R-fuzzy Logics with Additional Connectives and Their Validation Sets

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Abstract

The validation set of a formula in a fuzzy logic is the set of all truth values which this formula may achieve. We summarize and extend recent results on characterizations of validation sets in various fuzzy logics.

Keywords: fuzzy logic, many-valued logic, evaluation, validation set, Łukasiewicz logic, Gödel logic, product logic, product logic with involution, fuzzy logic with Δ.

1 Basic notions

One important way of representing vagueness of information is an enlargement of the set of truth values from $\{0,1\}$ to the whole unit interval $[0,1]$. This leads to fuzzy logics [3, 6, 11, 12, 13, 23, 27] which were successfully applied in many areas, especially in fuzzy control. These logics allow to violate the excluded middle law and achieve more degrees of satisfaction of a formula. E.g., the formula $\varphi = \neg p \vee p$ is a tautology in the classical logic, evaluated always by 1. Consider the same formula in a fuzzy logic where the negation and disjunction are interpreted by the standard operations considered by Zadeh [27], i.e., by the standard fuzzy negation ($x \mapsto 1-x$) and the maximum. Then φ may be evaluated by values less than one, but always at least $1/2$. More exactly, depending on the evaluation of p, the evaluation of φ may be any number from the interval $[1/2, 1]$. We express this fact by saying that $[1/2, 1]$ is the *validation set* of formula φ. In this paper we study the question of which validation sets may occur in various fuzzy logics.

Let us recall the basic notions used in the sequel.

Definition 1.1 A (propositional) fuzzy logic is an ordered pair $\mathcal{P} = (\mathcal{L}, \mathcal{Q})$ of a *language* (*syntax*) $\mathcal{L}$ and a *structure* (*semantics*) $\mathcal{Q}$ described as follows:

(i) The language of $\mathcal{P}$ is a pair $\mathcal{L} = (A, \mathcal{C})$, where A is a nonempty at most countable set of *atomic symbols* and $\mathcal{C}$ is a tuple of *connectives*.

(ii) The structure of $\mathcal{P}$ is a pair $\mathcal{Q} = ([0,1], \mathcal{M})$, where $[0,1]$ is the set of *truth values*, and the tuple $\mathcal{M}$ consists of the *interpretations* (*meanings*) of the connectives in $\mathcal{C}$.

The tuple of connectives always will contain at least a conjunction which is interpreted by a *triangular norm* (*t-norm* for short), i.e., a commutative, associative, non-decreasing operation $T\colon [0,1]^2 \to [0,1]$ with neutral element 1 (see [26, 17]). Three basic t-norms are the minimum $T_{\mathbf{G}}$, the product $T_{\mathbf{P}}$, and the Łukasiewicz t-norm $T_{\mathbf{L}}$ given, respectively, by $T_{\mathbf{G}}(x,y) = \min(x,y)$, $T_{\mathbf{P}}(x,y) = xy$, and $T_{\mathbf{L}}(x,y) = \max(0, x+y-1)$.

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A *triangular conorm* (*t-conorm* for short) is a commutative, associative, non-decreasing operation $S\colon [0,1]^2 \to [0,1]$ with neutral element 0. There is an obvious duality between t-norms and t-conorms. Let $N_{\mathbf{S}}\colon [0,1] \to [0,1]$ be the *standard negation* defined by $N_{\mathbf{S}}(x) = 1-x$. For each t-norm T, the function $S_T\colon [0,1]^2 \to [0,1]$ given by

$$S_T(x,y) = N_{\mathbf{S}}(T(N_{\mathbf{S}}(x), N_{\mathbf{S}}(y)))$$

is a t-conorm, called the *dual of* T. The duals of the three important t-norms are the maximum $S_{\mathbf{G}}$, the probabilistic sum $S_{\mathbf{P}}$, and the bounded sum $S_{\mathbf{L}}$ given, respectively, by $S_{\mathbf{G}}(x,y) = \max(x,y)$, $S_{\mathbf{P}}(x,y) = x+y-xy$, and $S_{\mathbf{L}}(x,y) = \min(1, x+y)$.

The class $\mathcal{F}_{\mathcal{P}}$ of *well-formed formulas* in a fuzzy logic $\mathcal{P}$ is defined in the standard way, starting from the atomic symbols and constructing new formulas using the connectives. For each function $e\colon A \to [0,1]$ which assigns a truth value to each atomic formula, there exists a unique *natural extension* of e to a *truth assignment* (*evaluation*) $\overline{e}\colon \mathcal{F}_{\mathcal{P}} \to [0,1]$.

Definition 1.2 Let φ be a formula and $\{p_1, \ldots, p_n\}$ be a set of atomic symbols occurring in φ. Then a function $f_\varphi : [0,1]^n \to [0,1]$ is called *truth function* of φ iff $f_\varphi(e(p_1), \ldots, e(p_n)) = \overline{e}(\varphi)$ for each evaluation e.

Here we concentrate on the properties of validations sets. The *validation set* of a formula φ is defined as

$$V_{\mathcal{P}}(\varphi) = \{\overline{e}(\varphi) \mid e \in [0,1]^A\}.$$

This paper deals with the question of which validation sets may occur in various fuzzy logics. The sections dealing with S-fuzzy and R-fuzzy logics summarize the results of [15] and [16] for comparison, while the sections on R_Δ-fuzzy logics and RS-fuzzy logics contain new results.

Proposition 1.3 *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be a fuzzy logics with tuples of connectives $\mathcal{C}_1$, $\mathcal{C}_2$ such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Let the set of truth values and the interpretations of the common connectives coincide in $\mathcal{P}_1$, $\mathcal{P}_2$. Then each validation set in $\mathcal{P}_1$ may occur as a validation set in $\mathcal{P}_2$.*

PROOF: It is obvious because $\mathcal{F}_{\mathcal{P}_1} \subseteq \mathcal{F}_{\mathcal{P}_2}$. □

The sets of all natural, rational, and real numbers will be denoted respectively by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$.

2 S-fuzzy logics

The following construction of propositional fuzzy logics was presented in [3]:

Definition 2.1 A *t-norm-based propositional fuzzy logic* (*S-fuzzy logic*) is a fuzzy logic (in the sense of Definition 1.1) in which the basic connectives are unary $\neg$ (*negation*) and binary $\wedge$ (*conjunction*), interpreted respectively by the standard fuzzy negation and a t-norm T.

The logics corresponding to the basic t-norms $T_{\mathbf{G}}, T_{\mathbf{L}}$ and $T_{\mathbf{P}}$ are *Gödel S-fuzzy logic*, *Łukasiewicz S-fuzzy logic*, and *product S-fuzzy logic*. Starting from the basic connectives we define binary disjunction, resp. implication, by the expression $a \vee b = \neg(\neg a \wedge \neg b)$, resp. $a \to b = \neg a \vee b$. The interpretation of the disjunction, resp. implication, is a t-conorm, resp. so-called *S-implication*.

Let us summarize results on validation sets from [3] and [15]:

Theorem 2.2 *The validation sets in Gödel S-fuzzy logic are of one of the following forms:*

$$[0, \tfrac{1}{2}]\,, \qquad [\tfrac{1}{2}, 1]\,, \qquad [0,1]\,.$$

The validation sets in product S-fuzzy logic are of one of the following forms:

$$[0,a]\,, \qquad [b,1]\,, \qquad [0,1]\,,$$

where $a, b \in \,]0,1[$. The validation sets in Łukasiewicz S-fuzzy logic are of one of the following forms:

$$\{0\}\,, \qquad \{1\}\,, \qquad [0,a]\,, \qquad [b,1]\,, \qquad [0,1]\,,$$

where $a, b \in \mathbb{Q} \cap \,]0,1[$. The possible values of the bounds a, b form a countable dense subset of $[0,1]$.

3 R-fuzzy logics

A reasonable way of constructing connectives in fuzzy logics is to start with a continuous t-norm T and to use the *residuum* (*R-implication*, see [7, 25]) defined by

$$R_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\} \ . \tag{1}$$

as the interpretation of the implication.

The following approach to fuzzy logics with residual implications is described in detail in [13].

Definition 3.1 A *residuum-based propositional fuzzy logic* (*R-fuzzy logic*) is a fuzzy logic (in the sense of Definition 1.1) in which the basic connectives are the nullary connective $\mathbf{0}$ (*false statement*) and the binary connectives $\wedge$ (*conjunction*) and $\rightarrow$ (*implication*) with respective interpretations 0, T, R_T, where T is a t-norm and R_T is the corresponding residuum.

The R-fuzzy logics corresponding to the basic t-norms $T_{\mathbf{G}}, T_{\mathbf{L}}$, and $T_{\mathbf{P}}$ are *Gödel R-fuzzy logic*, *Łukasiewicz R-fuzzy logic*, and *product R-fuzzy logic*.

Observe that the implication in Łukasiewicz S-fuzzy logic coincides with the implication in Łukasiewicz R-fuzzy logic. So the interpretation of logical connectives in Łukasiewicz S-fuzzy logic and Łukasiewicz R-fuzzy logic is identical (although not the same connectives are considered as the basic ones). One difference between Łukasiewicz S-fuzzy logic and R-fuzzy logic is that the nullary connective $\mathbf{0}$ is not considered as a formula in Łukasiewicz S-fuzzy logic. Nevertheless, it can be introduced as a derived logical connective putting, e.g., $\mathbf{0} = \neg\varphi \wedge \varphi$ for a fixed formula φ.

In Gödel R-fuzzy logic, the interpretation $R_{\mathbf{G}}$ of the implication is defined by

$$R_{\mathbf{G}}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{G}}$ (called the *Gödel implication*) is not continuous in the points (x,x) with $x \in [0,1[$.

In product R-fuzzy logic, we obtain the interpretation $R_{\mathbf{P}}$ of the implication defined by

$$R_{\mathbf{P}}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \dfrac{y}{x} & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{P}}$ (also called the *Goguen implication*) is not continuous in the point $(0,0)$.

Let us summarize results on validation sets from [16].

Theorem 3.2 *The validation sets in Łukasiewicz R-fuzzy logic are of one of the following forms:*

$$\{0\}\ , \qquad \{1\}\ , \qquad [0,a]\ , \qquad [b,1]\ , \qquad [0,1]\ ,$$

where $a, b \in \mathbb{Q} \cap \,]0,1[$.
The validation sets in Gödel and product R-fuzzy logic are of one of the following forms:

$$\{0\}\ , \qquad \{1\}\ , \qquad \{0,1\}\ , \qquad]0,1]\ , \qquad [0,1]\ .$$

4 R$_\triangle$-fuzzy logics

In this section we extend the language of R-fuzzy logics by a unary connective Δ which is interpreted as follows:

$$\triangle x = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This extension of R-fuzzy logics was introduced in [1].

Definition 4.1 A *residuum-based propositional fuzzy logic with a unary operation* Δ (*R_Δ-fuzzy logic*) is an R-fuzzy logic in which the set of basic connectives is extended by the unary connective Δ, with interpretation $\triangle$.

Using results of [16], we get the following theorem:

Theorem 4.2 *The validation sets in Gödel and product R_Δ-fuzzy logic are of one of the following forms:*

$$\{0\}\ , \qquad \{1\}\ , \qquad \{0,1\}\ , \qquad]0,1]\ , \qquad [0,1[\ , \qquad [0,1]\ .$$

PROOF: In comparison to Theorem 3.2, we have one additional form of validation set, $[0,1[$. In the first part, we have to prove that this validation set occurs. Let p be an atomic formula and let us take the formula $\varphi = p \wedge (\Delta p \to \mathbf{0})$. The validation set of this formula is $V(\varphi) = [0,1[$.

Now it is sufficient to show that this implication holds: *If φ is a formula and e is an evaluation such that $\overline{e}(\varphi) \in]0,1[$ then for each $b \in]0,1[$ there is an evaluation e_b such that $\overline{e}_b(\varphi) = b$.* The proof of this implication is analogous to the proof of [16, Theorem 3.3]. □

The situation in Łukasiewicz R$_\Delta$-fuzzy logic is more complex than the situation in previous logics. We know that all truth functions in Łukasiewicz R-fuzzy logic are McNaughton functions, i.e. piecewise linear continuous functions with integral coefficients (see for example [6]). We may divide the domain of each truth function into finitely many polyhedra so that the function is linear on each polyhedron. Now, if we add the connective Δ to our set of connectives, we obtain again piecewise linear functions with integral coefficients as truth functions but not necessarily continuous functions. However we can still divide the domain of each truth function into finitely many polyhedra such that the function is again linear on each polyhedron because of the interpretation of Δ.

Theorem 4.3 *A subset $V \subseteq [0,1]$ is a validation set of some formula in Łukasiewicz R_Δ-fuzzy logic if and only if it satisfies the following conditions:*

1. *$V \cap \{0,1\} \neq \emptyset$,*
2. *$V = \bigcup_{i=1}^{n} I_i$,*

where $n \in \mathbb{N}$ and I_i are intervals (possibly open, closed or half-closed). The possible bounds of I_i form the subset of rational numbers from $[0,1]$.

PROOF: The first condition only expresses the fact that Łukasiewicz R$_\Delta$-fuzzy logic works classically for crisp values 0 and 1. For the proof of the second condition, we have to prove that all above-mentioned cases may occur and furthermore we have to show that no other set may occur.

Firstly let p be an atomic formula. We will construct a formula τ of one variable p such that its truth function f_τ will be the characteristic function of the validation set V, i.e. $f_\tau(x) = 1$ if $x \in V$, $f_\tau(x) = 0$ otherwise. Let $m, n \in \mathbb{N}$. Let us consider a formula $\varphi_{m,n}$ of the variable p with the truth function

$$f_{\varphi_{m,n}}(x) = \begin{cases} 0, & \text{if } x \leq \frac{m-1}{n}\ , \\ nx - m + 1, & \text{if } \frac{m-1}{n} < x < \frac{m}{n}\ , \\ 1, & \text{if } x \geq \frac{m}{n}\ . \end{cases}$$

Such a formula exists because $f_{\varphi_{m,n}}$ is a McNaughton function. Then the truth function of the formula $\Delta\varphi_{m,n}$ is

$$f_{\Delta\varphi_{m,n}}(x) = \begin{cases} 1, & \text{if } x \geq \frac{m}{n}\ , \\ 0, & \text{if } x < \frac{m}{n}\ . \end{cases}$$

Thus we are able to construct the characteristic function of any interval $[m/n, 1]$. In the similar way, we can construct the characteristic function of any interval $[0, k/l]$, $k, l \in \mathbb{N}$, if we take a formula $\psi_{k,l}$ with the truth function $f_{\psi_{k,l}}(x) = -lx + k + 1$ for $x \in [k/l, (k+1)/l]$. The characteristic functions of the intervals $[m/n, k/l]\,,]m/n, k/l]\,, [m/n, k/l[\,,]m/n, k/l[$ can be obtained respectively by the formulas:

$$\begin{aligned} \Delta\varphi_{m,n} \quad &\wedge \quad \Delta\psi_{k,l}\ , \\ \neg\Delta\psi_{m,n} \quad &\wedge \quad \Delta\psi_{k,l}\ , \\ \Delta\varphi_{m,n} \quad &\wedge \quad \neg\Delta\varphi_{k,l}\ , \\ \neg\Delta\psi_{m,n} \quad &\wedge \quad \neg\Delta\varphi_{k,l}\ . \end{aligned}$$

Finally the formula τ can be constructed by taking the formulas with the truth functions representing the characteristic functions of all intervals I_i and using the connective $\vee$.

Now we have the formula τ of one variable p and we want to construct a formula with the validation set V. If we want the validation set to contain 0, resp. 1, then take a formula $p \wedge \tau$, resp. $p \vee \neg\tau$, and we obtain the desired result.

Secondly we have to show that no other form of validation set can occur. Since we know that the domain of each truth function f can be split into finitely many convex polyhedra and f is linear with integral coefficients on each polyhedron, the extreme point on each polyhedron has to lie at some vertex of the polyhedron. Moreover each vertex of the polyhedra is determined by a system of linear equations with integral coefficients. Therefore the coordinates of each vertex and the values of f in the vertices must be rational numbers. □

5 RS-fuzzy logics

In R-fuzzy logics in which conjunction is not interpreted by a nilpotent t-norm (i.e. in R-fuzzy logics different from the Łukasiewicz one), we have no disjunction dual to the conjunction. This has led recently to a new concept, an *R-fuzzy logic with an involutive negation* (see [8]). In this approach, negation $\neg$ becomes an additional basic connective interpreted by a strong fuzzy negation. (Without any loss of generality, we may interpret it by the standard fuzzy negation.) This negation can be used in the de Morgan formula defining a dual disjunction which is interpreted by the dual t-conorm. Here we call these logics *RS-fuzzy logics* in accordance with the preceding terminology—they possess both an R-implication (as a basic connective interpreted by the residuum) and an S-implication (as a derived connective $\neg(\varphi \wedge \neg\psi)$ using the involutive negation).

Definition 5.1 A *residuum-based propositional fuzzy logic with an involutive negation* (*RS-fuzzy logic*) is an R-fuzzy logic in which the set of basic connectives is extended by $\neg$ (*negation*) with interpretation $N_{\mathbf{S}}$.

Theorem 5.2 *A subset $V \subseteq [0,1]$ is a validation set of some formula in Gödel RS-fuzzy logic if and only if it satisfies the following conditions:*

1. $V \cap \{0,1\} \neq \emptyset$,
2. $V \cap \,]0,1[$ *is an arbitrary union of the following sets:* $\left]0,\frac{1}{2}\right[$, $\{\frac{1}{2}\}$, $\left]\frac{1}{2},1\right[$.

PROOF: First, we prove that all the above-mentioned cases occur. Let p be an atomic formula and let us consider these formulas:

$$\begin{aligned}
\varphi_1 &= p \to \mathbf{0}\,,\\
\varphi_2 &= \neg((\neg(\neg p \to p) \to \mathbf{0}) \vee (p \to \mathbf{0}))\,,\\
\varphi_3 &= (\neg(p \to \neg p) \to \mathbf{0}) \wedge (\neg(\neg p \to p) \to \mathbf{0})\,,\\
\varphi_4 &= \neg((\neg(p \to \neg p) \to \mathbf{0}) \vee (\neg p \to \mathbf{0}))\,,\\
\varphi_5 &= \neg p \to \mathbf{0}\,.
\end{aligned}$$

The truth functions of the formulas $\varphi_1, \ldots, \varphi_5$ coincide with the characteristic functions of the following sets $M_1 = \{0\}$, $M_2 = \left]0,\frac{1}{2}\right[$, $M_3 = \{\frac{1}{2}\}$, $M_4 = \left]\frac{1}{2},1\right[$, $M_5 = \{1\}$, respectively.

Now we can take formulas of the following form:

$$\psi_{I,J} = \Big(p \vee \bigvee_{i\in I} \varphi_i\Big) \wedge \bigwedge_{j\in J} \neg\varphi_j\,, \tag{2}$$

where $I, J \subseteq \{1,2,3,4,5\}$ and $I \cap J = \emptyset$. The evaluation of the formula $\psi_{I,J}$ gives us the desired results:

$$\overline{e}(\psi_{I,J}) = \begin{cases} 1 & \text{if } e(p) \in M_i,\ i \in I\,,\\ 0 & \text{if } e(p) \in M_j,\ j \in J\,,\\ e(p) & \text{otherwise}\,. \end{cases} \tag{3}$$

This proves that each of the above-mentioned cases may occur.

Second, we have to prove that all validation sets are of one of the above forms. For this, it is sufficient to prove the following implication:

If φ is a formula and e is an evaluation such that $\overline{e}(\varphi) \in \left]0, \frac{1}{2}\right[$ (resp. $\overline{e}(\varphi) \in \left]\frac{1}{2}, 1\right[$) then for each $b \in \left]0, \frac{1}{2}\right[$ (resp. $b \in \left]\frac{1}{2}, 1\right[$) there is an evaluation e_b such that $\overline{e}_b(\varphi) = b$.

The rest of the proof follows the method from [16, Theorem 3.3]; the only difference is that not all order automorphisms commute with the standard negation $\neg$. Nevertheless, there are such automorphisms, i.e. increasing bijections $h : [0,1] \to [0,1]$ such that $h(1-x) = 1 - h(x)$. □

Theorem 5.3 *A subset $V \subseteq [0,1]$ is a validation set of some formula in product RS-fuzzy logic if and only if it satisfies the following conditions:*

1. *$V \cap \{0,1\} \neq \emptyset$,*
2. *$V = \bigcup_{i=1}^{n} I_i$,*

where $n \in \mathbb{N}$ and $I_i \subseteq [0,1]$, $i = 1, \dots, n$, are intervals (open, closed or half-closed). The possible bounds of I_i form a countable dense subset of $[0,1]$.

Before proving this theorem, we will prepare several statements which will be useful in the sequel. Let p be an atomic formula, $k, n \in \mathbb{N}$. Let us consider the following formulas:

$$\psi_{n,k} = \bigwedge_{j=1}^{n} \neg \bigwedge_{i=1}^{k} p\,, \qquad n, k \in \mathbb{N}\,.$$

Their truth functions are

$$f_{\psi_{n,k}}(t) = (1 - t^k)^n\,.$$

Lemma 5.4 *For each $a, b, r \in \left]0,1\right[$, $a < b$, there are $k, n \in \mathbb{N}$ such that $f_{\psi_{n,k}}(r) \in \left]a, b\right[$.*

PROOF: For fixed k and r, the values $f_{\psi_{n,k}}(r) = (1 - r^k)^n$, $n \in \mathbb{N}$, form a geometric sequence with the quotient $1 - r^k < 1$. For a sufficiently large k we obtain $r^k < b - a$, so at least one element of this sequence belongs to the interval $\left]a, b\right[$. □

Proposition 5.5 *All forms of the validation sets mentioned in Th. 5.3 may occur.*

PROOF: It is obvious that every validation set must contain at least 0 or 1 because RS-fuzzy logic works classically on crisp values 0 and 1.

In the similar way as in the proof of the Theorem 4.3, we will construct a formula τ of one variable p such that its truth function f_τ will be the characteristic function of the validation set V, i.e. $f_\tau(x) = 1$ if $x \in V$, $f_\tau(x) = 0$ otherwise. Let us consider an arbitrary number $r \in \left]0,1\right[$ and assign $e(p) = r$. Due to Lemma 5.4, for any $\epsilon > 0$ we may find $k, n \in \mathbb{N}$ such that $\overline{e}(\psi_{n,k}) \in \left]r - \epsilon, r + \epsilon\right[$. Now let us take the following formulas:

$$\begin{aligned} \alpha &= p \to \mathbf{0}, \\ \beta_{n,k} &= \neg(\psi_{n,k} \to p) \to \mathbf{0}, \\ \gamma_{n,k} &= \neg(p \to \psi_{n,k}) \to \mathbf{0}, \\ \delta &= \neg p \to \mathbf{0}. \end{aligned}$$

The truth functions of the formulas α, $\beta_{n,k}$, $\gamma_{n,k}$, δ coincide with the characteristic functions of the following sets: $\{0\}$, $[0, r_{n,k}]$, $[r_{n,k}, 1]$, $\{1\}$, where $r_{n,k} \in \left]0,1\right[$, $k, n \in \mathbb{N}$, is an algebraic number which satisfies the equation $r_{n,k} = (1 - (r_{n,k})^k)^n$. Moreover, $r_{n,k}$ belongs to the interval $\left]r - \epsilon, r + \epsilon\right[$ because the truth function $f_{\psi_{n,k}}$ is decreasing.

Now, starting with formulas α, $\beta_{n,k}$, $\gamma_{n,k}$, δ for different values of n, k and using connectives $\wedge, \vee, \neg$, we can construct formula τ such that its truth function f_τ is a characteristic function of a finite union of the intervals I_i.

Finally we will construct a formula with the validation set V. If we want the validation set to contain 0, resp. 1, then take a formula $p \wedge \tau$, resp. $p \vee \neg\tau$, and we obtain the desired result. $\Box$

Further we have to prove that all validation sets are of one of the above forms. We will need several results and definitions from algebraic geometry (for details see [2]).

Definition 5.6 A semi-algebraic subset of $\mathbb{R}^n$ is a subset of the form

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{q_i} \{x \in \mathbb{R}^n | f_{i,j} *_{i,j} 0\},$$

where $f_{i,j}$ are polynomials in n variables and $*_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$ and $j = 1, \ldots, q_i$.

Note that the semi-algebraic sets are closed under finite intersections, finite unions and complements.

Proposition 5.7 *Let φ be a formula in product RS-fuzzy logic, A be a set of atomic symbols occurring in φ, and n be the number of elements in A. If f_φ is the truth function of φ, then its domain, $D = [0,1]^n$, can be written as a union of finitely many mutually disjoint connected semi-algebraic subsets $D_i \subseteq D$, $D = \bigcup_{i=1}^{m} D_i$. The function f_φ is continuous on D_i, $i = 1, \ldots, n$, and $f_\varphi | D_i = P_i/Q_i$, where P_i, Q_i are polynomials in n variables.*

PROOF: We will proceed by induction on the complexity of the formula. For $\varphi = p$, $p \in A$, $D = [0,1]$ is a connected semi-algebraic set and $f_\varphi(x) = x$ is continuous on D.

1. If $\varphi = \neg\psi$ then by induction assumptions the domain D of f_ψ can be split into finitely many connected semi-algebraic subsets, $D = \bigcup_{i=1}^{m} D_i$. We may also split the domain of f_φ in the same way as that of f_ψ because the interpretation of the negation is continuous. Let the truth function of ψ be
$$f_\psi | D_i = \frac{P_i}{Q_i}.$$
Then
$$f_\varphi | D_i = 1 - f_\psi | D_i = 1 - \frac{P_i}{Q_i} = \frac{Q_i - P_i}{Q_i},$$
and f_φ is continuous on each D_i, $i = 1, \ldots, n$.

2. Suppose that $\varphi = \psi_1 \wedge \psi_2$. Then we can split the domains E, F of f_{ψ_1}, f_{ψ_2} by induction assumptions,
$$E = \bigcup_{j=1}^{m_1} E_j, \qquad F = \bigcup_{k=1}^{m_2} F_k.$$
We may construct new semi-algebraic subsets D_i such that the domain D of f_φ can be written as $D = \bigcup_{i=1}^{m} D_i$. Let us take $D_i = E_j \cap F_k$ for every $j = 1, \ldots, m_1$, $k = 1, \ldots, m_2$, then D_i is semi-algebraic because it is an intersection of two semi-algebraic subsets which is again a semi-algebraic subset. Moreover, the subset D_i can be written as a finite union of connected semi-algebraic subsets due to [2, Theorem 2.4.5]. Let the truth functions of ψ_1 and ψ_2 be
$$f_{\psi_1} | E_j = \frac{P_j}{Q_j}, \qquad f_{\psi_2} | F_k = \frac{M_k}{N_k},$$
where P_i, Q_i, M_i, N_i are polynomials. As $D_i \subseteq E_j, F_k$, the function
$$f_\varphi | D_i = (f_{\psi_1} | D_i)(f_{\psi_2} | D_i) = \frac{P_j}{Q_j} \frac{M_k}{N_k}$$
is continuous on D_i.

3. Finally, let $\varphi = \psi_1 \to \psi_2$. Then we may construct new semi-algebraic subsets D_i such that the domain D of f_φ can be written as $D = \bigcup_{i=1}^m D_i$ in a similar way as in case 2. However, as the interpretation of the implication is not continuous, we have to show that $D_i = E_j \cap F_k$, $j = 1, \ldots, m_1$, $k = 1, \ldots, m_2$, can be written as a finite union of semi-algebraic subsets and f_φ is continuous on every subset of D_i. Let the truth functions of ψ_1 and ψ_2 be

$$f_{\psi_1}|E_j = \frac{P_j}{Q_j}\,, \qquad f_{\psi_2}|F_k = \frac{M_k}{N_k}\,.$$

According to the interpretation of the implication, we can write for each $(x_1, \ldots, x_n) \in D_i$

$$f_\varphi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if} \quad f_{\psi_1}(x_1, \ldots, x_n) \le f_{\psi_2}(x_1, \ldots, x_n), \\ \frac{f_{\psi_2}(x_1, \ldots, x_n)}{f_{\psi_1}(x_1, \ldots, x_n)} & \text{otherwise.} \end{cases}$$

The condition $f_{\psi_1}(x_1, \ldots, x_n) \le f_{\psi_2}(x_1, \ldots, x_n)$ divides D_i into two parts and it can be rewritten as the following

$$P_j(x_1, \ldots, x_n)N_k(x_1, \ldots, x_n) - M_k(x_1, \ldots, x_n)Q_j(x_1, \ldots, x_n) \le 0.$$

Since this condition is polynomial, both parts of D_i are semi-algebraic and it follows from [2, Theorem 2.4.5] that they can be split into finitely many connected semi-algebraic subsets. This completes the proof.

□

Lemma 5.8 *Let $f : D \to \mathbb{R}$ be a continuous bounded function and the domain D be semi-algebraic and connected. Then the range $f(D)$ of the function f is an interval $I \subset \mathbb{R}$.*

PROOF: The domain D is semi-algebraic and connected set. By using [2, Proposition 2.5.13] it can be shown that it is also path connected. It means that for each pair of points $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in D$ there exists a continuous mapping $g : [0, 1] \to D$ such that $g(0) = (x_1, \ldots, x_n)$ and $g(1) = (y_1, \ldots, y_n)$. Then the interval $[f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)]$ is a subset of I due to the intermediate value theorem. The set-theoretical supremum of the intervals obtained this way is I. □

Now we are able to finish the proof of the Theorem 5.3.

PROOF OF THEOREM 5.3: We know (due to Proposition 5.7) that if f_φ is an truth function of a formula φ, then its domain D can be split into finitely many connected semi-algebraic subsets D_i such that f_φ is continuous on D_i. If we apply Lemma 5.8 to each D_i then we obtain that $f_\varphi(D_i)$ is an interval and $f(D)$ has to be a union of finitely many intervals which finishes the proof. □

Remark 5.9 In [9], Godo, Esteva, and Montagna introduced so-called ŁΠ logic joining Łukasiewicz R-fuzzy logic and product R-fuzzy logic. This logic was further developed in [4]. The results about validation sets which we proved in the last section can be also used for ŁΠ logic because ŁΠ logic comprises product RS-fuzzy logic (i.e. all connectives in product RS-fuzzy logic can be defined in ŁΠ logic). As it was shown in [5], the standard semantics of RS-fuzzy logic and ŁΠ logic coincide. Thus we immediately obtain that the validation sets in ŁΠ logic are the same as in RS-fuzzy logic.

6 Concluding remarks

We studied four classes of frequently encountered fuzzy logics: S-fuzzy logics (where the basic connectives are negation and conjunction), R-fuzzy logics (where the basic connectives are conjunction, implication and the false statement), R_Δ-fuzzy logics (where the basic connectives are conjunction, implication, the false statement, and Δ), and RS-fuzzy logics (which combine connectives of the first and the second type). Each of these classes splits to numerous fuzzy logics depending on the interpretation of conjunction which we supposed to be a continuous t-norm. In all these fuzzy logics, we gave a characterization of validation sets of formulas. This gives us an information for comparison of the semantical richness of these logics and their ability to describe vagueness. Among other results, we observe that inclusion of an involutive negation increases substantially possible degrees of partial satisfaction of formulas.

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