

Finite Embeddability Property for Residuated Groupoids

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Abstract

A very simple proof of the finite embeddability property for residuated distributive-lattice-ordered groupoids and some related classes of algebras is presented. In particular, this gives an answer to the question posed in [3, Problem 4.2]. The presented construction allows for improvement of the upper bound on the complexity of the decision procedure for the universal theory of residuated distributive-lattice-ordered groupoids, given in [5]; for chains in the class, a tight bound can be obtained.

1 Introduction

A class of algebras \mathbb{K} has the *finite embeddability property* (FEP) if every finite partial subalgebra of a member of \mathbb{K} can be embedded into a finite member of \mathbb{K} . The importance of this property stems from the fact that it implies decidability of the universal theory of \mathbb{K} provided \mathbb{K} is a finitely axiomatizable class of algebra. The first result on the FEP for residuated structures appeared in [2] where it was proved that the class of partially ordered commutative residuated integral monoids (pocrims) has the FEP. Later, the proof was generalized to a non-associative and non-commutative setting in [3], namely it was shown that the class of integral residuated groupoids has the FEP. The proof uses Higman's lemma and relies heavily on the assumption of integrality.

This paper presents a very simple proof of the finite embeddability property (FEP) for residuated distributive-lattice-ordered groupoids (this class of algebras forms an equivalent algebraic semantics for distributive non-associative full Lambek calculus). One can also prove by the same method the FEP for residuated ordered groupoids, because every residuated ordered groupoid can be embedded into a distributive-lattice ordered one. Our proof does not use the integrality assumption, but it will not work for associative structures, nor will it work if an existing lattice ordering is not distributive. Our result thus answers the question whether the class of residuated ordered groupoids has the FEP, posed in [3, Problem 4.2]; this result is not new since an affirmative answer was already given in [10] using a proof-theoretic method. Still, our algebraic proof is considerably simpler. We also subsume the result proved in [5, 10] showing that the class of residuated distributive-lattice-ordered groupoids has the FEP. The decision procedure indicated in [5] can be improved by our method: their procedure operates in doubly exponential time, while our method puts the universal theory of residuated distributive-lattice-ordered groupoids into **coNEXP**. However, this bound might not be tight; the only lower bound we can give is that the theory is **coNP**-hard.

However, our method gives a tight upper bound on the complexity of the universal theory of totally ordered residuated ordered groupoids. The assumption that the initial algebra is totally ordered simplifies the construction described in Section 3, which then yields that the universal theory is **coNP**-complete.

2 Preliminaries

We review some notions from order theory and fix a notation thereto. Let $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ be a partially ordered set (*poset*). $Q \subseteq P$ is a *downset* on \mathbf{P} if for all $x \in Q$ and $y \in P$, $y \leq^{\mathbf{P}} x$ implies $y \in Q$. The system of all downsets on \mathbf{P} will be denoted $\mathcal{O}(\mathbf{P})$; for a $P' \subseteq P$, the downset generated by P' within \mathbf{P} will be denoted $\downarrow P'$. We say that $\langle P, \leq^{\mathbf{P}} \rangle$ is a *chain* if $\leq^{\mathbf{P}}$ is a total (linear) order. Considering two posets $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ and $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$, a map $\varphi: P \rightarrow Q$ is

- *order preserving* if $x \leq^{\mathbf{P}} y$ implies $\varphi(x) \leq^{\mathbf{Q}} \varphi(y)$ for all $x, y \in P$,
- *order reflecting* if $\varphi(x) \leq^{\mathbf{Q}} \varphi(y)$ implies $x \leq^{\mathbf{P}} y$ for all $x, y \in P$,
- an *order embedding* if it is order preserving and order reflecting.

All algebras considered in this paper will be of finite type. Let $\mathbf{A} = \langle A, \langle f_i^{\mathbf{A}} \rangle_{i \in I} \rangle$ be an algebra of any type and $B \subseteq A$. Then $\mathbf{B} = \langle B, \langle f_i^{\mathbf{B}} \rangle_{i \in I} \rangle$ is a *partial subalgebra* of \mathbf{A} where for every $n \in \mathbb{N}$, for every n -ary function symbol f_i with $i \in I$, and for every $b_1, \dots, b_n \in B$, one defines

$$f_i^{\mathbf{B}}(b_1, \dots, b_n) = f_i^{\mathbf{A}}(b_1, \dots, b_n)$$

if $f_i^{\mathbf{A}}(b_1, \dots, b_n) \in B$; otherwise, the value is not defined. If \mathbf{A} is ordered with a (partial) order $\leq^{\mathbf{A}}$ on A , then the partial subalgebra \mathbf{B} of \mathbf{A} is ordered as well with $\leq^{\mathbf{B}} = \leq^{\mathbf{A}} \upharpoonright B$, i.e., $b_1 \leq^{\mathbf{B}} b_2$ iff $b_1 \leq^{\mathbf{A}} b_2$ for all $b_1, b_2 \in B$.

Given an algebra \mathbf{C} of the same type as \mathbf{A} and a one-one map $g: B \rightarrow C$, we say that g is an *embedding* of \mathbf{B} into \mathbf{C} if for every $n \in \mathbb{N}$, for every n -ary function symbol f_i with $i \in I$, and for every $b_1, \dots, b_n \in B$, we have

$$g(f_i^{\mathbf{B}}(b_1, \dots, b_n)) = f_i^{\mathbf{C}}(g(b_1), \dots, g(b_n)),$$

whenever $f_i^{\mathbf{B}}(b_1, \dots, b_n)$ is defined. If \mathbf{B} and \mathbf{C} are ordered, then g is required to be an order embedding.

DEFINITION 2.1 ([9, 2]). *A class \mathbb{K} of (ordered) algebras has the finite embeddability property (FEP) if for each finite partial subalgebra \mathbf{B} of an algebra $\mathbf{A} \in \mathbb{K}$ there exists a finite algebra $\mathbf{C} \in \mathbb{K}$ such that \mathbf{B} embeds into \mathbf{C} .*

Let \mathbb{K} be a class of algebras. $\text{Th}_{\forall}(\mathbb{K})$ denotes the first-order theory of \mathbb{K} , i.e., the set of first-order universal sentences valid in \mathbb{K} . If \mathbb{K} has the FEP then $\text{Th}_{\forall}(\mathbb{K})$ is decidable provided that \mathbb{K} is finitely axiomatizable (see e.g. [3]). At the same time, the FEP as such gives no upper bound on the computational complexity of the universal fragment.

DEFINITION 2.2. *A residuated (partially) ordered groupoid (shortly r.o.g.) is a structure $\mathbf{A} = \langle A, \circ^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, 1^{\mathbf{A}}, \leq^{\mathbf{A}} \rangle$, or less explicitly $\langle A, \circ, \backslash, /, 1, \leq \rangle$, provided that $\langle A, \leq \rangle$ is a poset and for all $x, y, z \in A$ we have $1 \circ x = x = x \circ 1$ and*

$$x \circ y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z / y.$$

REMARK 2.3. *The unit 1 is not usually considered in the signature of residuated ordered groupoids. We include it here because our method works also if 1 is in the signature. Nevertheless, one can proceed in the same way without 1 just by omitting the parts in the proofs dealing with 1.*

The class of all residuated ordered groupoids (r.o.g.'s) is denoted \mathbb{ROG} . We consider subclasses of \mathbb{ROG} that satisfy some of the following properties. A r.o.g. \mathbf{A} is

- *distributive-lattice ordered* if $\langle A, \leq \rangle$ is a distributive lattice,
- *meet-semilattice ordered* if $\langle A, \leq \rangle$ is a meet-semilattice,
- *bounded* if $\langle A, \leq \rangle$ has a minimum \perp and a maximum \top .

We will use \mathbf{Prop} to denote all the properties listed above. For $Q \subseteq \mathbf{Prop}$, let \mathbb{ROG}^Q denote the class of r.o.g.'s satisfying all the properties from Q . The class \mathbb{BRDG} of r.o.g.'s having all the above properties, i.e., the class of all bounded residuated distributive-lattice-ordered groupoids (b.r.d.g.'s), forms a variety in the language $\{\wedge, \vee, \circ, \backslash, /, 1, \perp, \top\}$. Moreover, any r.o.g. can be embedded into a b.r.d.g.

THEOREM 2.4. *Let $Q \subseteq \mathbf{Prop}$ and $\mathbf{A} \in \mathbb{ROG}^Q$. Then \mathbf{A} can be order-embedded into $\mathbf{B} \in \mathbb{BRDG}$; the embedding preserves any meets, the minimum and the maximum existing in \mathbf{A} .*

Proof. If \mathbf{A} is a b.r.d.g., then we take \mathbf{B} to be identical to \mathbf{A} . If \mathbf{A} is a distributive-lattice ordered but not bounded, then it is sufficient to endow it with the missing bounds and extend the operations. Define \mathbf{B} by extending the domain A of \mathbf{A} with a new upper bound \top and a new lower bound \perp , extending the order \leq in the obvious way, and defining

- $x \circ \perp = \perp \circ x = \perp$ for all $x \in B$,
- $x \circ \top = \top \circ x = \top$ for all $x \in B \setminus \{\perp\}$,
- $x \backslash \top = \top / x = \top$ for all $x \in B$,
- $\perp \backslash x = x / \perp = \top$ for all $x \in B$,
- $\top \backslash x = \top / x = \perp$ for all $x \in B \setminus \{\top\}$,
- $x \backslash \perp = \perp / x = \perp$ for all $x \in B \setminus \{\perp\}$.

Then \mathbf{B} is a b.r.d.g. and \mathbf{A} is a subalgebra of \mathbf{B} .

If \mathbf{A} is not lattice ordered and does not contain a minimum, define \mathbf{B} by setting $B = \mathcal{O}(\mathbf{A})$; then $\langle B, \cap, \cup, \emptyset, A \rangle$ forms a bounded distributive lattice. Next, define for $X, Y \in \mathcal{O}(\mathbf{A})$:

$$\begin{aligned} X \odot Y &= \{z \in A \mid z \leq x \circ y \text{ for some } x \in X, y \in Y\}, \\ X \backslash\backslash Y &= \{z \in A \mid x \circ z \in Y \text{ for all } x \in X\}, \\ Y // X &= \{z \in A \mid z \circ x \in Y \text{ for all } x \in X\}. \end{aligned}$$

It is easy to check that $\mathbf{B} = \langle B, \cap, \cup, \odot, \backslash\backslash, //, \downarrow\{1\}, \emptyset, A \rangle$ is a b.r.d.g. If \mathbf{A} is not lattice ordered and contains a minimum \perp , then we construct \mathbf{B} in the same way as above considering only nonempty downsets and replacing \emptyset by \perp .

Finally, the map $x \mapsto \downarrow\{x\}$ is an order embedding which preserves the unit 1, any meets, the minimum and the maximum, provided they exist in \mathbf{A} . □

It follows from Theorem 2.4 that in order to establish the FEP for \mathbb{ROG}^Q , $Q \subseteq \mathbf{Prop}$, it suffices to prove it for \mathbb{BRDG} . Moreover, any complexity upper bound for $\text{Th}_{\forall}(\mathbb{BRDG})$ is also an upper bound for $\text{Th}_{\forall}(\mathbb{ROG}^Q)$.

3 FEP for BRDG

In this section we give a very simple algebraic proof of the FEP for the class \mathbb{BRDG} . Let \mathbf{P} be a poset. Recall that a map $\gamma: P \rightarrow P$ is called a *closure operator* on \mathbf{P} if it is expanding ($x \leq^{\mathbf{P}} \gamma(x)$ for all $x \in P$), order preserving, and idempotent ($\gamma(\gamma(x)) = \gamma(x)$ for all $x \in P$). The elements from the image of γ are called γ -*closed*. Dually, a map $\sigma: P \rightarrow P$ is called an *interior operator* on \mathbf{P} if it is contracting ($\sigma(x) \leq^{\mathbf{P}} x$ for all $x \in P$), order preserving, and idempotent. The elements from the image of σ are called σ -*open*.

Assume $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \circ^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, 1^{\mathbf{A}}, \perp^{\mathbf{A}}, \top^{\mathbf{A}} \rangle \in \mathbb{BRDG}$. Let \mathbf{B} be a finite partial subalgebra of \mathbf{A} . In order to show the FEP for \mathbb{BRDG} , we construct a finite algebra $\mathbf{D}(\mathbf{A}, \mathbf{B}) \in \mathbb{BRDG}$ into which \mathbf{B} embeds. Without any loss of generality we may assume that $1^{\mathbf{A}}, \perp^{\mathbf{A}}, \top^{\mathbf{A}} \in B$. Consider the bounded sublattice $\mathbf{D} = \langle D, \wedge^{\mathbf{D}}, \vee^{\mathbf{D}}, \perp^{\mathbf{D}}, \top^{\mathbf{D}} \rangle$ of \mathbf{A} generated by B , where $\perp^{\mathbf{D}} = \perp^{\mathbf{A}}$ and $\top^{\mathbf{D}} = \top^{\mathbf{A}}$. Since B is finite, \mathbf{D} is a finitely generated distributive lattice. Consequently, \mathbf{D} is finite and hence also complete.

We will define a closure operator γ on \mathbf{A} in such a way that D will be the set of its γ -closed elements. For each $x \in A$, define

$$\gamma(x) = \bigwedge \{y \in D \mid x \leq^{\mathbf{A}} y\}.$$

Note that the above meet is never empty since $\top^{\mathbf{A}} = \top^{\mathbf{D}} \in B \subseteq D$. In other words, $\gamma(x)$ is the least element in D above x . Similarly, one can define an interior operator σ on \mathbf{A} , whose image is D , by setting for each $x \in A$:

$$\sigma(x) = \bigvee \{y \in D \mid y \leq^{\mathbf{A}} x\}.$$

Again the join is never empty because $\perp^{\mathbf{D}} \in D$. Thus $\sigma(x)$ is the greatest element in D below x . Define $\mathbf{D}(\mathbf{A}, \mathbf{B}) = \langle D, \wedge^{\mathbf{D}}, \vee^{\mathbf{D}}, \circ^{\mathbf{D}}, \backslash^{\mathbf{D}}, /^{\mathbf{D}}, 1^{\mathbf{D}}, \perp^{\mathbf{D}}, \top^{\mathbf{D}} \rangle$ by setting:

$$1^{\mathbf{D}} = 1^{\mathbf{A}}, \quad x \circ^{\mathbf{D}} y = \gamma(x \circ^{\mathbf{A}} y), \quad x \backslash^{\mathbf{D}} y = \sigma(x \backslash^{\mathbf{A}} y), \quad x /^{\mathbf{D}} y = \sigma(x /^{\mathbf{A}} y).$$

LEMMA 3.1. *The algebra $\mathbf{D}(\mathbf{A}, \mathbf{B})$ belongs to \mathbb{BRDG} .*

Proof. In order to show that $\mathbf{D}(\mathbf{A}, \mathbf{B}) \in \mathbb{BRDG}$, it suffices to prove that $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is a residuated groupoid because $\langle D, \wedge^{\mathbf{D}}, \vee^{\mathbf{D}}, \perp^{\mathbf{D}}, \top^{\mathbf{D}} \rangle$ is a bounded distributive lattice. Note that we have $\gamma(x) = x = \sigma(x)$ for any $x \in D$. Let $x, y, z \in D$. Then we have the following chain of equivalences:

$$\begin{aligned} x \circ^{\mathbf{D}} y = \gamma(x \circ^{\mathbf{A}} y) \leq^{\mathbf{D}} z &\text{ iff } x \circ^{\mathbf{A}} y \leq^{\mathbf{A}} z \\ &\text{ iff } y \leq^{\mathbf{A}} x \backslash^{\mathbf{A}} z \\ &\text{ iff } y \leq^{\mathbf{D}} \sigma(x \backslash^{\mathbf{A}} z) = x \backslash^{\mathbf{D}} z. \end{aligned}$$

For $/^{\mathbf{D}}$, the argument is analogous.

Finally, for every $x \in D$ we have $1^{\mathbf{D}} \circ^{\mathbf{D}} x = \gamma(1^{\mathbf{A}} \circ^{\mathbf{A}} x) = \gamma(x) = x$. Similarly, $x \circ^{\mathbf{D}} 1^{\mathbf{D}} = x$. Thus $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is a b.r.d.g. \square

LEMMA 3.2. *The algebra \mathbf{B} embeds into $\mathbf{D}(\mathbf{A}, \mathbf{B})$.*

Proof. We claim that the identity map ι is the desired embedding. The identity map is clearly an order embedding. The map ι preserves meets, joins, top and bottom because $\mathbf{D}(\mathbf{A}, \mathbf{B})$ forms a sublattice of \mathbf{A} generated by B . Recall that we have $x = \gamma(x) = \sigma(x)$ for any $x \in B$. Thus we have $x \circ^{\mathbf{D}} y = \gamma(x \circ^{\mathbf{A}} y) = x \circ^{\mathbf{B}} y$ for $x, y, x \circ^{\mathbf{A}} y \in B$. Finally, let $x, y, x \setminus^{\mathbf{A}} y \in B$. Then $x \setminus^{\mathbf{D}} y = \sigma(x \setminus^{\mathbf{A}} y) = x \setminus^{\mathbf{B}} y$. Similarly, $x /^{\mathbf{D}} y = x /^{\mathbf{B}} y$ provided $x, y, x /^{\mathbf{A}} y \in B$. \square

Thus we have proved that \mathbb{BRDG} has the FEP. In combination with Theorem 2.4 we obtain the following result.

THEOREM 3.3. *The variety \mathbb{BRDG} has the FEP. Thus also $\mathbb{ROG}^{\mathbf{Q}}$ has the FEP for $\mathbf{Q} \subseteq \text{Prop}$.*

4 Succinct representation

In the following sections we are going to give an upper bound on the computational complexity of $\text{Th}_{\forall}(\mathbb{BRDG})$. From the previous section we know that if the cardinality of the finite partial subalgebra \mathbf{B} is n then the lattice reduct of the constructed finite b.r.d.g. $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is n -generated distributive lattice whose cardinality is known to be bounded by 2^{2^n} . Although the cardinality of $\mathbf{D}(\mathbf{A}, \mathbf{B})$ can be doubly exponential, it is possible to find a condensed representation of this algebra which will be useful at the sequel.

It is very well known that category \mathbb{FDL} of finite distributive lattices and $\{\perp, \top\}$ -preserving lattice homomorphisms is dually equivalent to the category \mathbb{FPOS} of finite posets and order-preserving maps [8].

$$\begin{array}{ccc} & \text{Stone} & \\ & \curvearrowright & \\ \mathbb{FDL} & & \mathbb{FPOS}^{\text{op}} \\ & \curvearrowleft & \\ & \text{Pred} & \end{array}$$

The functors $\text{Stone}, \text{Pred}$ defining this duality behave on object level as follows:

1. Given a finite bounded distributive lattice \mathbf{L} , $\text{Stone}(\mathbf{L})$ is its poset of join-irreducible elements $\mathcal{J}(\mathbf{L})$. Recall that an element $a \in L$ is *join-irreducible* provided that $a \neq \perp^{\mathbf{L}}$ and $a = b \vee^{\mathbf{L}} c$ for $b, c \in L$ implies $a = b$ or $a = c$.
2. Given a finite poset \mathbf{P} , $\text{Pred}(\mathbf{P})$ is the finite distributive lattice $\mathcal{O}(\mathbf{P})$ of all downsets on \mathbf{P} .

Moreover, the natural isomorphism from the identity functor on \mathbb{FDL} to PredStone is well known from Birkhoff's representation theorem for finite distributive lattices.

THEOREM 4.1. *A finite distributive lattice \mathbf{L} is isomorphic to $\mathcal{O}(\mathcal{J}(\mathbf{L}))$ via $\mu: \mathbf{L} \rightarrow \mathcal{O}(\mathcal{J}(\mathbf{L}))$ given by $\mu(x) = \mathcal{J}(\mathbf{L}) \cap \downarrow\{x\}$ for $x \in L$.*

Thus in order to represent a finite distributive lattice \mathbf{L} , it is sufficient to remember only the poset $\mathcal{J}(\mathbf{L})$ of its join-irreducible elements which is much smaller than \mathbf{L} itself. Namely, let $n \in \mathbb{N}$ and let \mathbf{L} be an n -generated distributive lattice. Then \mathbf{L} can be conceived as the homomorphic image of the free n -generated distributive lattice \mathbf{F}_n ; assuming the generators of \mathbf{F}_n are g_1, \dots, g_n , the set $\mathcal{J}(\mathbf{F}_n)$ of its join-irreducible elements is the dual of the poset of all nonempty proper subsets of $\{g_1, \dots, g_n\}$ (see e.g. [11]); so there are $2^n - 2$ join-irreducible

elements. Since \mathbf{L} is a homomorphic image of \mathbf{F}_n and surjective homomorphisms are $\{\perp, \top\}$ -preserving, $\mathcal{J}(\mathbf{L})$ is a subposet of $\mathcal{J}(\mathbf{F}_n)$ by the above duality, hence its cardinality is bounded by $2^n - 2$.

Now we want to represent finite b.r.d.g.'s similarly as finite distributive lattices. In order to do this, we have to enriched the poset of join-irreducible elements by a ternary relation capturing the groupoid operation and a unary relation capturing its neutral element. What follows it in fact a relational representation in the spirit of [1] restricted to the finite case.

DEFINITION 4.2. *A frame is a structure $\mathbf{W} = \langle W, \leq, R_\circ, U \rangle$ where $\langle W, \leq \rangle$ is a finite poset, $U \in \mathcal{O}(\mathbf{W})$ and $R_\circ \subseteq W^3$ such that for all $x, y, z, x', y', z' \in W$ we have*

- $x \leq x'$ and $R_\circ xyz$ implies $R_\circ x'yz$,
- $y \leq y'$ and $R_\circ xyz$ implies $R_\circ xy'z$,
- $z' \leq z$ and $R_\circ xyz$ implies $R_\circ xyz'$,
- $z \leq x$ iff there is $u \in U$ such that $R_\circ xuz$,
- $z \leq y$ iff there is $u \in U$ such that $R_\circ uyz$.

Given a finite b.r.d.g. $\mathbf{A} = \langle A, \wedge, \vee, \circ, \backslash, /, 1, \perp, \top \rangle$, we define the corresponding frame $\text{Stone}(\mathbf{A})$ as the poset of join irreducible elements $\mathcal{J}(\mathbf{A})$ together with a unary relation $U = \mu(1) = \mathcal{J}(\mathbf{A}) \cap \downarrow\{1\}$ and a ternary relation R_\circ defined for $x, y, z \in \mathcal{J}(\mathbf{A})$ as follows:

$$R_\circ xyz \quad \text{iff} \quad z \leq x \circ y.$$

It is easy to check that $\text{Stone}(\mathbf{A})$ is really a frame.

Conversely, given a frame $\mathbf{W} = \langle W, \leq, R_\circ, U \rangle$, we define the corresponding finite b.r.d.g. as $\text{Pred}(\mathbf{W}) = \langle \mathcal{O}(\mathbf{W}), \cap, \cup, \circ, \backslash, /, U, \emptyset, W \rangle$, where $\langle \mathcal{O}(\mathbf{W}), \cap, \cup, \emptyset, W \rangle$ is the bounded distributive lattice of downsets on \mathbf{W} and for $A, B, C \in \mathcal{O}(\mathbf{W})$ we define

$$\begin{aligned} A \circ B &= \{z \in P \mid \exists x \in A, \exists y \in B, R_\circ xyz\}, \\ A \backslash C &= \{y \in P \mid \forall z \in P, \forall x \in A, R_\circ xyz \implies z \in C\}, \\ C / B &= \{x \in P \mid \forall z \in P, \forall y \in B, R_\circ xyz \implies z \in C\}. \end{aligned} \tag{1}$$

It is again easy to check that $\text{Pred}(\mathbf{W})$ is really a b.r.d.g. Moreover, it is straightforward to check that one can obtain a representation theorem for finite b.r.d.g.'s analogous to Birkhoff's one.

THEOREM 4.3. *A finite b.r.d.g. \mathbf{A} is isomorphic to $\text{PredStone}(\mathbf{A})$ via $\mu: \mathbf{A} \rightarrow \text{PredStone}(\mathbf{A})$ given by $\mu(x) = \mathcal{J}(\mathbf{A}) \cap \downarrow\{x\}$ for $x \in A$.*

Thus, in order to represent a finite b.r.d.g. \mathbf{A} whose lattice reduct is n -generated, it is sufficient to store its poset of join-irreducibles of size $m \leq 2^n - 2$, a ternary relation R_\circ of size m^3 and a unary relation of size m .

5 Decision procedures for BRDG

In this section we exploit our proof of the FEP for bounded residuated distributive-lattice-ordered groupoids in order to obtain an upper bound on the computational complexity of their universal theory.

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. $\mathbf{NTIME}(f)$ is the class of decision problems P such that there is a nondeterministic Turing machine M that accepts P and operates in time $O(f)$. In particular,

$$\mathbf{NP} = \bigcup_{k \in \mathbb{N}} \mathbf{NTIME}(n^k) \quad \text{and} \quad \mathbf{NEXP} = \bigcup_{k \in \mathbb{N}} \mathbf{NTIME}(2^{n^k}).$$

Moreover, \mathbf{coNP} (\mathbf{coNEXP}) is the class of complements of problems in \mathbf{NP} (\mathbf{NEXP}).

It is quite common to think of \mathbf{NP} in more logical terms: a problem P is in \mathbf{NP} if $P = \{x \mid \exists y \langle x, y \rangle \in R\}$ for some binary relation R such that

- $\langle u, v \rangle \in R$ implies $|v| \leq p(|u|)$ for some polynomial p ,
- R is decidable in time polynomial in the size of the given tuple.

Analogously, a problem P is in \mathbf{NEXP} if $P = \{x \mid \exists y \langle x, y \rangle \in R\}$ for some binary relation R such that

- $\langle u, v \rangle \in R$ implies $|v| \leq 2^{p(|u|)}$ for some polynomial p ,
- R is decidable in time polynomial in the size of the given tuple.

We will now argue that $\overline{\text{Th}_{\forall}(\mathbf{BRDG})}$, the complement of the universal fragment of the first-order theory of \mathbf{BRDG} , is in \mathbf{NEXP} . Consider a well-formed open formula Φ in the language of b.r.d.g.'s. Thus Φ is a Boolean combination of identities $I(\Phi) = \{s_i \approx t_i\}_{i=1}^k$, where s_i, t_i are b.r.d.g.-terms for $i = 1, \dots, k$; we may further assume w.l.o.g. that the Boolean connectives in Φ are limited to negation, conjunction, and disjunction, and that negation only occurs in front of the identities in $I(\Phi)$. For the *size of* Φ we take the total number of occurrences of all (sub)terms in Φ ; it will be denoted n . Note that one may conceive Φ as a binary tree whose leaves are literals (identities or their negations); then n is an upper bound on the number of leaves, and the tree has no more than $2n$ nodes. Moreover, if $s_i \approx t_i$ is an identity in Φ , then s_i and t_i can be conceived as binary trees also, where the number of nodes in *all* such trees in $I(\Phi)$ is bounded by n . Thus our definition of a formula size is reasonably robust without being too technical. Moreover, the set of all (sub)terms occurring in Φ will be denoted $\tau(\Phi)$; we will later need to consider $\tau(\Phi)$ as a finite *list* without repetitions, in a fixed order. To that purpose, one can consider the lexicographic order, relying on an enumeration of object variables, and establishing an order on the function symbols (e.g., $\wedge, \vee, \circ, \setminus, /, 1, \perp, \top$).

Let $\mathbf{A} \in \mathbf{BRDG}$. Let e be an evaluation of b.r.d.g.-terms. Denote $e(\tau(\Phi)) = \{e(t) \mid t \in \tau(\Phi)\}$. Clearly, $e(\tau(\Phi))$ together with $1^{\mathbf{A}}$, the top and the bottom of \mathbf{A} determines a finite partial subalgebra \mathbf{B} of \mathbf{A} of cardinality at most $n + 3$. Suppose $\mathbf{A} \not\models \Phi[e]$. Then, as shown in Section 3, Φ is not valid in the algebra $\mathbf{D}(\mathbf{A}, \mathbf{B})$, whose lattice reduct is at most $(n + 3)$ -generated bounded distributive lattice; the set of its join-irreducible elements is bounded in cardinality by $2^{n+3} - 2$, and the cardinality of the whole lattice is at most $2^{2^{n+3}}$.

LEMMA 5.1. *Let Φ be a universal formula in the language of b.r.d.g.'s, of size n . Φ is valid in the class \mathbf{BRDG} iff it is valid in the class of all finite b.r.d.g.'s whose lattice reduct is an n' -generated distributive lattice for $n' \leq n + 3$.*

The finite algebra $\mathbf{D}(\mathbf{A}, \mathbf{B})$ can be given as a finite list of finite tables defining all its operations on its finite domain. However, here we are going to use the succinct representation of $\mathbf{D}(\mathbf{A}, \mathbf{B})$ described in Section 4, i.e., we will use its corresponding frame $\text{Stone}(\mathbf{D}(\mathbf{A}, \mathbf{B})) = \langle \mathcal{J}(\mathbf{D}), \leq, R_{\circ}, U \rangle$ where \mathbf{D} stands for the lattice reduct of $\mathbf{D}(\mathbf{A}, \mathbf{B})$.

\mathbf{D} is an at most $(n + 3)$ -generated bounded distributive lattice, $\mathcal{J}(\mathbf{D})$ is the poset of its join-irreducible elements. Denote $m = |\mathcal{J}(\mathbf{D})|$; we know $m \leq 2^{n+3} - 2$. The poset $\mathcal{J}(\mathbf{D})$ can thus be given as a binary array of size m^2 . The ternary relation R_\circ can be given as a binary array of size m^3 and U as a binary array of size m .

With each universal formula Φ , each $\mathbf{A} \in \text{BRDG}$, and each evaluation e such that $\mathbf{A} \not\models \Phi[e]$, one can consider the condensed representation of $\mathbf{D}(\mathbf{A}, \mathbf{B})$ described above. We fix the format of the representation for $m = |\mathcal{J}(\mathbf{D})|$: a binary array of size m^2 , a binary array of size m^3 and a binary array of size m . We will refer to it as a ‘fixed format representation’ of $\mathbf{D}(\mathbf{A}, \mathbf{B})$. Moreover, we shall consider $\tau(\Phi)$ in a fixed (lexicographic) order, and we shall consider a complete evaluation of terms in Φ , i.e., a list of length $|\tau(\Phi)| \leq n$ of downsets in $\mathcal{J}(\mathbf{D})$, specifying $e(t)$ for each $t \in \tau(\Phi)$ in the fixed order. We will refer to it as a ‘fixed-order list of term values’.

We shall consider a binary relation R defined as follows: for each universal formula Φ , each $\mathbf{A} \in \text{BRDG}$ and each evaluation e of b.r.d.g.-terms such that $\mathbf{A} \not\models \Phi[e]$, Φ is in the relation R with the tuple consisting of a fixed format representation of $\mathbf{D}(\mathbf{A}, \mathbf{B})$ and a fixed-order list of term values for $e(\tau(\Phi))$. Note that if the size of Φ is n then the size of the fixed format representation of $\mathbf{D}(\mathbf{A}, \mathbf{B})$ and the fixed-order list of term values for $e(\tau(\Phi))$ is $m^2 + m^3 + m + |\tau(\Phi)|m \leq 2^{3n+11}$. Thus there is a polynomial p such that $\langle u, v \rangle \in R$ implies $|v| \leq 2^{p(|u|)}$.

LEMMA 5.2. *The relation R is decidable in polynomial time.*

Proof. We describe a decision procedure that determines, for a given triple F, L, E , whether F is an open b.r.d.g.-formula, L is a fixed-format representation of a finite b.r.d.g.-algebra of size at most $2^{p(n)}$, and E is a complete evaluation of b.r.d.g.-terms occurring in the formula F in the algebra given by L such that F does not hold under E .

The procedure is described below as a sequence of (informally rendered) steps that test certain properties of F , L , and E . Let us agree that if a given test is successful, the procedure proceeds to the next step; if not, it returns ‘no’ and halts. Bounds on time needed in particular steps are given for a random access machine (see [12]).

- Check that F is an open b.r.d.g.-formula. (This is a routine polynomial affair.) We shall refer to the formula as Φ and denote n its size.
- Check that L is a triple consisting of binary arrays J_0, R_0, U_0 of size respectively m^2, m^3, m . Check that $m \leq 2^{p(n)}$.
- Check that E is an array of length $|\tau(\Phi)|$ and that each element is a characteristic vector of length m (a subset of $\{1, \dots, m\}$).
- Check that the binary relation given by J_0 is a partial order; this can be done in time $O(m^3)$. We shall denote the poset given by J_0 as $\mathbf{J} = \langle J, \leq \rangle$, with $J = \{1, \dots, m\}$. The poset \mathbf{J} determines a finite distributive lattice $\mathbf{D} = \langle \mathcal{O}(\mathbf{J}), \cap, \cup, \emptyset, \mathbf{J} \rangle$.
- Check that U_0 is a downset on \mathbf{J} .
- Check that R_0 and U_0 satisfies all the conditions from Definition 4.2. The first three conditions can be checked in $O(m^4)$ steps. The last two in time $O(m^3)$. If all the checks up to now ended successfully, L is a fixed-format representation of a b.r.d.g. We have shown in (1) how to compute operation values in a such a representation of a b.r.d.g.
- Check that each element of the array E is a downset in \mathbf{J} .
- Check that E is a sound complete evaluation of terms in Φ . This involves, for each term $t \in \tau(\Phi)$ distinct from a variable or a constant, evaluating the corresponding operation

on the two subterms, and checking that the result of the operation on subterms equals the value given by E for t . An operation evaluation can be performed in time linear in m for the operations \cap and \cup ; for the remaining operations, the evaluation takes $O(m^3)$ steps.

- For each identity $s_i \approx t_i$ in $I(\Phi)$, determine whether it is valid under E . This gives a Boolean evaluation v of all identities in $I(\Phi)$.
- Compute the Boolean value of Φ under v . If $v(\Phi) = 1$, return ‘no’; otherwise return ‘yes’.

Clearly, the above procedure works in time polynomial in the size of its input. □

THEOREM 5.3. *The universal theory of \mathbb{BRDG} is in \mathbf{coNEXP} . The same holds also for $\mathbb{ROG}^{\mathbf{Q}}$ where $\mathbf{Q} \subseteq \mathbf{Prop}$.*

6 Modifications

The above proof can be also easily modified in order to show the \mathbf{coNEXP} upper bound for the universal theory of b.r.d.g.’s satisfying any combinations of the following properties corresponding to the well known structural rules:

- commutativity, i.e., $x \circ y = y \circ x$,
- contractivity, i.e., $x \leq x^2$,
- integrality, i.e., $x \leq 1$.

Moreover, one can observe that if one limits one’s attention to totally ordered b.r.d.g.’s, then for each such chain \mathbf{A} , the size of $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is at most $n + 3$, and this algebra is a chain. Thus one can get a much better upper bound on complexity of the universal theory of this variety by considering the following modification of Lemma 5.1.

LEMMA 6.1. *Let Φ be a universal formula in the language of b.r.d.g.’s, of size n . Φ is valid in the class of chains in \mathbb{BRDG} iff it is valid in the class of finite totally ordered b.r.d.g.’s whose lattice reduct is a chain of cardinality at most $n + 3$.*

Since all elements of $\mathbf{D}(\mathbf{A}, \mathbf{B})$ except the bottom are join-irreducible, we have $m \leq n + 2$ in the above algorithm, and with each universal formula that is not valid, we can provide a witness (an algebra and a complete evaluation of terms) of size polynomial in n . Consequently, the universal theory of totally ordered algebras in \mathbb{BRDG} is in \mathbf{coNP} .

To establish a lower bound for our problem, recall the result of [4], saying that the equational theory of the two-element distributive lattice on $\{0, 1\}$ is \mathbf{coNP} hard. Since the two-element distributive lattice generates the variety of distributive lattice \mathbb{DL} , its equational theory $\text{Th}_{\text{Eq}}(\mathbb{DL})$ is \mathbf{coNP} hard: in fact, \mathbf{coNP} complete. Finally, it is not difficult to realize that the two-element Boolean algebra viewed as a b.r.d.g. belongs to \mathbb{BRDG} . Thus the equational theory of lattice reducts of b.r.d.g.’s is exactly $\text{Th}_{\text{Eq}}(\mathbb{DL})$.

THEOREM 6.2. *The universal theory of totally ordered algebras in \mathbb{BRDG} is \mathbf{coNP} complete.*

Theorem 6.2 can be used in order to put the quasi-equational theory of semilinear b.r.d.g.'s into **coNP**. Recall that a b.r.d.g. \mathbf{A} is *semilinear* if it is a subdirect product of totally ordered b.r.d.g.'s (see [7]). The class of semilinear b.r.d.g.'s forms a variety (whose axiomatization was given in [6]) which is generated by totally ordered members as a quasi-variety. Consequently, we obtain the following corollary.

COROLLARY 6.3. *The quasi-equational theory of semilinear b.r.d.g.'s is **coNP** complete.*

7 Conclusion

To conclude, let us mention a remaining open problem. All the proofs in this paper rely on distributivity of the lattice reducts. Using this assumption, we are able to prove the FEP not only for **BRDG** but also for residuated ordered groupoids and residuated meet-semilattice ordered groupoids. However, we are unable to settle the following question.

PROBLEM 7.1. *Does the class of residuated lattice ordered groupoids have the FEP?*

Acknowledgment

The work of the authors was supported by the grant P202/11/1632 of the Czech Science Foundation.

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