,\bar{0}\}}$	bounded BCK-algebra	[34]
$\mathrm{BCK}^{\{\to,\&\}}$	BCK^s-algebra, reversed left BCK(RP)-algebra	[30, 33]
	pocrim, ordered quasihoop	[6, 25]
$\mathrm{BCK}^{\{\to,\wedge,\vee,\&\}}$	integral commutative residuated lattice, semi-hoop	[36, 27]
$\mathrm{FBCK}^{\{\to\}}$	FBCK-algebra, representable BCK-algebra	[8, 53]
$\mathrm{FBCK}^{\{\to,\&\}}$	ordered basic quasihoop	[25]
$\mathrm{FBCK}^{\{\to,\wedge,\vee,\&\}}$	representable integral commutative residuated lattice,	[36]
	prelinear (basic) semi-hoop, MTLH-algebra	[45, 27]
$\mathrm{FBCK}^{\{\to,\wedge,\vee,\&,\bar{0}\}}$	MTL-algebra	[13]

Theorem 3.17 (Strong Completeness) *Let $\mathcal{L}$ be a propositional language and let $\mathbf{L}$ be any axiomatic extension of* $\mathrm{FBCK}^{\mathcal{L}}$*. Then for each theory T and each formula φ the following are equivalent:*

- $T \vdash_{\mathbf{L}} \varphi$.
- $e(\varphi) = \bar{1}$ *for each* $\mathbf{L}$*-algebra* $\mathbf{B}$ *and each* $\mathbf{B}$*-model e of T.*
- $e(\varphi) = \bar{1}$ *for each* $\mathbf{L}$*-chain* $\mathbf{B}$ *and each* $\mathbf{B}$*-model e of T.*

Proof: From 3.12 we know that all logics under the consideration are implicative fuzzy logics (i.e., satisfy the Prelinear Property). Thus [8, Theorem 3] completes the proof. □

As a consequence of [8, Theorem 3] we also obtain:

Theorem 3.18 (Subdirect decomposition) *Let $\mathcal{L}$ be a propositional language and let $\mathbf{L}$ be any axiomatic extension of* $\mathrm{FBCK}^{\mathcal{L}}$*. Then each* $\mathbf{L}$*-algebra is isomorphic to a subdirect product of* $\mathbf{L}$*-chains.*

Observe that BCK-chains and FBCK-chains coincide. Thus the following known result on BCK-algebras (see e.g. [16, 47]) is an instance of the latter theorem for $\mathbf{L} = \mathrm{FBCK}$.

Theorem 3.19 *The subdirect products (and their isomorphic images) of linearly ordered* BCK*-algebras are exactly those* BCK*-algebras fulfilling the identity:*

$$((x \to y) \to z) \to (((y \to z) \to z) \to z) = \bar{1}\,.$$

Now we introduce the definition of the (finite) strong standard completeness and recall which of our logics enjoy these properties (for a general approach to these issues, see [9]).

Definition 3.20 *Let $\mathcal{L}$ be a propositional language and let $\mathbf{L}$ be any axiomatic extension of* $\mathrm{FBCK}^{\mathcal{L}}$*. We say that* $\mathbf{L}$ *enjoys the (finite) strong standard completeness if for each (finite) theory T and each formula φ the following are equivalent:*

- $T \vdash_{\mathbf{L}} \varphi$.
- $e(\varphi) = \bar{1}$ *for each standard* $\mathbf{L}$*-chain* $\mathbf{B}$ *and each* $\mathbf{B}$*-model e of T.*

Theorem 3.21 ([22, 12, 29, 35]) *The logics* MTL*,* IMTL*,* SMTL*, and* G *enjoy the strong standard completeness. The logics* ΠMTL*,* BL*,* SBL*,* Ł*, and* Π *enjoy the finite strong standard completeness (but not the strong standard completeness).*

We conclude this section by two simple lemmas and one useful construction we will need in the upcoming sections. The reader familiar with the ordinal sums of MTL-chains can notice that the construction is in fact the same as the ordinal sum of two element Boolean algebra and $\mathbf{A}$.

Lemma 3.22 *Let* $\mathbf{A} = (A, \rightarrow, \overline{0}, \overline{1})$ *be an* $\mathrm{FBCK}^{\{\rightarrow, \overline{0}\}}$*-chain, where* (SBL) *is a tautology. Then we have:*

1. $x \leq \neg\neg x$.
2. *If* $x > \overline{0}$, *then* $\neg x < x$.
3. *If* $x > \overline{0}$, *then* $\neg x = \overline{0}$.

Definition 3.23 *Let* $\mathbf{A}$ *be an* $\mathrm{FBCK}^{\mathcal{L}}$*-chain,* $b \notin A$. *We define an algebra* $\mathbf{2} \oplus \mathbf{A}$ *whose universe is* $A \cup \{b\}$, $b \leq x$ *for all* $x \in A$, $\overline{1}_{\mathbf{2}\oplus\mathbf{A}} = \overline{1}_{\mathbf{A}}$, *and the operations are defined as follows:*

$$x \rightarrow_{\mathbf{2}\oplus\mathbf{A}} y = \begin{cases} x \rightarrow_{\mathbf{A}} y & \text{if } x, y \in A\,, \\ \overline{1}_{\mathbf{2}\oplus\mathbf{A}} & \text{if } x = b\,, \\ b & \text{if } x \in A \text{ and } y = b\,. \end{cases}$$

If $\& \in \mathcal{L}$ *then the new monoidal operation is defined as follows:*

$$x \&_{\mathbf{2}\oplus\mathbf{A}} y = \begin{cases} x \&_{\mathbf{A}} y & \text{if } x, y \in A\,, \\ b & \text{if } x = b \text{ or } y = b\,. \end{cases}$$

Finally, if $\overline{0} \in \mathcal{L}$ *then it is interpreted by* b. *If some of the lattice connectives is in* $\mathcal{L}$ *then it is interpreted by means of the linear order in* $\mathbf{A}$.

Lemma 3.24 *Let* $\mathbf{A}$ *be an* $\mathrm{FBCK}^{\mathcal{L}}$*-chain. Then* $\mathbf{2} \oplus \mathbf{A}$ *is an* $\mathrm{FBCK}^{\mathcal{L}\cup\{\overline{0}\}}$*-chain and* (SBL) *is a* $\mathbf{2} \oplus \mathbf{A}$*-tautology.*

4 Fragments and their axiomatic systems

In this section we study fragments of fuzzy logics introduced in Subsection 2.3. Our goal is to find axiomatic systems for these fragments. Recall that we want to study fragments with respect to 16 different languages for each of our 9 logics (MTL, IMTL, SMTL, ΠMTL, BL, SBL, Ł, G, Π). This gives 144 logics (many of them are termwise equivalent, see Section 6 for more details). We give axiomatic systems for nearly all of these logics. After basic general lemmata we start with fragments of MTL (in Subsection 4.2) and then we proceed with other logics (in Subsection 4.3).

4.1 General results

The following lemma is known to hold in a much wider context (see e.g. [5]). However, we formulate it in a form sufficient for our needs.

Lemma 4.1 *Let* $\mathcal{L}_1 \subseteq \mathcal{L}_2$ *be propositional languages and* $\mathbf{L}_i$ *an axiomatic extension of* $\mathrm{BCK}^{\mathcal{L}_i}$. *Finally, let* $\mathbf{L}_i$ *be strongly complete w.r.t. a class* $\mathcal{K}_i$ *of* $\mathbf{L}_i$*-algebras. Then the following holds:*

1. $\mathbf{L}_2$ *is an expansion of* $\mathbf{L}_1$ *and each* $\mathbf{L}_1$*-algebra from the class* $\mathcal{K}_1$ *is a subreduct of an* $\mathbf{L}_2$*-algebra iff* $\mathbf{L}_1$ *is the* $\mathcal{L}_1$*-fragment of* $\mathbf{L}_2$.
2. *If* $\mathbf{L}_1$ *is the* $\mathcal{L}_1$*-fragment of* $\mathbf{L}_2$, *then* $\mathbf{L}_1$ *is strongly complete w.r.t. the class of* $\mathcal{L}_1$*-reducts of* $\mathbf{L}_2$*-algebras from the class* $\mathcal{K}_2$.

As we are dealing mainly with chains in this paper we need a stronger version of the right-to-left part of the first claim of the lemma above.

Lemma 4.2 *Let* $\mathcal{L}_1 \subseteq \mathcal{L}_2$ *be propositional languages and* $\mathbf{L}_i$ *an axiomatic extension of* $\mathrm{FBCK}^{\mathcal{L}_i}$ *such that* $\mathbf{L}_1$ *is the* $\mathcal{L}_1$*-fragment of* $\mathbf{L}_2$. *Then each* $\mathbf{L}_1$*-chain is a subreduct of an* $\mathbf{L}_2$*-chain.*

Proof: Let $\mathbf{B}$ be an $\mathbf{L}_1$-chain and $\{v_a \mid a \in B\}$ be a set of mutually distinct propositional variables and let T be its diagram, i.e.,

$$T = \{v_a \to c(v_{b_1}, c_{b_2}), c(v_{b_1}, c_{b_2}) \to v_a \mid c \in \{\to, \wedge, \vee, \&\} \cap \mathcal{L}_1; a, b_1, b_2 \in B, \text{ and } a = c_{\mathbf{B}}(b_1, b_2)\}.$$

Observe that $T \not\vdash_{\mathbf{L}_1} v_d$ (in $\mathbf{B}$ define the $\mathbf{B}$-evaluation $e(v_a) = a$) for each $d < \bar{1}$. Thus from our assumption we obtain that $T \not\vdash_{\mathbf{L}_2} v_d$ for each $d < \bar{1}$. Let us take all theories T' such that $T \subseteq T'$ and $T' \not\vdash_{\mathbf{L}_2} v_d$ for each $d < \bar{1}$. Let $\hat{T}$ be a maximal such theory (using Zorn's lemma).

Assume that neither $\varphi \to \psi \in \hat{T}$ nor $\psi \to \varphi \in \hat{T}$. Thus $\hat{T}, \varphi \to \psi \vdash_{\mathbf{L}_2} v_b$ and $\hat{T}, \psi \to \varphi \vdash_{\mathbf{L}_2} v_c$ for some $b, c < \bar{1}$. Assume (w.l.o.g.) that $b \leq c$, thus $T \vdash_{\mathbf{L}_1} v_b \to v_c$ and so $\hat{T} \vdash_{\mathbf{L}_2} v_b \to v_c$. So we have $\hat{T}, \varphi \to \psi \vdash_{\mathbf{L}_2} v_c$ and so using the prelinearity property $\hat{T} \vdash_{\mathbf{L}_2} v_c$—a contradiction. Thus $\hat{T}$ is a linear theory[5] (i.e., $\hat{T} \vdash_{\mathbf{L}_2} \varphi \to \psi$ or $\hat{T} \vdash_{\mathbf{L}_2} \psi \to \varphi$ for each pair φ, ψ) and so its Lindenbaum algebra $\mathbf{C}$ is an $\mathbf{L}_2$-chain (see [8]). Define $f(a) = [v_a]_{\hat{T}}$, obviously f is a morphism from $\mathbf{B}$ to $\mathbf{C}$ and we can show that f is one-one. Assume that $b < a$, so $a \to b < \bar{1}$, observe that $f(a \to b) = [v_{a\to b}]_{\hat{T}} < [\bar{1}]_{\hat{T}}$ (otherwise we obtain $\hat{T} \vdash v_{a\to b}$—a contradiction) and so $f(a) < f(b)$. □

Lemma 4.1 has an important corollary, which demonstrates together with Theorem 3.21 that all fragments of all our nine logics have some form of standard completeness.

Theorem 4.3 (Strong Standard Completeness) *Let* $\mathbf{L}$ *be an axiomatic extension of* MTL *and* $\mathcal{L}$ *a propositional language. If* $\mathbf{L}$ *enjoys the (finite) strong standard completeness then also* $\mathbf{L}{\restriction}\mathcal{L}$ *enjoys the (finite) strong standard completeness.*

Proof: We use the second part of Lemma 4.1 (or its analogy for the case of finite strong standard completeness) and the fact that each $\mathcal{L}$-reduct of a standard $\mathbf{L}$-algebra is a standard $\mathbf{L}{\restriction}\mathcal{L}$-algebra. (Notice that we do not claim the converse statement, i.e., there could be a standard $\mathbf{L}{\restriction}\mathcal{L}$-algebra which is not a reduct of some standard $\mathbf{L}$-algebra.) □

For example this allows us to say that MTL$\restriction\{\to\}$ (or FBCK as we show later) is *the logic of residua of left-continuous t-norms.*

Now we demonstrate one of the reasons for restricting to axiomatic extensions. We know that if $\mathbf{L}$ is an axiomatic extension of MTL then for each propositional language $\mathcal{L}$ the logic $\mathbf{L}{\restriction}\mathcal{L}$ is an extension of MTL$\restriction\mathcal{L}$. The following proposition shows that $\mathbf{L}{\restriction}\mathcal{L}$ is even an *axiomatic* extension of MTL$\restriction\mathcal{L}$.

Proposition 4.4 *Let* $\mathcal{L} \subseteq \mathcal{L}'$ *be propositional languages. If* $\mathbf{L}$ *is an axiomatic extension of* MTL$\restriction\mathcal{L}'$ *then* $\mathbf{L}{\restriction}\mathcal{L}$ *is an axiomatic extension of* MTL$\restriction\mathcal{L}$.

Proof: From Theorem 3.7 we know that $\mathbf{L}$ has the local deduction theorem. If we show that this is also the case for $\mathbf{L}{\restriction}\mathcal{L}$ the proof is done by means of [8, Corollary 8] (we also need to show that $\mathbf{L}{\restriction}\mathcal{L}$ is finitary, but we leave this to the reader). We write a chain of equivalencies: $T, \varphi \vdash_{\mathbf{L}\restriction\mathcal{L}} \psi$ iff $T, \varphi \vdash_{\mathbf{L}} \psi$ iff there is n such that $T \vdash_{\mathbf{L}} \varphi \to^n \psi$ iff there is n such that $T \vdash_{\mathbf{L}\restriction\mathcal{L}} \varphi \to^n \psi$. □

4.2 Fragments of MTL

We start with a crucial lemma. Although this lemma already follows from known results, we present a proof for readers convenience. In fact our proof is a modified and simplified version of the proof of analogous statement for BCK-algebras and pocrims, proven by Fleischer in [17] (the construction we use is essentially the same), see also [6, 46].

Lemma 4.5 *Each* FBCK$^{\{\to,\wedge,\vee,\bar{0}\}}$*-chain is a subreduct of an* MTL*-chain.*

[5] Also called a complete theory in [22].

Proof: We take an FBCK$^{\{\rightarrow,\wedge,\vee,\overline{0}\}}$-chain $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \overline{0}, \overline{1})$. Let $A^{<\omega}$ denote the set of finite sequences of elements of A. Given $\mathbf{f}$ and $\mathbf{g}$ from $A^{<\omega}$, we denote their concatenation by $\mathbf{f}, \mathbf{g}$. For each $\mathbf{y} \in A^{<\omega}$ we set $[\mathbf{y}] = \{x \mid \mathbf{y} \rightarrow x = \overline{1}\}$ (for the meaning of the term $\mathbf{y} \rightarrow x$ see the analogous definition for $\bar{\varphi} \rightarrow \psi$); notice that $[\mathbf{y}]$ are upper sets in $\mathbf{A}$. Furthermore, we define $B = \{[\mathbf{y}] \mid \mathbf{y} \in A^{<\omega}\}$ and C as the set of all non-empty upper sets in B w.r.t. $\subseteq$. Finally, we define $\mathbf{C} = (C, \cap, \cup, \&, \Rightarrow, \overline{0}_{\mathbf{C}}, \overline{1}_{\mathbf{C}})$, where $\cup$ and $\cap$ are the usual set-theoretical operations,

$$F \& G = \{[\mathbf{a}] \in B \mid \text{there are } \mathbf{f}, \mathbf{g} \text{ such that } [\mathbf{f}] \in F, [\mathbf{g}] \in G, \text{ and } [\mathbf{f}, \mathbf{g}] \subseteq [\mathbf{a}]\},$$

$F \Rightarrow G = \bigcup\{H \mid H \& F \subseteq G\}$, $\overline{0}_{\mathbf{C}} = \{[\overline{0}]\} = \{A\}$, and $\overline{1}_{\mathbf{C}} = B$.

We show that $\mathbf{C}$ is an MTL-chain. First, observe that the operations are well-defined (i.e., $F \& G$ and $F \Rightarrow G$ are upper sets in $\mathbf{B}$). Now notice that $F \& G \subseteq F$ (trivial consequence of $[\mathbf{f}] \subseteq [\mathbf{f}, \mathbf{g}]$, and properties of upper sets). Let us denote the set $\{[\mathbf{b}] \mid [\mathbf{a}] \subseteq [\mathbf{b}]\}$ as $[\mathbf{a}]^{\uparrow}$ and observe that $[\mathbf{a}]^{\uparrow} \& [\mathbf{b}]^{\uparrow} = [\mathbf{a}, \mathbf{b}]^{\uparrow}$.

We have to show the validity of defining conditions of an MTL-algebra (cf. Definition 2.10): condition (1) is trivial. Condition (2): commutativity (see Proposition 2.6) and associativity are easy, to show that $F \& \overline{1}_{\mathbf{C}} = F$ we first notice that $[\mathbf{f}, \overline{1}] \subseteq [\mathbf{f}]$ and $[\overline{1}] \in \overline{1}_{\mathbf{C}}$ and so $F \subseteq F \& \overline{1}_{\mathbf{C}}$. Condition (3):

$$M \subseteq \bigcup\{H \mid H \& F \subseteq G\} \ \text{ iff } \ M \& F \subseteq G\,.$$

The right-left direction is simple. We prove the second one: if $[\mathbf{a}] \in M \& F$ then there is $[\mathbf{m}] \in M$ and $[\mathbf{f}] \in F$ such that $[\mathbf{m}, \mathbf{f}] \subseteq [\mathbf{a}]$. From our assumption we know that there is H such that $[\mathbf{m}] \in H$ and $H \& F \subseteq G$. Thus $[\mathbf{a}] \in H \& F$ and so $[\mathbf{a}] \in G$. Condition (4) holds trivially as $\mathbf{C}$ is linearly ordered.

Now we define a mapping $f : \mathbf{A} \rightarrow \mathbf{C}$ as $f(a) = [a]^{\uparrow}$. Observe that f is one-one, $f(\overline{1}) = \overline{1}_{\mathbf{C}}$, and $f(\overline{0}) = [\overline{0}]^{\uparrow} = \{[\overline{0}]\} = \overline{0}_{\mathbf{C}}$. Further, we need to show that $f(a) \Rightarrow f(b) = f(a \rightarrow b)$. To prove the $\supseteq$-direction notice that $[a \rightarrow b]^{\uparrow} \& [a]^{\uparrow} = [a \rightarrow b, a]^{\uparrow} \subseteq [b]^{\uparrow}$ (the first equality is clear, the second inequality follows from $b \in [a, a \rightarrow b]$) and the residuation property completes the proof of this direction.

The $\subseteq$-direction is more complicated: assume $[\mathbf{c}] \in [a]^{\uparrow} \Rightarrow [b]^{\uparrow}$, i.e., there is H such that $[\mathbf{c}] \in H$ and $H \& [a]^{\uparrow} \subseteq [b]^{\uparrow}$. Thus $[\mathbf{c}, a] \in [b]^{\uparrow}$ (as $[\mathbf{c}, a] \in H \& [a]^{\uparrow}$), i.e., $[b] \subseteq [\mathbf{c}, a]$. Thus $\mathbf{c} \rightarrow (a \rightarrow b) = \overline{1}$ and so $[a \rightarrow b] \subseteq [\mathbf{c}]$. Thus finally $[\mathbf{c}] \in [a \rightarrow b]^{\uparrow}$.

To complete the proof, observe that f is order-preserving, i.e., a lattice homomorphism. Suppose that $a \leq b$. Then $f(a) \Rightarrow f(b) = f(a \rightarrow b) = f(\overline{1}) = \overline{1}_{\mathbf{C}}$ which is equivalent to $f(a) \leq f(b)$. □

Lemma 4.6 *Let $\mathcal{L}$ be a propositional language. Then each* FBCK$^{\mathcal{L}}$*-chain is a subreduct of an* MTL*-chain.*

Proof: Firstly, we show that each FBCK$^{\mathcal{L}}$-chain $\mathbf{A}$ is a subreduct of an FBCK$^{\mathcal{L} \cup \{\wedge,\vee,\overline{0}\}}$-chain $\mathbf{B}$. If $\mathcal{L}$ does not contain some of the lattice connectives then we can define it using the linear order. Then it can be easily shown that the axioms $(A4a\text{–}c)$ and $(A4\vee a\text{–}c)$ are tautologies. If $\overline{0}$ is missing then take $\mathbf{B} = \mathbf{2} \oplus \mathbf{A}$.

Secondly, if $\mathcal{L} \cup \{\wedge, \vee, \overline{0}\} = \{\rightarrow, \&, \wedge, \vee, \overline{0}\}$ then $\mathbf{B}$ is already an MTL-chain and we are done. If $\& \notin \mathcal{L}$ then, by Lemma 4.5, $\mathbf{B}$ is a subreduct of an MTL-chain and the proof is done. □

The following theorem is an easy consequence of the previous lemma and of the first part of Lemma 4.1 (and the fact that FBCK$^{\mathcal{L}}$ is strongly complete w.r.t. FBCK$^{\mathcal{L}}$-chains).

Theorem 4.7 *Let $\mathcal{L}$ be a propositional language. Then* FBCK$^{\mathcal{L}}$ *is the $\mathcal{L}$-fragment of* MTL*, i.e., the $\mathcal{L}$-fragment of* MTL *is axiomatized by $\mathcal{AX}(\mathcal{L})$.*

Thus we can write FBCK$^{\mathcal{L}}$ = MTL$\restriction\mathcal{L}$.

4.3 Fragments of other fuzzy logics

In the previous subsection we have axiomatized the 16 fragments of MTL out of the 144 fragments considered in this paper. Thus there are 135 other fragments left. The upcoming lemma simplifies our work in a substantial way. First, we demonstrate that it is enough to find axioms of $\mathcal{L}$-fragments for propositional languages containing & or $\overline{0}$ only (besides $\to$, of course), then we obtain axiomatization for $\mathcal{L} \cup \{\wedge\}$, $\mathcal{L} \cup \{\vee\}$, and $\mathcal{L} \cup \{\wedge, \vee\}$-fragments for free by simply adding the usual axioms for these extra connectives. Second, we show that "reverse" claim holds as well. We formalize this idea as follows:

Lemma 4.8 *Let $\mathcal{L}$ be a propositional language, $\mathcal{L}' \subseteq \{\wedge, \vee\}$, $\mathbf{L}$ an axiomatic extension of* MTL*, and $\mathcal{A}$ a set of axioms in $\mathcal{L}$. Then $\mathcal{AX}(\mathcal{L}) + \mathcal{A}$ is an axiomatic system of $\mathbf{L}\restriction\mathcal{L}$ iff $\mathcal{AX}(\mathcal{L} \cup \mathcal{L}') + \mathcal{A}$ is an axiomatic system of $\mathbf{L}\restriction\mathcal{L} \cup \mathcal{L}'$.*

Proof: Left–to-right direction: let us denote the logic axiomatized by $\mathcal{AX}(\mathcal{L} \cup \mathcal{L}') + \mathcal{A}$ as $\mathbf{S}$. Obviously $\mathbf{L}\restriction\mathcal{L} \cup \mathcal{L}'$ is an extension of $\mathbf{S}$. We need to show that they equal.

Let T be an $\mathcal{L}\cup\mathcal{L}'$-theory and φ an $\mathcal{L}\cup\mathcal{L}'$-formula, such that $T \nvdash_{\mathbf{S}} \varphi$. Then there is an $\mathbf{S}$-chain $\mathbf{B}$ such that $T \not\models_{\mathbf{B}} \varphi$ (see Theorem 3.17). Obviously the $\mathcal{L}$-reduct of $\mathbf{B}$ is an $\mathbf{L}\restriction\mathcal{L}$-chain, let us denote it $\mathbf{B}\restriction\mathcal{L}$. Thus $\mathbf{B}\restriction\mathcal{L}$ is a subreduct of some $\mathbf{L}$-chain $\mathbf{C}$ (by Lemma 4.2, let us call the corresponding embedding f). Clearly $\mathbf{B}$ is a subreduct of $\mathbf{C}$ as well (from $f(x \to_{\mathbf{B}\restriction\mathcal{L}} y) = f(x) \to_{\mathbf{C}} f(y)$ and $f(\overline{1}_{\mathbf{B}\restriction\mathcal{L}}) = \overline{1}_{\mathbf{C}}$ we obtain that f is an order-preserving mapping and thus it preserves the lattice operations since the domain of f is a chain) and so $T \nvdash_{\mathbf{L}} \varphi$. Thus finally $T \nvdash_{\mathbf{L}\restriction\mathcal{L}\cup\mathcal{L}'} \varphi$.

Right-to-left direction: let us denote the logic axiomatized by $\mathcal{AX}(\mathcal{L}) + \mathcal{A}$ as $\mathbf{S}$. Obviously $\mathbf{L}\restriction\mathcal{L}$ is an extension of $\mathbf{S}$. We need to show that they equal. Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-formula, such that $T \nvdash_{\mathbf{S}} \varphi$. Then there is an $\mathbf{S}$-chain $\mathbf{B}$ such that $T \not\models_{\mathbf{B}} \varphi$ and obviously we can define lattice operations from $\mathcal{L}'$ in $\mathbf{B}$ and obtain $\mathbf{L}\restriction\mathcal{L} \cup \mathcal{L}'$-chain $\mathbf{B}'$ where also $T \not\models_{\mathbf{B}'} \varphi$. So $T \nvdash_{\mathbf{L}\restriction\mathcal{L}\cup\mathcal{L}'} \varphi$ and thus $T \nvdash_{\mathbf{L}\restriction\mathcal{L}} \varphi$. □

Observe that no additional axiom of our logics involves lattice connectives (see Table 1). Thus using the right-to-left direction of the lemma above we already know the axiomatizations of their $\{\to, \&, \overline{0}\}$-fragments. Since the axiomatizations of $\overline{0}$-free fragments of our logics (with the exception of ΠMTL; see the next theorem) were studied in [27], we also know axiomatizations of (nearly) all $\{\to, \&\}$-fragments.

Theorem 4.9 ([27])

- *The $\overline{0}$-free fragments of* MTL, IMTL, *and* SMTL *are axiomatized by $\mathcal{AX}(\to, \&, \wedge, \vee)$.*
- *The $\overline{0}$-free fragment of* BL *and* SBL *are axiomatized by $\mathcal{AX}(\to, \&, \wedge, \vee) + (Div)$.*
- *The $\overline{0}$-free fragment of* Ł *is axiomatized by $\mathcal{AX}(\to, \&, \wedge, \vee) + (Waj)$.*
- *The $\overline{0}$-free fragment of* G *is axiomatized by $\mathcal{AX}(\to, \&, \wedge, \vee) + (Cont)$.*
- *The $\overline{0}$-free fragment of* Π *is axiomatized by $\mathcal{AX}(\to, \&, \wedge, \vee) + (P1\text{–}3)$.*

The additional axioms $(P1$–$3)$ are defined as follows:

$(P1)$	$(\chi \to \chi\&\chi) \to [(\varphi\&(\varphi \to (\varphi \to \chi))) \to \chi]$
$(P2)$	$[((\varphi \to \chi) \to \chi)\&(\varphi\&\alpha \to \varphi\&\beta)\&(\chi\&\alpha \to \chi\&\beta)] \to (\alpha \to \beta)$
$(P3)$	$((\varphi \to \psi) \to \psi) \to [((\psi \to \chi) \to ((\psi \to \varphi) \to \varphi)) \to ((\psi \to \varphi) \to \varphi)]$

The axiomatic systems of $\overline{0}$-free fragment of ΠMTL was left open in [27]. We solve this problem by giving an axiomatic system for ΠMTL$\restriction\{\to, \&\}$ and using left-to-right direction of Lemma 4.8.

Theorem 4.10 *The logic* ΠMTL$\restriction\{\to, \&\}$ *is axiomatized by $\mathcal{AX}(\to, \&)$ plus the following axioms:*

$(C1)$	$[(\psi \to \chi) \to ((\psi \to \varphi\&\psi) \to \varphi)] \to ((\psi \to \varphi\&\psi) \to \varphi)$
$(C2)$	$[((\psi \to \varphi\&\psi) \to \varphi) \to (\psi \to \chi)] \to (\psi \to \chi)$

Proof: Obviously formula $(C1) \wedge (C2)$ is equivalent to $(\psi \to \chi) \vee ((\psi \to \varphi\&\psi) \to \varphi)$ in MTL (due to the definition of $\vee$) and $(\psi \to \chi) \vee ((\psi \to \varphi\&\psi) \to \varphi)$ is a theorem of ΠMTL. Thus $(C1)$ and $(C2)$ are provable in ΠMTL. Hence to complete the proof (using Lemma 4.1) we only have to show that each FBCK$^{\{\to,\&\}}$-chain $\mathbf{A} = (A, \&, \to, \overline{1}_{\mathbf{A}})$ satisfying the axioms $(C1), (C2)$ is a subreduct of a ΠMTL-chain. Since $\mathbf{A}$ is linearly ordered, we can easily define $\wedge$ and $\vee$. Thus $\mathbf{A}$ can be extended to $\mathbf{B} = (B, \wedge, \vee, \&, \to, \overline{1}_{\mathbf{B}})$ which is an $\overline{0}$-free subreduct of an MTL-chain. Moreover the formula $(\psi \to \chi) \vee ((\psi \to \varphi\&\psi) \to \varphi)$ is obviously a $\mathbf{B}$-tautology.

Now there are two cases. Firstly, assume that $\mathbf{B}$ has a minimum, say m. Then if we interpret $\overline{0}_{\mathbf{B}}$ as m, $\mathbf{B}$ becomes an MTL-chain. Since $(y \to z) \vee ((y \to x\&y) \to x) = \overline{1}_{\mathbf{B}}$ we obtain by setting $z = \overline{0}_{\mathbf{B}}$ that $\mathbf{B}$ satisfies $\neg y \vee ((y \to x\&y) \to x) = \overline{1}_{\mathbf{B}}$. Thus $\mathbf{B}$ must be a ΠMTL-chain as it was shown in [45, Corollary 19]. Secondly, suppose that $\mathbf{B}$ has no minimum. Then $\mathbf{B}$ is a subreduct of $\mathbf{2} \oplus \mathbf{B}$. We have to show again that $\mathbf{2} \oplus \mathbf{B}$ is a ΠMTL-chain, i.e., $\neg y \vee ((y \to x\&y) \to x) = \overline{1}_{\mathbf{2}\oplus\mathbf{B}}$. If $y = \overline{0}_{\mathbf{2}\oplus\mathbf{B}}$ then $\neg y = \overline{1}_{\mathbf{2}\oplus\mathbf{B}}$. If $y > \overline{0}_{\mathbf{2}\oplus\mathbf{B}}$ then there is $m \in B$ such that $\overline{0}_{\mathbf{2}\oplus\mathbf{B}} < m < y$ since we suppose that B has no minimum. Recall that $(y \to z) \vee ((y \to x\&y) \to x) = \overline{1}_{\mathbf{B}}$ is valid in $\mathbf{B}$. If we set $z = m$ then $y \to m < \overline{1}_{\mathbf{B}}$. Thus we obtain $((y \to x\&y) \to x) = \overline{1}_{\mathbf{B}}$ showing that $\neg y \vee ((y \to x\&y) \to x) = \overline{1}_{\mathbf{2}\oplus\mathbf{B}}$. □

Corollary 4.11 *The $\overline{0}$-free fragment of* ΠMTL *is axiomatized by* $\mathcal{AX}(\to, \&, \wedge, \vee) + (C1) + (C2)$.

Finally, we find axiomatizations of $\{\to\}$ and $\{\to, \overline{0}\}$-fragments of all our logics (with the exception of ΠMTL). Thus we can find axiomatizations of all other fragments by using the left-to-right direction of Lemma 4.8.

Recall that BL$\upharpoonright\{\to\}$ coincides with SBL$\upharpoonright\{\to\}$ (see Theorem 4.9) and that implicative fragments IMTL and SMTL coincide with MTL$\upharpoonright\{\to\}$ (again by Theorem 4.9) whose axiomatic systems were described in the previous section. Axiomatizations of $\{\to\}$-fragments of logics extending BL were given in paper [2]. Thus ΠMTL is the only logic whose axiomatization of the implicational fragment remains unknown. Let us summarize the known results:

Theorem 4.12 ([2])

- *The $\{\to\}$-fragments of* MTL, IMTL, *and* SMTL *are axiomatized by* $\mathcal{AX}(\to)$.
- *The $\{\to\}$-fragments of* BL *and* SBL *are axiomatized by* $\mathcal{AX}(\to) + (Div)$.
- *The $\{\to\}$-fragment of* Ł *is axiomatized by* $\mathcal{AX}(\to) + (Waj)$.
- *The $\{\to\}$-fragment of* G *is axiomatized by* $\mathcal{AX}(\to) + (Cont)$.
- *The $\{\to\}$-fragment of* Π *is axiomatized by* $\mathcal{AX}(\to) + (P3)$.

To complete our task we need to study the $\{\to, \overline{0}\}$-fragments. In Section 6 we will see that $\{\to, \overline{0}\}$-fragment of IMTL is termwise equivalent to its $\{\to, \&, \overline{0}\}$-fragment, and the $\{\to, \overline{0}\}$-fragment of Ł is termwise equivalent with Ł, thus we deal with the remaining fragments only (again with except of ΠMTL):

Theorem 4.13

1. *The $\{\to, \overline{0}\}$-fragment of* SMTL *is axiomatized by* $\mathcal{AX}(\to, \overline{0}) + (SBL)$.
2. *The $\{\to, \overline{0}\}$-fragment of* SBL *is axiomatized by* $\mathcal{AX}(\to, \overline{0}) + (Div), (SBL)$.
3. *The $\{\to, \overline{0}\}$-fragment of* G *is axiomatized by* $\mathcal{AX}(\to, \overline{0}) + (Cont)$.
4. *The $\{\to, \overline{0}\}$-fragment of* BL *is axiomatized by* $\mathcal{AX}(\to, \overline{0}) + (Div)$.
5. *The $\{\to, \overline{0}\}$-fragment of* Π *is axiomatized by* $\mathcal{AX}(\to, \overline{0}) + (P3) + (SBL)$.

Proof: To prove this theorem we use the first part of Lemma 4.1 and the fact that each logic **L** under consideration is complete w.r.t. the class of **L**-chains (cf. Theorem 3.17).

Part 1.: we show that each FBCK$^{\{\to,\overline{0}\}}$-chain **A** for which (SBL) is a tautology is a subreduct of an SMTL-chain. We take its $\overline{0}$-free reduct $\mathbf{A}'$, it is obviously an FBCK-chain and so we know from Lemma 4.6 that it is a subreduct of an MTL-chain **B**. Let us denote the embedding $\mathbf{A}' \to \mathbf{B}$ by f. Let $\mathbf{B}' = \mathbf{2} \oplus \mathbf{B}$. Observe that $\mathbf{B}'$ is an SMTL-chain. We define a mapping f' as $f'(\overline{0}_{\mathbf{A}}) = \overline{0}_{\mathbf{B}'}$ and $f'(x) = f(x)$ if $x \neq \overline{0}_{\mathbf{A}}$ (recall that $\overline{0}_{\mathbf{A}} \in \mathbf{A}'$). We show that f' is an embedding of **A** into $\mathbf{B}'$ and the proof is done. Nearly all the cases are simple, we show the only non-trivial case: assume that $\overline{0}_{\mathbf{A}} \neq x$. Thus $f'(x) \neq \overline{0}_{\mathbf{B}'}$ and $x \to \overline{0}_{\mathbf{A}} = \overline{0}_{\mathbf{A}}$, notice that in the last equality we used validity of (SBL) in **A** (see Lemma 3.22). Consequently, $f'(x \to \overline{0}_{\mathbf{A}}) = f'(\overline{0}_{\mathbf{A}}) = \overline{0}_{\mathbf{B}'} = f'(x) \to f'(\overline{0}_{\mathbf{A}})$.

Part 2. is proven analogously, we just start with an FBCK$^{\{\to,\overline{0}\}}$-chain **A** for which (Div) and (SBL) are tautologies. Using Theorem 4.12 we obtain that $\overline{0}$-free reduct $\mathbf{A}'$ is an SBL$\restriction\{\to\}$-chain. Lemma 4.2 gives us that $\mathbf{A}'$ is a subreduct of an SBL-chain **B**. The rest of the proof is the same as in the previous case.

Part 3. can be proved analogously as Part 2. (because (SBL) is a consequence of $(Cont)$).

We prove Part 4. differently. Let $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{B}$, and $f : \mathbf{A}' \to \mathbf{B}$ be defined in analogy with Part 1.; here **A** is an arbitrary FBCK$^{\{\to,\overline{0}\}}$-chain for which (Div) is a tautology, $\mathbf{A}'$ is its $\overline{0}$-free reduct, **B** is a BL-chain, and f is an embedding $\mathbf{A}' \to \mathbf{B}$ (the existence of **B** and f can be justified as in Part 2.). We define $\mathbf{B}'$ in the following way: its domain is the interval $[f(\overline{0}_{\mathbf{A}}), \overline{1}_{\mathbf{B}}]$ and its operations are: $\overline{0}_{\mathbf{B}'} = f(\overline{0}_{\mathbf{A}})$, $\overline{1}_{\mathbf{B}'} = \overline{1}_{\mathbf{B}}$, lattice connectives and implication are defined as in **B** (this interval is closed under these operations), and finally we set $x \&_{\mathbf{B}'} y = x \&_{\mathbf{B}} y$ if $x \&_{\mathbf{B}} y \geq \overline{0}_{\mathbf{B}'}$ or $\overline{0}_{\mathbf{B}'}$ otherwise (the reader familiar with the interval BL-chains [11] will observe that $\mathbf{B}' = \mathbf{B}[f(\overline{0}_{\mathbf{A}})]$). It is easy to observe that $\mathbf{B}'$ is a BL-chain and f is an embedding of **A** into $\mathbf{B}'$.

Finally, we prove Part 5. Let **A** be an FBCK$^{\{\to,\overline{0}\}}$-chain for which $(P3)$ and (SBL) are tautologies. The $\overline{0}$-free reduct $\mathbf{A}'$ of **A** must be a $\Pi\restriction\{\to\}$-chain (by Theorem 4.12). Thus, by Lemma 4.2, $\mathbf{A}'$ is a subreduct of a Π-chain **B**. Let $f : A \to B$ be the corresponding embedding. If we show that $f(\overline{0}_{\mathbf{A}}) = \overline{0}_{\mathbf{B}}$ then **A** is a subreduct of **B** as well. Suppose that $f(\overline{0}_{\mathbf{A}}) > \overline{0}_{\mathbf{B}}$. If **A** is a two-element chain then **A** is obviously a reduct of the two-element Boolean algebra which is also a Π-chain. Thus assume that A has more than two elements. Let $\overline{0}_{\mathbf{A}} < a < \overline{1}_{\mathbf{A}}$. Then $f(\overline{0}_{\mathbf{A}}) < f(a) < f(\overline{1}_{\mathbf{A}})$ since f is one-one. Further, by Lemma 3.22, we obtain $a \to \overline{0}_{\mathbf{A}} = \overline{0}_{\mathbf{A}}$. Since f is a homomorphism w.r.t. $\to$, we have $f(a) \to f(\overline{0}_{\mathbf{A}}) = f(a \to \overline{0}_{\mathbf{A}}) = f(\overline{0}_{\mathbf{A}})$. Since $\overline{0}_{\mathbf{B}} < y < x < \overline{1}_{\mathbf{B}}$ implies $x \to y > y$ in any Π-chain, we obtain $f(a) \to f(\overline{0}_{\mathbf{A}}) > f(\overline{0}_{\mathbf{A}})$ (a contradiction). □

As we will see in Section 6 the connective $\vee$ is definable in all fragments containing $\wedge$. Thus the following are the only remaining open problems[6].

Problem 4.14 *Let $\mathcal{L}$ be either empty or $\{\vee\}$ or $\{\wedge\}$. Then the axiomatic systems for the following logics seem to be unknown:* ΠMTL$\restriction\{\to\} \cup \mathcal{L}$ *and* ΠMTL$\restriction\{\to,\overline{0}\} \cup \mathcal{L}$.

Now we can summarize our results in Table 2. In the entry given by a logic **L** and a language $\mathcal{L}$ we list the axioms we need to add to MTL$\restriction\mathcal{L}$ to obtain an axiomatic system of $\mathbf{L}\restriction\mathcal{L}$. We restrict ourselves to $\{\to\}$, $\{\to,\overline{0}\}$, $\{\to,\&\}$, and $\{\to,\&,\overline{0}\}$-fragments as the axiomatizations of other fragments (those including lattice connectives) are obtained by means of Lemma 4.8.

Another way how to summarize our results is using the notion of the so-called strongly separable axiomatic systems. First, we give a formal definition of this notion in a form sufficient for us in this paper. Second, we form Table 3 containing the majority of our results achieved in this section.

Definition 4.15 *Let* **L** *be an axiomatic extension of* MTL *and let $\mathcal{A}$ be an axiomatic system of* **L**. *We say that $\mathcal{A}$ is* strongly separable *if for each propositional language $\mathcal{L}$ the $\mathcal{L}$-fragment of* **L** *is axiomatized by formulas from $\mathcal{A}$ which contain the connectives from $\mathcal{L}$ only.*

[6] Notice that those six open problems are strongly interrelated, first by means of Lemma 4.8 and second by the fact that the construction given for Π in the previous theorem would work for ΠMTL as well. Thus e.g. finding axiomatization for ΠMTL$\restriction\{\to\}$ would solve all the problems.

Table 2: Axiomatic systems of fragments of some prominent fuzzy logics

	$\to$	$\to, \overline{0}$	$\to, \&$	$\to, \&, \overline{0}$
MTL	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
IMTL	$\emptyset$	(DN)	$\emptyset$	(DN)
SMTL	$\emptyset$	(SBL)	$\emptyset$	(SBL)
ΠMTL	?	?	$(C1), (C2)$	$(\Pi 1), (SBL)$
BL	(Div)	(Div)	(Div)	(Div)
SBL	(Div)	$(Div), (SBL)$	(Div)	$(Div), (SBL)$
G	$(Cont)$	$(Cont)$	$(Cont)$	$(Cont)$
Ł	(Waj)	(Waj)	(Waj)	(Waj)
Π	$(Div), (P3)$	$(Div), (SBL), (P3)$	$(Div), (P1–3)$	$(Div), (\Pi 1), (SBL)$

Table 3: Strongly separable axiomatic systems for some prominent fuzzy logics

Logic	strongly separable axiomatic system
MTL	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0})$
IMTL	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0}) + (DN)$
SMTL	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0}) + (SBL)$
BL	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0}) + (Div)$
SBL	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0}) + (Div), (SBL)$
G	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0}) + (Cont)$
Ł	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0}) + (Waj)$
Π	$\mathcal{AX}(\to, \&, \wedge, \vee, \overline{0}) + (Div), (P1–3), (SBL)$

At the end of this section we tackle a problem which of our 144 logics are really different. In this section we answer this question for logics in the same propositional language and Section 6 is dedicated to study of definability of connectives and thus solving this problem in general.

Theorem 4.16 *Let $\mathcal{L}$ be a propositional language and $\mathbf{L}_1, \mathbf{L}_2$ be two different logics from the set* $\{\mathrm{MTL}, \mathrm{SMTL}, \mathrm{IMTL}, \Pi\mathrm{MTL}, \mathrm{BL}, \mathrm{SBL}, \mathrm{G}, \text{Ł}, \Pi\}$. *Then $\mathbf{L}_1 \restriction \mathcal{L}$ and $\mathbf{L}_2 \restriction \mathcal{L}$ are equal iff $\overline{0} \notin \mathcal{L}$ and one of the following conditions is fulfilled:*

- $\mathbf{L}_i \in \{\mathrm{MTL}, \mathrm{IMTL}, \mathrm{SMTL}\}$ *for* $i \in \{1, 2\}$
- $\mathbf{L}_i \in \{\mathrm{BL}, \mathrm{SBL}\}$ *for* $i \in \{1, 2\}$

Proof: We start by two simple observations. First, if $\mathbf{L}_1 \restriction \mathcal{L}$ and $\mathbf{L}_2 \restriction \mathcal{L}$ are equal then $\mathbf{L}_1 \restriction \mathcal{L}'$ and $\mathbf{L}_2 \restriction \mathcal{L}'$ are equal for each $\mathcal{L}' \subseteq \mathcal{L}$. This observation together with Theorem 4.9 gives us right-to-left direction. Second, we just reverse the first observation: if logics $\mathbf{L}_1 \restriction \{\to\}$ and $\mathbf{L}_2 \restriction \{\to\}$ are different then so are logics $\mathbf{L}_1 \restriction \mathcal{L}$ and $\mathbf{L}_2 \restriction \mathcal{L}$ (analogously for $\{\to, \overline{0}\}$ and $\mathcal{L} \cup \{\overline{0}\}$). It is quite obvious (as we know the corresponding axiomatic systems) that $\{\to\}$-fragments of logics MTL, BL, G, Ł, and Π are mutually distinct and so are $\{\to, \overline{0}\}$-fragments of logics MTL, SMTL, IMTL, BL, and SBL. Thus if we show that $\{\to\}$-fragments of logics ΠMTL and MTL are distinct the proof is done.

Suppose that the implicational fragments of MTL and ΠMTL are the same. It follows from Lemma 4.2 that the $\{\to\}$-reduct of any MTL-chain $\mathbf{A}$ can be embedded into the $\{\to\}$-reduct of a ΠMTL-chain $\mathbf{B}$. We construct an MTL-chain whose $\{\to\}$-reduct cannot be embedded into the $\{\to\}$-reduct of any ΠMTL-chain. In this way we obtain a contradiction with our original assumption. For this purpose we use the totally ordered monoid from [15, Section 5] which is not formally integral (for details see [15]). Let $(\mathbb{N}, +, 0)$ be the additive monoid on natural numbers.

For any $a, b, c, d \in \mathbb{N}$, $\langle a, b, c \rangle$ will denote the submonoid of $(\mathbb{N}, +, 0)$ generated by a, b, c, and $\langle a, b, c \rangle / d$ will denote the totally ordered monoid obtained by identifying with infinity all elements of $\langle a, b, c \rangle$ that are greater than or equal to d.

Let $S = \{32^*\} \cup \langle 9, 12, 16 \rangle / 30$ denote the totally ordered monoid obtained from $\langle 9, 12, 16 \rangle / 30$ by adding one additional element, denoted by 32^*. This element satisfies $16 + 16 = 32^*$, $32^* + z = \infty$ for $z \neq 0$, and the whole monoid is to be ordered as follows:

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty\,.$$

All the relations that do not involve 32^* are as in $\langle 9, 12, 16 \rangle / 30$, so we have to only check that $x \leq y$ implies $x + z \leq y + z$. If x or y or z is equal to 32^* then it is easy to see. If $x + z = 32^*$ and $x, z \neq 32^*$ then $x = z = 16$. Thus $32^* = 16 + 16 \leq y + 16$ because if $y > x$ then $y + 16 = \infty$.

Now since we want to make from this monoid an MTL-chain $\mathbf{A}$, we reverse the order:

$$0 > 9 > 12 > 16 > 18 > 21 > 24 > 25 > 27 > 28 > 32^* > \infty\,.$$

It is clear that a residuum exists since A is finite. Thus $\mathbf{A} = (A, \wedge, \vee, +, \rightarrow, \infty, 0)$ is an MTL-chain.

Now let us introduce the following identity:

$$(x_1 \& z_1 \rightarrow y_1 \& z_2) \vee (x_2 \& z_2 \rightarrow y_2 \& z_1) \vee (y_1 \& y_2 \rightarrow x_1 \& x_2) = \overline{1} \tag{1}$$

This identity is not valid in A. Indeed, let

$$\begin{array}{lll} x_1 = 16\,, & y_1 = 18\,, & z_1 = 16\,, \\ x_2 = 12\,, & y_2 = 9\,, & z_2 = 12\,. \end{array}$$

Then we obtain the following:

$$\begin{aligned} x_1 + z_1 \rightarrow y_1 + z_2 &= 32^* \rightarrow \infty = 9\,, \\ x_2 + z_2 \rightarrow y_2 + z_1 &= 24 \rightarrow 25 = 9\,, \\ y_1 + y_2 \rightarrow x_1 + x_2 &= 27 \rightarrow 28 = 9\,. \end{aligned}$$

Thus

$$(x_1 + z_1 \rightarrow y_1 + z_2) \vee (x_2 + z_2 \rightarrow y_2 + z_1) \vee (y_1 + y_2 \rightarrow x_1 + x_2) = 9 \neq 0\,.$$

Let $f : A \rightarrow B$ be an embedding of the $\{\rightarrow\}$-reduct of $\mathbf{A}$ into the $\{\rightarrow\}$-reduct of a ΠMTL-chain $\mathbf{B} = (B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1})$. We will show that the identity (1) is not valid also in $\mathbf{B}$. Since $0 > x_1 + z_1 \rightarrow y_1 + z_2 = x_1 \rightarrow (z_1 \rightarrow y_1 + z_2)$, we obtain

$$\overline{1} = f(0) > f(x_1) \rightarrow (f(z_1) \rightarrow f(y_1 + z_2)) = f(x_1) \& f(z_1) \rightarrow f(y_1 + z_2)\,.$$

Thus $f(x_1) \& f(z_1) > f(y_1 + z_2) \geq f(y_1) \& f(z_2)$, the second inequality holds in general: indeed form $x \leq y \rightarrow x + y$ and the fact that f is order-preserving we obtain $f(x) \leq f(y \rightarrow x + y) = f(y) \rightarrow f(x + y)$. Similarly we can obtain $f(x_2) \& f(z_2) > f(y_2) \& f(z_1)$ and $f(y_1) \& f(y_2) > f(x_1) \& f(x_2)$. Thus the identity (1) is not valid in $\mathbf{B}$ since

$$(f(x_1) \& f(z_1) \rightarrow f(y_1) \& f(z_2)) \vee (f(x_2) \& f(z_2) \rightarrow f(y_2) \& f(z_1)) \vee (f(y_1) \& f(y_2) \rightarrow f(x_1) \& f(x_2)) < \overline{1}\,.$$

However, it can be shown that the identity (1) is valid in any ΠMTL-chain $\mathbf{B} = (B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1})$. There are two cases.

1. It is clear that if one of the inequalities $x_1 \& z_1 \leq y_1 \& z_2$, $x_2 \& z_2 \leq y_2 \& z_1$, $y_1 \& y_2 \leq x_1 \& x_2$ is satisfied then the identity (1) is obviously valid.

2. Suppose that $x_1 \& z_1 > y_1 \& z_2$ and $x_2 \& z_2 > y_2 \& z_1$. In addition, we can assume that $x_1, x_2, y_1, y_2, z_1, z_2 > \overline{0}$ otherwise one of the inequalities from the previous point would be satisfied. Since we have $x_1 \& x_2 \& z_1 \& z_2 > y_1 \& y_2 \& z_1 \& z_2$, we obtain $x_1 \& x_2 > y_1 \& y_2$ by cancellativity. Thus the identity (1) is valid in this case as well.

Summing up, f cannot be an embedding of the $\{\rightarrow\}$-reduct of $\mathbf{A}$ into the $\{\rightarrow\}$-reduct of any ΠMTL-chain $\mathbf{B}$—a contradiction.

$\square$

5 Variety problem

In this section we decide for each of the 144 logics considered in this paper, whether its class of algebras is a variety or not. In fact we prove several general results, subsuming known results about BCK-algebras. We start by well-known positive results proven in [30].

Theorem 5.1 *Let $\mathcal{L}$ be a propositional language and $\mathcal{L} \cap \{\vee, \wedge\} \neq \emptyset$. Then the class of* $\mathrm{BCK}^{\mathcal{L}}$*-algebras is a variety.*

Another positive result is proven in [39] (see also [2]). It shows that whenever **L** proves (*Div*) then the class of its algebras is a variety.

Theorem 5.2 *Let $\mathcal{L}$ be a propositional language and* **L** *a logic in $\mathcal{L}$ such that* **L** *is an axiomatic extension of* $\mathrm{BL}\restriction\mathcal{L}$*. Then the class of* **L***-algebras is a variety.*

Thus the negative results can be only achieved in logics strictly weaker that BL. Such negative results for some expansions of BCK are known, e.g. the result by Wronski that BCK-algebras do not form a variety (see [58]) and the result by Higgs (see [28]) that (bounded) pocrims do not form a variety. We obtain these results as corollaries of our new stronger result. In particular we prove that $\Pi\mathrm{MTL}\restriction\{\to, \&\}$-algebras do not form a variety (notice that the example by Higgs from [28] would not work here since it is not cancellative).

Lemma 5.3 *The class of* $\Pi\mathrm{MTL}\restriction\{\to, \&\}$*-algebras is not a variety.*

Proof: Let $\mathbb{Z}^2_{lex}$ be the lexicographic product of two copies of integers. The set $\mathbb{Z}^2_{lex}$ forms a linearly ordered group with the multiplication defined component-wise. Consider its negative cone $\mathrm{N}(\mathbb{Z}^2_{lex})$. Let $\mathbb{Z}^-$ be the set of non-positive integers. We define the following set:

$$A = \mathrm{N}(\mathbb{Z}^2_{lex}) \setminus \{\langle 0, 2k-1\rangle \mid k \in \mathbb{Z}^-\}\,.$$

The set A clearly forms a submonoid of $\mathrm{N}(\mathbb{Z}^2_{lex})$. Let us denote the monoidal operation in A by &. Since A is a submonoid of the group $\mathbb{Z}^2_{lex}$, A must be cancellative. Even it is a residuated lattice with the residuum defined as follows:

$$\langle a,b\rangle \to \langle c,d\rangle = \begin{cases} \langle 0,0\rangle & \text{if } \langle a,b\rangle \leq \langle c,d\rangle\,,\\ \langle 0,\lfloor d-b\rfloor\rangle & \text{if } a=c \text{ and } b>d\,,\\ \langle c-a, d-b\rangle & \text{if } a>c\,,\end{cases}$$

where $\lfloor b\rfloor$ is the greatest even number less or equal to b. Thus $\mathbf{A} = (A, \wedge, \vee, \&, \to, \langle 0,0\rangle)$ forms a cancellative integral residuated chain and $\mathbf{2} \oplus \mathbf{A}$ is a ΠMTL-chain.

Now let us define the following relation:

$$\langle a,b\rangle \sim \langle c,d\rangle \text{ if } a=c \text{ and } [b=d \mod 2]\,,$$

where $b = d \mod 2$ stands for b equals d modulo 2. Thus b and d must be simultaneously even or odd. Observe that $\langle 0,b\rangle \sim \langle 0,d\rangle$ since $\langle 0,x\rangle \in A$ iff x is even. Obviously, $\sim$ is symmetric and reflexive. The transitivity can be easily checked. Thus $\sim$ is an equivalence. We show that $\sim$ is a congruence relation w.r.t. the monoidal operation & and the residuum $\to$.

Suppose that $\langle a,b\rangle \sim \langle a',b'\rangle$. Then $a = a'$. Thus it is sufficient to show that $\langle a,b\rangle\&\langle c,d\rangle \sim \langle a,b'\rangle\&\langle c,d\rangle$. If $a+c=0$ then it is obvious. Thus let us suppose that $a+c\neq 0$. Then we have to prove that $b+d = b'+d \mod 2$. There are several cases: (1) b,b',d are odd. Then $b+d$ and $b'+d$ are even. (2) b,b' are odd and d is even. Then $b+d$ and $b'+d$ are both odd. (3) b,b',d are even. Then $b+d$ and $b'+d$ are even. (4) b,b' are even and d is odd. Then $b+d$ and $b'+d$ are both odd.

Now we show that $\langle a,b\rangle \to \langle c,d\rangle \sim \langle a,b'\rangle \to \langle c,d\rangle$. If $a \leq c$ then the first components in $\langle a,b\rangle \to \langle c,d\rangle$, $\langle a,b'\rangle \to \langle c,d\rangle$ are zero. Thus $\langle a,b\rangle \to \langle c,d\rangle \sim \langle a,b'\rangle \to \langle c,d\rangle$ in this case. Assume that $a>c$. Then $\langle a,b\rangle \to \langle c,d\rangle = \langle c-a, d-b\rangle$ and similarly $\langle a,b'\rangle \to \langle c,d\rangle = \langle c-a, d-b'\rangle$. Since $c-a\neq 0$, we have to show that $d-b = d-b' \mod 2$.

1. $a = 0$. Then b, b' are even since the only pairs in A with the first component equal to 0 are of the form $\langle 0, 2k\rangle$, $k \in \mathbb{Z}^-$. If d is odd then $d - b$, $d - b'$ are odd otherwise even.

2. $a \neq 0$. Then $b = b' \mod 2$ by the definition of $\sim$. Again odd minus even is odd, even minus even is even, odd minus odd is even, and even minus odd is odd.

Finally we have to prove that $\langle c, d\rangle \to \langle a, b\rangle \sim \langle c, d\rangle \to \langle a, b'\rangle$. Again it is trivially valid for $c \leq a$ since the first components are 0 in this case. Thus suppose that $c > a$. Then $\langle c, d\rangle \to \langle a, b\rangle = \langle a - c, b - d\rangle$ and $\langle c, d\rangle \to \langle a, b'\rangle = \langle a - c, b' - d\rangle$. Since $0 \geq c > a$, we obtain $a - c \neq 0$ and $a \neq 0$. Thus $b = b' \mod 2$, by the definition of $\sim$. Again by arguing whether a subtraction of even and odd numbers is even or odd we finish the proof.

Now we know that $\sim$ is a congruence w.r.t. & and $\to$. Thus we can factorize A by $\sim$ and we obtain an algebra belonging to the variety generating by the $\{\&, \to\}$-subreducts of ΠMTL-chains. Recall that the class of $\{\&, \to\}$-subreducts of ΠMTL-algebras form a quasi-variety and the following quasi-identity is valid there:

$$\text{if } x \to y = y \to x = \overline{1} \text{ then } x = y.$$

If we prove that this quasi-identity is not valid in $A/\sim$ the proof is done.

Consider the equivalence classes $x = [\langle -1, 0\rangle]_\sim$ and $y = [\langle -1, -1\rangle]_\sim$ which are not equal. Then

$$[\langle -1, 0\rangle]_\sim \to [\langle -1, -1\rangle]_\sim = [\langle -1, 0\rangle \to \langle -1, -1\rangle]_\sim = [\langle 0, -2\rangle]_\sim = [\langle 0, 0\rangle]_\sim\,,$$
$$[\langle -1, -1\rangle]_\sim \to [\langle -1, 0\rangle]_\sim = [\langle -1, -1\rangle \to \langle -1, 0\rangle]_\sim = [\langle 0, 0\rangle]_\sim\,.$$

Thus $x \to y$ and $y \to x$ equal to the top element $[\langle 0, 0\rangle]_\sim$ but $x \neq y$. □

Corollary 5.4 *The classes of* ΠMTL$\restriction\{\to, \&, \overline{0}\}$*-algebras,* ΠMTL$\restriction\{\to\}$*-algebras, and* ΠMTL$\restriction\{\to, \overline{0}\}$*-algebras are not varieties.*

Proof: To prove the first claim we modify the proof of the lemma above. Let $\mathbf{A}$ and $\sim$ be the same as in the previous lemma. Then observe that $\sim \cup \{\langle \overline{0}_{\mathbf{2}\oplus\mathbf{A}}, \overline{0}_{\mathbf{2}\oplus\mathbf{A}}\rangle\}$ is a congruence in $\mathbf{2} \oplus \mathbf{A}$ w.r.t. & and $\to$. The other claims are obvious. □

Corollary 5.5 *The class of* IMTL$\restriction\{\to, \&, \overline{0}\}$*-algebras is not a variety.*

Proof: Let $\mathbf{A}$ be the cancellative integral residuated chain from the proof of Lemma 5.3. Then, by the construction from [18, Section 6] (also known as the disconnected rotation), we can make from $\mathbf{A}$ an IMTL-chain $\mathbf{A}'$. Let θ be the same congruence as in the proof of Lemma 5.3 and $\theta' = \{\langle a', b'\rangle \mid \langle a, b\rangle \in \theta\}$. Then it is proved in [18, Proposition 6.2] that $\theta \cup \theta'$ is a congruence w.r.t. the monoidal operation and the residuum. Again as in the proof of Lemma 5.3, $\mathbf{A}/\theta\cup\theta'$ does not satisfy the quasi-identity: if $x \to y = y \to x = \overline{1}$ then $x = y$. Thus IMTL$\restriction\{\to, \&, \overline{0}\}$-algebras does not form a variety. □

Now we formulate a final theorem of this section. Notice that it contains the aforementioned results by Wronski and Higgs.

Theorem 5.6 *Let $\mathcal{L}$ be a propositional language and $\mathbf{L}$ a logic in $\mathcal{L}$ such that $\mathbf{L}$ is an axiomatic extension of* BCK$^{\mathcal{L}}$ *weaker than* ΠMTL$\restriction\mathcal{L}$ *or* IMTL$\restriction\mathcal{L}$*. Then the class of $\mathbf{L}$-algebras is a variety iff $\mathcal{L} \cap \{\vee, \wedge\} \neq \emptyset$.*

Proof: First assume that $\mathcal{L} \cap \{\vee, \wedge\} \neq \emptyset$, then use Theorem 5.1 to obtain the result. Now assume that $\mathcal{L} \cap \{\vee, \wedge\} = \emptyset$. Further assume that $\mathbf{L}$ is weaker than ΠMTL$\restriction\mathcal{L}$. Then the result is a corollary of Lemma 5.3 and Corollary 5.4. Finally, assume that $\mathbf{L}$ is weaker than IMTL$\restriction\mathcal{L}$. If $\overline{0} \notin \mathcal{L}$ then $\mathbf{L}$ is also weaker than ΠMTL$\restriction\mathcal{L}$ (because IMTL$\restriction\mathcal{L}$ = MTL$\restriction\mathcal{L}$ in this case) and we can use the previous case. In Lemma 6.5 we will see that & is definable in IMTL$\restriction\{\to, \overline{0}\}$ and so there is only one remaining case to prove: when $\mathbf{L}$ is weaker than IMTL$\restriction\{\to, \&, \overline{0}\}$. This case is solved by Corollary 5.5. □

6 Definability problem

Now we start with the investigation of termwise equivalence of our fragments. We start by some definitions.

Definition 6.1 *Let $\mathcal{L}$ be a propositional language, $c \in \{\&, \wedge, \vee\}$ and $\mathbf{L}$ a logic in $\mathcal{L}$. We say that c is* definable *in $\mathbf{L}$ if there is an $\mathcal{L}$-formula φ of two variables, such that if we abbreviate $\varphi(p,q)$ as $c(p,q)$ we obtain $\vdash_{\mathbf{L}} \mathcal{AX}(c)$.*

We could formulate the analogous definition for $\overline{0}$ but it is quite obvious that $\overline{0}$ is not definable in any logic not containing it in its signature. Observe that if $c \in \mathcal{L}$ and $\mathbf{L}$ extends $\mathrm{BCK}^{\mathcal{L}}$ then c is trivially definable in $\mathbf{L}$. Now we formulate one lemma concerning definability in weaker or stronger logics. It is of course a rather trivial lemma but it explicates the way we prove our results.

Lemma 6.2 *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be propositional languages and $\mathbf{L}_i$ a logic in $\mathcal{L}_i$ such that $\mathbf{L}_2$ is an expansion of $\mathbf{L}_1$. If c is definable in $\mathbf{L}_1$ then it is definable in $\mathbf{L}_2$.*

Notice that given a propositional language $\mathcal{L}$ and a logic $\mathbf{L}$ in $\mathcal{L}$, if c is definable in $\mathbf{L}$ then we could understand $\mathbf{L}$ as a logic in the language $\mathcal{L} \cup \{c\}$.

Definition 6.3 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be propositional languages and $\mathbf{L}_i$ a logic in $\mathcal{L}_i$. We say that $\mathbf{L}_1$ and $\mathbf{L}_2$ are* termwise equivalent *if each $c \in \mathcal{L}_1 \setminus \mathcal{L}_2$ is definable in $\mathbf{L}_2$, each $c \in \mathcal{L}_2 \setminus \mathcal{L}_1$ is definable in $\mathbf{L}_1$, and $\mathbf{L}_1$ and $\mathbf{L}_2$ (understood as logics in $\mathcal{L}_1 \cup \mathcal{L}_2$) are equal.*

Of course if the logics are equal then they are also termwise equivalent. The next lemma is rather self-evident.

Lemma 6.4 *Let $\mathcal{L}$ be a propositional language, $c \in \{\&, \wedge, \vee\}$, and $\mathbf{L}$ an axiomatic extension of* MTL *such that c is definable in $\mathbf{L}{\restriction}\mathcal{L}$. Then $\mathbf{L}{\restriction}\mathcal{L}$ and $\mathbf{L}{\restriction}\mathcal{L} \cup \{c\}$ are termwise equivalent.*

Lemma 6.5

1. *$\wedge$ is definable in* $\mathrm{FBCK}^{\{\to,\&,\vee\}}$.
2. *$\vee$ is definable in* $\mathrm{FBCK}^{\{\to,\wedge\}}$.
3. *$\wedge$ is definable in* $\mathrm{BL}{\restriction}\{\to,\&\}$.
4. *$\vee$ is definable in* $\mathrm{Ł}{\restriction}\{\to\}$.
5. *& is definable in* $\mathrm{IMTL}{\restriction}\{\to,\overline{0}\}$.
6. *& is definable in* $\mathrm{G}{\restriction}\{\to,\wedge\}$.

Proof: 1. Set $\varphi \wedge \psi = (\varphi \& (\varphi \to \psi)) \vee (\psi \& (\psi \to \varphi))$. The proof of axiom $(A4c)$ is obvious, to prove $(A4b)$ we use the theorems $\varphi \& (\varphi \to \psi) \to \varphi$ and $\psi \& (\psi \to \varphi) \to \varphi$ and axiom $(A4{\vee}a)$. Finally we give a formal proof of $(A4a)$:

(i)	$\varphi \to \psi \vdash \varphi \to \varphi \wedge \psi$	simple
(ii)	$(\chi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \varphi))$	simple
(iii)	$\varphi \to \psi \vdash (\chi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \varphi \wedge \psi))$	(i),(ii), and transitivity
(iv)	$\psi \to \varphi \vdash (\chi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \varphi \wedge \psi))$	analogously
(v)	$\vdash (\chi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \varphi \wedge \psi))$	(iii), (iv), and prelinearity

2. Set $\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. The proofs are similar to the previous case.

3. Set $\varphi \wedge \psi = \varphi \& (\varphi \to \psi)$. The proof is well-known, see e.g. [27].

4. Set $\varphi \vee \psi = (\varphi \to \psi) \to \psi$. Again the proof is well-known, see e.g. [34].
5. Set $\varphi \& \psi = \neg(\varphi \to \neg\psi)$. Again the proof is well-known, see e.g. [13].
6. We set $\& = \wedge$. The proof is a part of the folklore, however for the sake of completeness we present it. First, we need to show that $\mathrm{G}\restriction\{\to, \wedge\}$ proves $(A5a)$:

   (i) $\vdash ((\varphi \to (\psi \to \chi)) \to (\varphi \wedge \psi \to (\varphi \wedge \psi \to \chi))$ — $(A4b), (A4c)$ and transitivity

   (ii) $\vdash ((\varphi \to (\psi \to \chi)) \to (\varphi \wedge \psi \to \chi)$ — (i) and $(Cont)$.

   Now we need to show that $\mathrm{G}\restriction\{\to, \wedge\}$ proves $(A5b)$:

   (i) $\vdash (\psi \to \varphi) \to ((\psi \to \psi) \to (\psi \to \varphi \wedge \psi))$ — $(A4a)$

   (ii) $\vdash \varphi \to (\psi \to \varphi \wedge \psi)$ — (i) and $(A2)$.

   (iii) $\vdash (\varphi \wedge \psi \to x) \to (\varphi \to (\psi \to \chi))$ — (ii) and transitivity.

□

Lemma 6.5 together with Lemma 6.2 tells us about definability of connectives in our logics. Using Lemma 6.4 we obtain the following important theorem which summarizes the positive results.

Theorem 6.6 *Let* $\mathbf{L}$ *be an axiomatic extension of* MTL *and* $\mathcal{L}$ *a propositional language. Then the following holds:*

1. *The logic* $\mathbf{L}\restriction\mathcal{L} \cup \{\wedge\}$ *is termwise equivalent with* $\mathbf{L}\restriction\mathcal{L} \cup \{\wedge, \vee\}$.
2. *The logics* $\mathbf{L}\restriction\mathcal{L} \cup \{\wedge, \&\}$, $\mathbf{L}\restriction\mathcal{L} \cup \{\vee, \&\}$, *and* $\mathbf{L}\restriction\mathcal{L} \cup \{\wedge, \vee, \&\}$ *are termwise equivalent.*
3. *If* $\vdash_{\mathbf{L}} (DIV)$ *then* $\mathbf{L}\restriction\mathcal{L} \cup \{\&\}$ *are* $\mathbf{L}\restriction\mathcal{L} \cup \{\wedge, \&\}$ *are termwise equivalent.*
4. *If* $\vdash_{\mathbf{L}} (DN)$ *then* $\mathbf{L}\restriction\mathcal{L} \cup \{\overline{0}\}$ *and* $\mathbf{L}\restriction\mathcal{L} \cup \{\overline{0}, \&\}$ *are termwise equivalent.*
5. *If* $\vdash_{\mathbf{L}} (Waj)$ *then* $\mathbf{L}\restriction\mathcal{L}$ *and* $\mathbf{L}\restriction\mathcal{L} \cup \{\vee\}$ *are termwise equivalent.*
6. *If* $\vdash_{\mathbf{L}} (Waj)$ *then* $\mathbf{L}\restriction\mathcal{L} \cup \{\overline{0}\}$ *and* $\mathbf{L}$ *are termwise equivalent.*
7. *If* $\vdash_{\mathbf{L}} (Cont)$ *then* $\mathbf{L}\restriction\mathcal{L} \cup \{\&\}$, $\mathbf{L}\restriction\mathcal{L} \cup \{\wedge\}$, *and* $\mathbf{L}\restriction\mathcal{L} \cup \{\wedge, \&\}$ *are termwise equivalent.*

In the upcoming lemma we present basic negative results concerning definability.

Lemma 6.7

1. $\overline{0}$ *is undefinable in any logic in any language not containing it.*
2. $\wedge$ *is undefinable in* $\Pi\mathrm{MTL}\restriction\{\to, \&\}$ *and* $\Pi\mathrm{MTL}\restriction\{\to, \&, \overline{0}\}$.
3. $\wedge$ *is undefinable in* $\mathrm{IMTL}\restriction\{\to, \&, \overline{0}\}$.
4. $\wedge$ *is undefinable in three-valued* $\mathrm{G}\restriction\{\to, \overline{0}, \vee\}$*-algebra.*
5. $\&$ *is undefinable in* $\Pi\restriction\{\to, \wedge, \vee, \overline{0}\}$ *and* $\Pi\restriction\{\to, \wedge, \vee\}$.
6. $\vee$ *is undefinable in three-valued* $\mathrm{G}\restriction\{\to, \overline{0}\}$*-algebra.*
7. $\wedge$ *is undefinable in* $\text{Ł}\restriction\{\to, \vee\}$.
8. $\&$ *is undefinable in* $\text{Ł}\restriction\{\to, \vee, \wedge\}$.

Proof: Part 1. is obvious since $\overline{0}$ is not definable even in the classical logic. Parts 2. and 3. are easy consequences of Theorem 5.6. For the proof of Part 4. see [52]. We present the proof of Part 5: take any formula φ of one variable p in the language $\{\rightarrow, \wedge, \vee, \overline{0}\}$. Then take the standard product algebra and an evaluation $e(p) = \frac{1}{2}$. Observe that $e(\varphi) \in \{0, \frac{1}{2}, 1\}$, while $e(p\&p) = \frac{1}{4}$.

Next we prove[7] Part 6. Let us recall the semantics of $\rightarrow$ in this logic:

$\rightarrow$	0	u	1
0	1	1	1
u	0	1	1
1	0	u	1

Now take any formula φ of two variable p and q in the language $\{\rightarrow, \overline{0}\}$. By $\varphi(x, y)$ denote the value of φ in evaluation $e(p) = x$ and $e(q) = y$. We show that either $\varphi(0, u)$ or $\varphi(u, 0)$ is not u (thus none such formula defines $\vee$). We show it by induction: observe that formulas p, q, and $\overline{0}$ has this property. Now take formula $\varphi \rightarrow \psi$. Recall that either $\psi(0, u)$ or $\psi(u, 0)$ is not u and observe that in this case the same holds for formula $\varphi \rightarrow \psi$.

Part 7. is an easy consequence of the well-known fact that $\wedge$ is not definable from $\rightarrow$ and $\vee$ even in the classical logic.

Finally to prove Part 8. take three-valued Ł$\restriction\{\rightarrow, \vee, \wedge\}$-algebra. The interpretation of $\rightarrow$ is:

$\rightarrow$	0	u	1
0	1	1	1
u	u	1	1
1	0	u	1

Observe that the value of any formula φ in (u, u) is either u or 1 and $u\&u = 0$. □

The proof of the following lemma is trivial. It allows us to extend the non-definability results from the three-valued Gödel logic to the logics weaker than $\Pi\restriction\{\rightarrow, \wedge, \vee, \overline{0}\}$.

Lemma 6.8 *Three valued* $\mathrm{G}\restriction\{\rightarrow, \overline{0}\}$*-algebra is also an* $\Pi\restriction\{\rightarrow, \wedge, \vee, \overline{0}\}$*-algebra (if we define* $\wedge$ *and* $\vee$ *in the obvious way).*

Using Lemma 6.2 we can extend the negative results to the remaining fragments. Note that one of our results, namely that $\wedge$ is undefinable in $\mathrm{MTL}\restriction\{\rightarrow, \&, \overline{0}\}$, was recently proven in [55]. Some other results are also known, namely those dealing with definability in Łukasiewicz logic in fragments containing $\overline{0}$, see [57].

Theorem 6.9

1. $\wedge$ *is undefinable in* $\mathbf{L}\restriction\{\rightarrow, \&\}$ *and* $\mathbf{L}\{\rightarrow, \&, \overline{0}\}$ *for* $\mathbf{L}$ *being one of the* MTL*,* SMTL*,* IMTL *and* ΠMTL*.*
2. $\wedge$ *is undefinable in* $\mathbf{L}\restriction\{\rightarrow, \vee, \overline{0}\}$*,* $\mathbf{L}\{\rightarrow, \vee\}$*,* $\mathbf{L}\{\rightarrow, \overline{0}\}$*, and* $\mathbf{L}\{\rightarrow\}$ *for* $\mathbf{L}$ *being one of the* MTL*,* SMTL*,* ΠMTL*,* BL*,* SBL*,* G*, and* Π*.*
3. $\wedge$ *is undefinable in* $\mathrm{IMTL}\restriction\{\rightarrow, \vee\}$*,* $\mathrm{IMTL}\{\rightarrow, \overline{0}\}$*,* $\mathrm{IMTL}\{\rightarrow\}$*.*
4. $\wedge$ *is undefinable in* Ł$\restriction\{\rightarrow, \vee\}$ *and* Ł$\{\rightarrow\}$*.*
5. $\vee$ *is undefinable in* $\mathbf{L}\restriction\{\rightarrow, \&\}$ *and* $\mathbf{L}\{\rightarrow, \&, \overline{0}\}$ *for* $\mathbf{L}$ *being one of the* MTL*,* SMTL*,* IMTL *and* ΠMTL*.*
6. $\vee$ *is undefinable* $\mathbf{L}\restriction\{\rightarrow, \overline{0}\}$ *and* $\mathbf{L}\{\rightarrow\}$ *for* $\mathbf{L}$ *being one of the* MTL*,* SMTL*,* IMTL*,* ΠMTL*,* BL*,* SBL*,* G*, and* Π*.*
7. $\&$ *is undefinable in* $\mathbf{L}\restriction\mathcal{L}$ *for* $\& \notin \mathcal{L}$ *and* $\mathbf{L}$ *being one of the* MTL*,* SMTL*,* ΠMTL*,* BL*,* SBL*,* G*, and* Π*.*
8. $\&$ *is undefinable in* $\mathbf{L}\restriction\mathcal{L}$ *for* $\&, \overline{0} \notin \mathcal{L}$ *and* $\mathbf{L}$ *being one of the* IMTL *and* Ł*.*

Table 4: Fragments without $\overline{0}$

Logic	$\to$	$\to,\wedge$	$\to,\vee$	$\to,\wedge,\vee$	$\to,\&$	$\to,\&,\vee$	$\to,\&,\wedge$	$\to,\&,\wedge,\vee$
MTL		$\vee$				$\wedge$	$\vee$	
IMTL		$\vee$				$\wedge$	$\vee$	
SMTL		$\vee$				$\wedge$	$\vee$	
ΠMTL		$\vee$				$\wedge$	$\vee$	
BL		$\vee$			$\wedge,\vee$	$\wedge$	$\vee$	
SBL		$\vee$			$\wedge,\vee$	$\wedge$	$\vee$	
G		$\&,\vee$		$\&$	$\wedge,\vee$	$\wedge$	$\vee$	
Ł	$\vee$	$\vee$			$\wedge,\vee$	$\wedge$	$\vee$	
Π		$\vee$			$\wedge,\vee$	$\wedge$	$\vee$	

Table 5: Fragments with $\overline{0}$

Logic	$\to,\overline{0}$	$\to,\wedge,\overline{0}$	$\to,\vee,\overline{0}$	$\to,\wedge,\vee,\overline{0}$	$\to,\&,\overline{0}$	$\to,\&,\vee,\overline{0}$	$\to,\&,\wedge,\overline{0}$
MTL		$\vee$				$\wedge$	$\vee$
IMTL	$\&$	$\&,\vee$	$\&,\wedge$	$\&$		$\wedge$	$\vee$
SMTL		$\vee$				$\wedge$	$\vee$
ΠMTL		$\vee$				$\wedge$	$\vee$
BL		$\vee$			$\wedge,\vee$	$\wedge$	$\vee$
SBL		$\vee$			$\wedge,\vee$	$\wedge$	$\vee$
G		$\&,\vee$		$\&$	$\wedge,\vee$	$\wedge$	$\vee$
Ł	$\&,\wedge,\vee$	$\&,\vee$	$\&,\wedge$	$\&$	$\wedge,\vee$	$\wedge$	$\vee$
Π		$\vee$			$\wedge,\vee$	$\wedge$	$\vee$

We summarize our results in Tables 4 and 5. Each entry of the table is determined by a logic $\mathbf{L}$ and a language $\mathcal{L}$. Let us by $C(\mathbf{L},\mathcal{L})$ denote the set of connectives appearing in the corresponding entry (if the entry is empty we set $C(\mathbf{L},\mathcal{L}) = \emptyset$). Observe that we have not included column for $\mathcal{L} = \{\to,\&,\wedge,\vee,\overline{0}\}$, in this case let $C(\mathbf{L},\{\to,\&,\wedge,\vee,\overline{0}\}) = \emptyset$. Now we can formulate all our definability results in one theorem.

Theorem 6.10 *The connective c is definable in $\mathbf{L}\restriction\mathcal{L}$ iff c is in $C(\mathbf{L},\mathcal{L}) \cup \mathcal{L}$.*

We can also use these tables to formulate all results about termwise equivalence in one theorem. The rather complicated formulation of this theorem is caused by the fact that some fragments of different logics are in fact the same (see Theorem 4.16).

Theorem 6.11 *For $i \in \{1,2\}$ let $\mathbf{L}_i \in \{\mathrm{MTL},\mathrm{SMTL},\mathrm{IMTL},\Pi\mathrm{MTL},\mathrm{BL},\mathrm{SBL},\mathrm{G},\text{Ł},\Pi\}$ and $\mathcal{L}_i$ be a propositional language. Then $\mathbf{L}_1\restriction\mathcal{L}_1$ and $\mathbf{L}_2\restriction\mathcal{L}_2$ are termwise equivalent iff $C(\mathbf{L}_1,\mathcal{L}_1)\cup\mathcal{L}_1 = C(\mathbf{L}_2,\mathcal{L}_2)\cup\mathcal{L}_2$ and at least one of the following conditions is fulfilled:*

- $\mathbf{L}_1 = \mathbf{L}_2$
- $\overline{0} \notin \mathcal{L}_i$ *and* $\mathbf{L}_i \in \{\mathrm{MTL},\mathrm{IMTL},\mathrm{SMTL}\}$ *for* $i \in \{1,2\}$
- $\overline{0} \notin \mathcal{L}_i$ *and* $\mathbf{L}_i \in \{\mathrm{BL},\mathrm{SBL}\}$ *for* $i \in \{1,2\}$

[7] We wish to thank to Vítězslav Švejdar for providing this elegant proof in a personal communication.

Using this theorem we can find out that there are exactly 57 mutually distinct logics (out of 144). To see this inspect Tables 4 and 5. In Table 4, remembering that the corresponding fragments of MTL, SMTL, and IMTL are the same, delete the rows for (say) SMTL and IMTL; also delete the row for SBL. Then compute the number of empty entries—you get 24. In Table 5 nothing is to be deleted, but add the last column corresponding to the full language with all entries empty. Then compute the number of empty entries—you get 33. The sum is the resulting number of mutually distinct logics.

7 Predicate logics

In this section we deal with a problem of axiomatizing predicate logics in fragments where $\vee$ is not definable(we have seen on the previous section that there are many such logics). We assume that the reader is familiar with the syntax and semantics of the predicate versions of our fuzzy logics (see [13, 22]). We just recall the definition of the axiomatic system:

Definition 7.1 *Let* $\mathbf{L}$ *be an axiomatic extension of* MTL. *The logic* $\mathbf{L}\forall$ *has the axioms:*

(P)	*the axioms resulting from the axioms of* $\mathbf{L}$ *by the substitution of propositional variables with formulas of the predicate language* $\mathbf{\Gamma}$,
$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$, *where* t *is substitutable for* x *in* φ,
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$, *where* t *is substitutable for* x *in* φ,
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where* x *is not free in* χ,
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where* x *is not free in* χ,
$(\forall 3)$	$(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$, *where* x *is not free in* χ.

The deduction rules are modus ponens and generalization: from φ *infer* $(\forall x)\varphi$.

We give an alternative definition with an axiom where $\vee$ does not occur. Then we show the completeness theorem and the fact that both definitions coincide in the case that $\vee$ is in the language.

Before we start we recall a useful abbreviation. Let $\varphi \prec \psi$ stand for the formula $(\varphi \to \psi) \to \psi$ (cf. [24]; in [44] the notation $\varphi \uparrow \psi$ is used).

Definition 7.2 *Let* $\mathcal{L}$ *be a propositional language and* $\mathbf{L}$ *an axiomatic extension of* MTL$\restriction\mathcal{L}$. *The logic* $\mathbf{L}\forall'$ *has the same axioms as* $\mathbf{L}\forall$ *with axiom* $(\forall 3)$ *replaced by:*

$$(\forall 3') \quad (\forall x)(\nu \prec \varphi) \to [(\forall x)(\varphi \prec \nu) \to ((\forall x)\varphi \prec \nu)],$$

where x *is not free in* ν.

The following lemma is an easy consequence of quantifiers shifts valid in fuzzy logics (see e.g. [22, Theorems 5.1.14, 5.1.16]).

Proposition 7.3 *Let* $\mathcal{L}$ *be a propositional language and* $\mathbf{L}$ *an axiomatic extension of* MTL$\restriction\mathcal{L}$. *Then the following formula is provable in* $\mathbf{L}\forall'$*:*

$$(\forall x)(\nu \prec \varphi) \to (\nu \prec (\forall x)\varphi),$$

where x *is not free in* ν.

Theorem 7.4 (Local Deduction Theorem) *Let* $\mathcal{L}$ *be a propositional language and let* $\mathbf{L}$ *be any axiomatic extension of* MTL$\restriction\mathcal{L}$. *Then for each pair of sentences* φ *and* ψ *and a theory* T *over* $\mathbf{L}\forall'$ *we have:* $T \cup \{\varphi\} \vdash_{\mathbf{L}\forall'} \psi$ *iff there is* n *such that* $T \vdash_{\mathbf{L}\forall'} \varphi \to^n \psi$.

Proof: The proof is analogous to the propositional case. □

Now we can use this deduction theorem together with the axiom (A6) and show that the predicate logics also fulfils the *Prelinearity Property*, i.e.,

- for each theory T, sentences φ, ψ, and a formula χ it holds: if $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ then $T \vdash \chi$.

Theorem 7.5 (Completeness theorem) *Let $\mathcal{L}$ be a propositional language and* $\mathbf{L}$ *an axiomatic extension of* MTL$\restriction\mathcal{L}$. *Let T be a theory over $\mathbf{L}\forall'$ and φ be a formula. Then $T \vdash_{\mathbf{L}\forall'} \varphi$ iff $\mathbf{M} \models \varphi$ for each* $\mathbf{L}$*-chain* $\mathbf{A}$ *and each* $\mathbf{A}$*-model* $\mathbf{M}$ *of T.*

Proof: The soundness is easy (see the proof of analogous lemma—[22, Lemma 5.1.9]). We just inspect the proof of completeness of BL$\forall$ [22, Section 5.2] and construct a theory $\hat{T} \supseteq T$ which is complete, Henkin and $\hat{T} \not\vdash \varphi$. Moreover, it gives us a model of T over the Lindenbaum algebra of $\hat{T}$, which is an $\mathbf{L}$-chain. The places to be changed are Case 1 (handling a pair α, β of sentences) and Case 2 (handling $(\forall x)\chi(x)$) of the proof of Lemma 5.2.7. For Case 1 show that if $T_n \not\vdash \varphi$ then $T \cup \{\beta \to \alpha\} \not\vdash \varphi$ or $T \cup \{\alpha \to \beta\} \not\vdash \varphi$ using the Prelinearity Property. For Case 2 proceed as follows: given T_n, α_n and $\chi(x)$, distinguish three subcases (c being a new constant).

(i) $T_n \not\vdash \alpha_n \prec \chi(c)$: let $T_{n+1} = T_n$ and $\alpha_{n+1} = \alpha_n \prec \chi(c)$.

(ii) $T_n \not\vdash \chi(c) \prec \alpha_n$: let $T_{n+1} = T_n$ and $\alpha_{n+1} = \chi(c) \prec \alpha_n$.

(iii) T_n proves both $\chi(c) \prec \alpha_n$ and $\alpha_n \prec \chi(c)$, hence T_n proves $(\forall x)(\chi(x) \prec \alpha_n)$ and $(\forall x)(\alpha_n \prec \chi(x))$ (since T_n assumes nothing on c). By $(\forall 3')$, $T_n \vdash (\forall x)\chi(x) \prec \alpha_n$.

Define $T_{n+1} = T_n, (\alpha_n \to (\forall x)\chi(x))$ and $\alpha_{n+1} = \alpha_n$. Observe that $T_n, (\forall x)\chi(x) \to \alpha_n \vdash \alpha_n$ and therefore $T_{n+1} \not\vdash \alpha_n$ (use the Prelinearity Property and the fact $T_n \not\vdash \alpha_n$).

Since $T_n \vdash (\forall x)(\alpha_n \prec \chi(x))$ we can use Proposition 7.3 to obtain $T_n \vdash \alpha_n \prec (\forall x)\chi(x)$ and so $T_{n+1} \vdash (\forall x)\chi$.

Finally observe that $\hat{T} = \bigcup_i T_i$ is Henkin. If $\hat{T} \not\vdash (\forall x)\chi$ and χ was processed in the step n then we used either the case (i) or (ii) (recall that in the case (iii) we would obtain that $T_{n+1} \vdash (\forall x)\chi$). As both $\varphi \to (\varphi \prec \psi)$ and $\psi \to (\varphi \prec \psi)$ are theorems we obtain that $T_{n+1} \not\vdash \chi(c)$ and the proof is done. □

Now we can use this completeness theorem together with the completeness theorem for logics with $\vee$ in the language and obtain the promised result:

Theorem 7.6 *Let $\mathcal{L}$ be a propositional language such that $\vee \in \mathcal{L}$ and* $\mathbf{L}$ *an axiomatic extension of* MTL$\restriction\mathcal{L}$. *Then $\mathbf{L}\forall$ and $\mathbf{L}\forall'$ are equal.*

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