Archimedean Classes in Integral Commutative Residuated Chains

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1 Introduction

The problem of characterization of the structure of MTL-algebras, which form an equivalent algebraic semantics for Monoidal T-norm Based Logic (see [9]) in the sense of Blok and Pigozzi (see [3]), is still far from being solved. Since MTL-algebras are in fact subdirect products of chains, it suffices to investigate only the structure of MTL-chains if we want to characterize the structure of MTL-algebras. Thus a closely related problem already discussed in the literature [11, 12, 13, 24] is the same task for totally ordered monoids since each MTL-chain forms a totally ordered monoid. As was pointed out in [11], the characterization of the structure of totally ordered monoids could be split into two steps: (1) determine the structure of an arbitrary Archimedean totally ordered monoid; (2) determine the ways in which a given chain of Archimedean totally ordered monoids can be assembled to form a totally ordered monoid having the elements of the chain as its Archimedean classes. In order to solve these two steps, it is clear that the notion of an Archimedean class is crucial.

Another closely related problem is to understand better the structure of the lattice of subvarieties of MTL-algebras. It is quite natural to ask whether it is possible to express the number of Archimedean classes in an MTL-chain by an identity. Unfortunately, this is not possible in general. Indeed, there are product chains with arbitrary number of Archimedean classes but the only nontrivial subvariety of product algebras is the variety of Boolean algebras. However, in some cases it is possible as we are going to show in this paper. The obtained results also shed some light on the structure of MTL-chains. The original motivation of our results comes from [15] where the author posed a question whether the variety of ΠMTL-algebras (i.e. the class of cancellative MTL-algebras) is generated by Archimedean ΠMTL-chains. The paper [15] offers only a partial answer by showing it is not generated as a quasivariety. More precisely, the author shows that the quasi-identity

$$(p \to q) \to q \approx \mathbf{1} \quad \Rightarrow \quad p \vee q \vee \neg q \approx \mathbf{1}$$

is valid in all Archimedean ΠMTL-chains but there are ΠMTL-chains where this quasi-identity is not valid. In this paper we prove that the answer to this question is negative. As a byproduct we further describe a strictly increasing chain of subvarieties of ΠMTL-algebras.

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However, our results are applicable also to MTL-chains. In order to present the results more generally, we prove our results for zero-free subreducts of MTL-chains and then translate their consequences for particular subvarieties of MTL-algebras.

2 Preliminaries

A *commutative residuated lattice* (CRL) $\mathbf{L} = (L, *, \to, \wedge, \vee, \mathbf{1})$ is an algebraic structure, where $(L, *, \mathbf{1})$ is a commutative monoid, $(L, \wedge, \vee)$ is a lattice, and $(*, \to)$ forms an adjoint pair, i.e.,

$$x * y \leq z \text{ iff } x \leq y \to z\,.$$

The operation $\to$ is called a *residuum*. It follows from the definition that $*$ is order-preserving, i.e., $a \leq b$ implies $a*c \leq b*c$. The residuum is decreasing in the first argument and increasing in the second one. Further, the inequality $a * x \leq b$ has a greatest solution for x (namely $a \to b$). In particular, the residuum is uniquely determined by $*$ and $\leq$. It is well known that the class $\mathcal{CRL}$ of all commutative residuated lattices forms a variety (see e.g. [21]). The existence of the residuum also implies that $*$ distributes over all existing joins, i.e. if $\bigvee X$ exists for $X \subseteq L$, then $\bigvee_{x \in X} a * x$ exists for $a \in L$ and is equal to $a * \bigvee X$. Observe also that if a CRL $\mathbf{L}$ possesses a bottom element $\mathbf{0}$ then we have $a * \mathbf{0} = \mathbf{0}$ for any $a \in L$.

Now we introduce several well-known subvarieties of residuated lattices.

- An *integral* CRL (ICRL) $\mathbf{L}$ is a CRL such that $\mathbf{1}$ is the top element of L. In this case we have that $x \leq y$ implies $x \to y = \mathbf{1}$. The class $\mathcal{ICRL}$ of ICRLs is easily seen to be a variety defined by $x \leq \mathbf{1}$.

- A CRL $\mathbf{L}$ is said to be *cancellative* if for any $x, y, z \in L$, $x * z = y * z$ implies $x = y$. It was shown in [2] that the class $\mathcal{C}an\mathcal{CRL}$ of cancellative CRLs forms a variety defined by $x \approx y \to (y * x)$.

- Finally, a totally ordered CRL is referred to as a *commutative residuated chain* (CRC). Analogously as before we write ICRC instead of integral CRC. A residuated lattice is called *representable* if it is a subdirect product of CRCs. This is equivalent to the demand that it satisfies $\mathbf{1} \approx ((x \to y) \wedge \mathbf{1}) \vee ((y \to x) \wedge \mathbf{1})$ (see [21]), whence the representable CRLs also form a variety. In the case of representable ICRLs this identity can be simplified to $\mathbf{1} \approx (x \to y) \vee (y \to x)$.

The class of all ICRCs will be denoted by $\mathcal{ICRC}$ and the variety generated by this class by $\mathcal{ICRL}^{\mathcal{C}}$ (i.e., the variety of representable ICRLs). The superscript $\mathcal{C}$ refers to the fact that this variety is generated by chains. Similarly the class of cancellative ICRCs is denoted by $\mathcal{C}an\mathcal{ICRC}$ and the variety generated by this class $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$.

The classes of algebras introduced above are closely connected to the algebraic semantics of fuzzy logics. Let us recall this connection.

Definition 2.1 An algebra $\mathbf{L} = (L, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is called an MTL-*algebra* if it satisfies the following conditions:

1. $(L, *, \to, \wedge, \vee, \mathbf{1})$ is a representable ICRL and

2. $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice.

A totally ordered MTL-algebra is called an MTL-*chain*. The variety of MTL-algebras is denoted by $\mathcal{MTL}$.

There are several well-known subvarieties of $\mathcal{MTL}$. We recall some of them.

- MTL-algebras satisfying $x \wedge \neg x \approx \mathbf{0}$ are called SMTL-*algebras*.
- The subvariety of $\mathcal{MTL}$ axiomatized by $x \wedge y \approx x * (x \to y)$ is the variety of BL-*algebras*.
- The class of MTL-algebras satisfying $\neg x \vee ((x \to x * y) \to y) \approx \mathbf{1}$ is the variety of ΠMTL-*algebras*. It is known that ΠMTL-algebras form a subvariety of the variety of SMTL-algebras.
- Finally, the subvariety of $\mathcal{MTL}$ defined by $x \vee \neg x \approx \mathbf{1}$ is the variety of Boolean algebras.

The corresponding varieties are denoted respectively by $\mathcal{SMTL}$, $\mathcal{BL}$, $\mathcal{PMTL}$, and $\mathcal{BA}$.

MTL-algebras were introduced by Esteva and Godo in [9] as the algebras of truth values for the monoidal t-norm based logic (MTL). It is well known that zero-free subreducts of MTL-algebras are exactly representable ICRLs[1] (see [10]). The class of zero-free subreducts of BL-algebras is the class of representable ICRLs satisfying the divisibility law, i.e. $x \wedge y = x * (x \to y)$. These algebras are usually called *basic hoops* (see [10]).

It can be shown that there is a connection between ICRCs and SMTL-chains. First, we need the notion of an ordinal sum. This notion was introduced in [1] for totally ordered basic hoops and it was generalized to the case of ICRCs in [23].

Definition 2.2 Let $\langle I, \leq \rangle$ be a totally ordered set. Let $\{\mathbf{A}_i \mid i \in I\}$ be a family of ICRCs sharing the same top element, say $\mathbf{1}$, and such that for $i \neq j$, $A_i \cap A_j = \{\mathbf{1}\}$. Then $\bigoplus_{i \in I} \mathbf{A}_i$ (the *ordinal sum* of the family) is the ICRC whose universe is $\bigcup_{i \in I} A_i$ and whose operations are:

$$x * y = \begin{cases} x *_{A_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{\mathbf{1}\} \text{ with } i > j, \\ x & \text{if } x \in A_i \setminus \{\mathbf{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

$$x \to y = \begin{cases} x \to_{A_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j, \\ \mathbf{1} & \text{if } x \in A_i \setminus \{\mathbf{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

For every $i \in I$, $\mathbf{A}_i$ is called a *component* of the ordinal sum.

If in addition I has a minimum, say i_0, and $\mathbf{A}_{i_0}$ is bounded, then the ordinal sum $\bigoplus_{i \in I} \mathbf{A}_i$ forms an MTL-chain.

We say that an ICRC is *indecomposable* if it cannot be written as the ordinal sum of two nontrivial ICRCs. For totally ordered basic hoops the indecomposable chains are characterized (see [1]). Recall that a basic hoop $\mathbf{L}$ is called *Wajsberg* if $(x \to y) \to y \approx (y \to x) \to x$ holds in $\mathbf{L}$. Then [1, Theorem 3.6] says that a totally ordered basic hoop $\mathbf{L}$ is indecomposable iff $\mathbf{L}$ is Wajsberg. Further, we have the following theorem.

[1] Representable ICRLs are known also under different names, e.g. like prelinear semihoops, basic semihoops, or MTLH-algebras.

Theorem 2.3 ([1]) *Every totally ordered basic hoop* $\mathbf{L}$ *is isomorphic to an ordinal sum of Wajsberg hoops. If* $\mathbf{L}$ *is a BL-chain, then the first component of the ordinal sum is bounded.*

Using the construction of ordinal sum, one can easily prove the following lemma.

Lemma 2.4 *Let* $\mathbf{L}$ *be a nontrivial SMTL-chain. Then* $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ *where* $\mathbf{C}$ *is the zero-free subreduct of* $\mathbf{L}$ *whose domain is* $L \setminus \{\mathbf{0}\}$*.*

Conversely, let $\mathbf{A}$ *be an ICRC. Then* $\mathbf{2} \oplus \mathbf{A}$ *is an SMTL-chain.*

Analogous lemma holds also for ΠMTL-chains.

Lemma 2.5 ([14, 19]) *Let* $\mathbf{L}$ *be a nontrivial* ΠMTL*-chain. Then* $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ *where* $\mathbf{C}$ *is the zero-free subreduct of* $\mathbf{L}$ *whose domain is* $L \setminus \{\mathbf{0}\}$*.*

Conversely, let $\mathbf{A}$ *be a cancellative ICRC. Then* $\mathbf{2} \oplus \mathbf{A}$ *is a* ΠMTL*-chain.*

Further, we need also to recall several facts on congruences and Archimedean classes. Let $\mathbf{L}$ be an ICRL. A subalgebra $\mathbf{S}$ of $\mathbf{L}$ is called *convex* if for all $x, y \in S$ and $z \in L$ we have: $x \leq z \leq y$ implies $z \in S$. It is easy to see that the class of all convex subalgebras ordered by set-inclusion forms a lattice.

Theorem 2.6 ([16]) *Let* $\mathbf{L}$ *be an ICRL. Then its congruence lattice* Con $\mathbf{L}$ *is isomorphic to the lattice of all convex subalgebras of* $\mathbf{L}$*. The isomorphism is established via the assignments* $\theta \mapsto F_\theta$ *and* $F \mapsto \theta_F$*, where*

$$F_\theta = \{a \in L \mid \langle a, \mathbf{1}\rangle \in \theta\}\,,$$

and

$$\theta_F = \{\langle a, b\rangle \in L \times L \mid a \to b \in F \text{ and } b \to a \in F\}\,.$$

In the case when a bottom element $\mathbf{0}$ is in the signature (in particular if $\mathbf{L}$ is an MTL-algebra) then Theorem 2.6 still holds but F_θ need not be a subalgebra. Nevertheless, it is a subalgebra of the zero-free reduct of $\mathbf{L}$. Thus we will call F_θ rather a *filter*. In other words:

> When $\mathbf{L}$ is an ICRL or an MTL-algebra, then the *filters* of $\mathbf{L}$ are just the upward closed submonoids of $(L, *, \leq, \mathbf{1})$.

Let $\mathbf{L}$ be an ICRL or MTL-algebra. The set of all filters of $\mathbf{L}$ will be denoted by $\mathcal{F}_\mathbf{L}$. Let $F \in \mathcal{F}_\mathbf{L}$. Then we write simply $\mathbf{L}/F$ instead of $\mathbf{L}/\theta_F$. The equivalence class containing an element $x \in L$ with respect to a filter F will be denoted by $[x]_F = \{a \in L \mid a\ \theta_F\ x\}$. Observe also that if $\mathbf{L}$ is totally ordered then only one of the terms in the definition of θ_F is important because for all x, y either $x \leq y$ or $y \leq x$, thus either $x \to y = \mathbf{1}$ or $y \to x = \mathbf{1}$. Moreover, $\mathcal{F}_\mathbf{L}$ forms a chain in this case since filters are upward closed.

The next easy result characterizes the *principal* filters, i.e., the filters generated by a single element. A principal filter F generated by b will be denoted by $F(b)$. The set of all principal filters of an ICRL $\mathbf{L}$ will be denoted by $\mathcal{P}_\mathbf{L}$.

Lemma 2.7 *Let* $\mathbf{L}$ *be an ICRL and* $b \in L$*. Then the principal filter generated by* b *is of the form:*

$$F(b) = \{z \in L \mid (\exists n \in \mathbb{N})(b^n \leq z)\}\,.$$

As will be seen later on, the filters in ICRCs are also related to the so-called Archimedean classes (see [12]).

Definition 2.8 Let $\mathbf{L}$ be an ICRC, a, b elements of L, and $\sim$ an equivalence on L defined as follows:

$a \sim b$ iff there exists $n \in \mathbb{N}$ such that $a^n \leq b \leq a$ or $b^n \leq a \leq b$.

Then for any $a \in L$ the equivalence class $[a]_\sim$ is called an *Archimedean class*.

Observe that $\{\mathbf{1}\}$ is always an Archimedean class in an ICRC. Such Archimedean class is called *trivial*. An ICRC containing only one nontrivial Archimedean class is referred to as *Archimedean* ICRC, i.e. if $x \leq y < \mathbf{1}$, then there exists $n \in \mathbb{N}$ such that $y^n \leq x$. Observe that due to the connection between filters and congruences, an ICRC is Archimedean iff it is simple. It is also useful to note that $a \sim b$ iff $F(a) = F(b)$.

Lemma 2.9 *Let* $\mathbf{L}$ *be an ICRC and* $a, b \in L$. *Then we have:*

1. $[a]_\sim$ *is closed under* $*$.
2. $[a]_\sim$ *is convex.*
3. $[a * b]_\sim = [a \wedge b]_\sim$.

Let $\mathbf{L}$ be an ICRC. Thanks to the previous lemma the collection $\mathcal{C}_\mathbf{L}$ of all Archimedean classes of $\mathbf{L}$ can be totally ordered as follows:

$[a]_\sim < [b]_\sim$ iff $a < b$ and $[a]_\sim \cap [b]_\sim = \emptyset$.

As we mentioned in the previous section, the Archimedean classes are related to filters. This connection is described by the next proposition.

Proposition 2.10 ([18]) *Let* $(\mathcal{C}_\mathbf{L}, \leq)$ *be the chain of all Archimedean classes of an ICRC* $\mathbf{L}$. *Then* $\mathcal{C}_\mathbf{L}$ *is dually-isomorphic to the chain of all principal filters* $\mathcal{P}_\mathbf{L}$. *Let* $C \in \mathcal{C}_\mathbf{L}$. *The order-isomorphism* $\phi : \mathcal{C}_\mathbf{L} \to \mathcal{P}_\mathbf{L}$ *is defined as follows:*

$$\phi(C) = F(b)\,, \text{ for any } b \in C\,.$$

For the inverse of ϕ *we have* $\phi^{-1}(F(b)) = F(b) \setminus F$ *where* F *is the predecessor of* $F(b)$.

Originally this proposition was proved for MTL-chains but the same proof works for ICRCs as well.

Corollary 2.11 *Let* $\mathbf{L}$ *be an ICRC such that* $\mathcal{P}_\mathbf{L}$ *is finite. Then* $\mathcal{C}_\mathbf{L}$ *is dually-isomorphic to* Con $\mathbf{L}$.

PROOF: Recall that $\mathcal{F}_\mathbf{L}$ is a chain. Let $F \in \mathcal{F}_\mathbf{L}$. Then $F = \bigcup_{a \in F} F(a)$. Since we assume that there are only finitely many principal filters, $F = F(a)$ for some $a \in L$. Thus $\mathcal{F}_\mathbf{L} = \mathcal{P}_\mathbf{L}$ and the claim follows by Proposition 2.10 since $\mathcal{F}_\mathbf{L} \cong$ Con $\mathbf{L}$.

At the end of this section we recall another useful construction, which we will need for the construction of non-Archimedean cancellative ICRCs.

Definition 2.12 Let $\mathbf{A} = (A, *_A, \to_A, \leq_A, \mathbf{1}_A)$ and $\mathbf{B} = (B, *_B, \to_B, \leq_B, \mathbf{1}_B)$ be cancellative ICRCs. Then the *lexicographic product* of $\mathbf{A}$ and $\mathbf{B}$ is the algebra $\mathbf{A} \overrightarrow{\times} \mathbf{B} = (A \times B, *, \to, \leq, \langle \mathbf{1}_A, \mathbf{1}_B \rangle)$ where $\leq$ is the lexicographic order, i.e., $\langle a, b\rangle \leq \langle c, d\rangle$ iff $a <_A c$ or $[a = c$ and $b \leq_B d]$ and the operations are defined as follows:

$$\langle a, b\rangle * \langle c, d\rangle = \langle a *_A c, b *_B d\rangle \,,$$

$$\langle a, b\rangle \to \langle c, d\rangle = \begin{cases} \langle a \to_A c, \mathbf{1}_B\rangle & \text{if } a * (a \to_A c) <_A c\,, \\ \langle a \to_A c, b \to_B d\rangle & \text{otherwise.} \end{cases} .$$

Proposition 2.13 ([19]) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be cancellative ICRCs. Then* $\mathbf{A} \overrightarrow{\times} \mathbf{B}$ *is a cancellative ICRC.*

Example 2.14 Let $\mathbf{R}^- = (\mathbb{R}^-, +, \Rightarrow, \min, \max, 0)$ be the cancellative ICRC arising from the negative cone of the additive group of reals, i.e., $x \Rightarrow y = (y - x) \wedge 0$. Similarly, let $\mathbf{Z}^- = (\mathbb{Z}^-, +, \Rightarrow, \min, \max, 0)$ be the cancellative ICRC arising from the negative cone of the additive group of integers. Then $\mathbf{Z}^- \overrightarrow{\times} \mathbf{R}^-$ is a cancellative ICRC which is not Archimedean since $\langle -1, 0\rangle < \langle 0, -1\rangle^n$ for all $n \in \mathbb{N}$.

3 Subvarieties of $\mathcal{ICRL^C}$ and $\mathcal{SMTL}$

As we mentioned already in the introduction, we derive all our results for ICRCs. Then we present their consequences for particular subvarieties of $\mathcal{MTL}$, namely for $\mathcal{MTL}$, $\mathcal{SMTL}$, $\mathcal{PMTL}$, and $\mathcal{BL}$. In order to do this for $\mathcal{SMTL}$ and $\mathcal{PMTL}$ we need the result showing that the lattice of subvarieties of $\mathcal{ICRL^C}$ can be embedded into the lattice of subvarieties of $\mathcal{SMTL}$.

Let $\mathcal{V}$ be a subvariety of $\mathcal{ICRL^C}$. Then $\mathcal{V}$ is generated by its totally ordered members since $\mathcal{ICRL^C}$ is. Let $\mathcal{V}_{\mathrm{ch}}$ be the class of totally ordered members of $\mathcal{V}$. We define a subvariety $\mathbf{2} \oplus \mathcal{V}$ of $\mathcal{SMTL}$ as the variety generated by $\mathbf{2} \oplus \mathcal{V}_{\mathrm{ch}}$, where $\mathbf{2} \oplus \mathcal{V}_{\mathrm{ch}}$ is the class consisting of $\mathbf{2} \oplus \mathbf{C}$ for all $\mathbf{C} \in \mathcal{V}_{\mathrm{ch}}$. In [14, Theorem 9.66, Lemma 9.67] it is proved that the mapping defined by $\mathcal{V} \mapsto \mathbf{2} \oplus \mathcal{V}$ is in fact a lattice isomorphism[2].

Theorem 3.1 ([14]) *Let* $\mathbf{\Lambda}(\mathcal{ICRL^C})$ *and* $\mathbf{\Lambda}(\mathcal{SMTL})$ *be the lattices of subvarieties of* $\mathcal{ICRL^C}$ *and* $\mathcal{SMTL}$ *respectively. Then* $\mathbf{\Lambda}(\mathcal{ICRL^C})$ *is isomorphic to the interval* $[\mathcal{BA}, \mathcal{SMTL}]$ *of* $\mathbf{\Lambda}(\mathcal{SMTL})$ *via the mapping* $\mathcal{V} \mapsto \mathbf{2} \oplus \mathcal{V}$.

According to the latter theorem we have $\mathcal{SMTL} = \mathbf{2} \oplus \mathcal{ICRL^C}$. Thus we can bijectively assign to each subvariety $\mathcal{V}$ of $\mathcal{ICRL^C}$ the subvariety $\mathbf{2} \oplus \mathcal{V}$ of $\mathcal{SMTL}$. In addition, we are able to give an axiomatization of $\mathbf{2} \oplus \mathcal{V}$ as we show in the rest of this section.

Let Γ be the set of defining identities for $\mathcal{ICRL^C}$. Then the variety of SMTL-algebras is axiomatized by $\Gamma \cup \{\mathbf{0} \wedge x \approx \mathbf{0}, x \wedge \neg x \approx \mathbf{0}\}$. Recall that the logic of ICRLs is algebraizable in the sense of Blok and Pigozzi if we translate a formula φ as $\varphi \approx \mathbf{1}$ and an identity $\varphi \approx \psi$ as $\varphi \leftrightarrow \psi$. Thus we can assume that each identity is of the form $\varphi \approx \mathbf{1}$ where φ is a term. The fact that a term φ contains exactly n variables $v_1, \ldots, v_n$ will be denoted by $\varphi(v_1, \ldots, v_n)$. Let Δ be a set of identities. Then $\mathrm{Mod}(\Delta)$ stands for the variety of algebras defined by Δ.

[2] In order to see that [14, Theorem 9.66, Lemma 9.67] really applies to our case, one has to note that the identity $x \wedge \neg x \approx \mathbf{0}$ in the axiomatization of $\mathcal{SMTL}$ (cf. Definition 2.1) can be alternatively replaced by the identity $\neg\neg x \vee \neg x \approx \mathbf{1}$.

Theorem 3.2 *Let $\mathcal{V}$ be a subvariety of $\mathcal{ICRL}^{\mathcal{C}}$ and $\Gamma \cup \Sigma$ the set of its defining identities. Then $\mathbf{2} \oplus \mathcal{V}$ is axiomatized by $\Delta = \Gamma \cup \{\mathbf{0} \wedge x \approx \mathbf{0}, x \wedge \neg x \approx \mathbf{0}\} \cup \Sigma_0$, where*

$$\Sigma_0 = \{\neg v_1 \vee \cdots \vee \neg v_n \vee \varphi(v_1, \ldots, v_n) \approx \mathbf{1} \mid \varphi(v_1, \ldots, v_n) \approx \mathbf{1} \in \Sigma\}.$$

PROOF: Clearly all the generators of $\mathbf{2} \oplus \mathcal{V}$ are SMTL-chains. Thus they satisfy each identity from $\Gamma \cup \{\mathbf{0} \wedge x \approx \mathbf{0}, x \wedge \neg x \approx \mathbf{0}\}$. Let $\mathbf{2} \oplus \mathbf{C}$ be a generator of $\mathbf{2} \oplus \mathcal{V}$ and $\neg v_1 \vee \cdots \vee \neg v_n \vee \varphi(v_1, \ldots, v_n) \approx \mathbf{1}$ an element of Σ_0. Then $\varphi(v_1, \ldots, v_n) \approx \mathbf{1}$ is valid in $\mathbf{C}$. Let $a_1, \ldots, a_n$ be interpretations of $v_1, \ldots, v_n$. If $a_i > \mathbf{0}$ for all i, then $\neg a_1 \vee \cdots \vee \neg a_n \vee \varphi(a_1, \ldots, a_n) = \varphi(a_1, \ldots, a_n) = \mathbf{1}$. If at least one $a_i = \mathbf{0}$, then $\neg a_1 \vee \cdots \vee \neg a_n \vee \varphi(a_1, \ldots, a_n) = \neg a_i = \mathbf{1}$. Thus $\mathbf{2} \oplus \mathcal{V} \subseteq \mathrm{Mod}(\Delta)$.

On the other hand, assume that $\mathbf{C} \in \mathrm{Mod}(\Delta)$. Then $\mathbf{C}$ is an SMTL-algebra. Consequently, $\mathbf{C}$ is a subdirect product of SMTL-chains $\mathbf{C}_i$. Thus it is sufficient to prove that each $\mathbf{C}_i \in \mathbf{2} \oplus \mathcal{V}$. Clearly each $\mathbf{C}_i$ belongs to $\mathrm{Mod}(\Delta)$ as well. By Lemma 2.4 we have $\mathbf{C}_i \cong \mathbf{2} \oplus \mathbf{A}$ for an ICRC $\mathbf{A}$. We have to show that $\mathbf{A} \in \mathcal{V}$. Let $\varphi(v_1, \ldots, v_n) \approx \mathbf{1}$ be an identity in Σ and let $a_1, \ldots, a_n$ be interpretations of $v_1, \ldots, v_n$ such that $a_1, \ldots, a_n \in A$. Since $\neg a_i = \mathbf{0}$ for all i, we obtain

$$\varphi(a_1, \ldots, a_n) = \neg a_1 \vee \cdots \vee \neg a_n \vee \varphi(a_1, \ldots, a_n) = \mathbf{1}.$$

□

Thanks to the latter theorem we have $\mathcal{PMTL} = \mathbf{2} \oplus \mathcal{CanICRL}^{\mathcal{C}}$. To see this, observe that the identity $\neg x \vee ((x \to x * y) \to y) \approx \mathbf{1}$ defining $\mathcal{PMTL}$ is equivalent to $\neg x \vee \neg y \vee ((x \to x * y) \to y) \approx \mathbf{1}$ for MTL-chains. Indeed, the first one clearly implies the second one. Conversely, if y is interpreted by $\mathbf{0}$, then the both identities become the same, namely $\neg x \vee \neg\neg x \approx \mathbf{1}$. If $y > \mathbf{0}$, then $\neg y < \mathbf{1}$ implies $\neg x \vee ((x \to x * y) \to y) \approx \mathbf{1}$.

4 Main results

Recall that the original motivation for our results was to show that $\mathcal{PMTL}$ is not generated by its subclass of Archimedean ΠMTL-chains[3]. As was shown in [15] there is a quasi-identity showing that this class does not generate it as a quasivariety. In order to show that it is not generated as a variety, it suffices to prove that it is possible to replace this quasi-identity by an identity. This will be in fact our main goal in this section. We will introduce a quasi-identity (Q_1) in the framework of ICRCs and then we will prove that it is possible to replace it by an identity (A_1^2). The above-mentioned quasi-identity and identity are defined as follows:

$$\begin{array}{ll} (Q_1) & (p \to q) \to q \approx \mathbf{1} \Rightarrow p \vee q \approx \mathbf{1} \\ (A_1) & (p \to q) \to q \leq p \vee q \\ (A_1^2) & ((p \to q) \to q)^2 \leq p \vee q \end{array}$$

The quasi-variety of representable ICRLs satisfying (Q_1) will be denoted by $\mathcal{Q}_1$. The varieties of representable ICRLs satisfying (A_1), (A_1^2) resp. will be denoted by $\mathcal{A}_1$, $\mathcal{A}_1^2$ respectively.

Observe that (A_1) and (A_1^2) hold trivially if $p \leq q$ or $p = \mathbf{1}$. Thus the only interesting case is for $\mathbf{1} > p > q$. Observe that (A_1) is in fact equivalent to the well-known Wajsberg axiom

$$(p \to q) \to q \leq (q \to p) \to p,$$

[3]The notion of an Archimedean ΠMTL-chain used by Hájek in [15] is not in accordance with our definition of an Archimedean ICRC. Since the only Archimedean ΠMTL-chain in our sense is $\mathbf{2}$, Hájek calls a ΠMTL-chain $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ Archimedean if $\mathbf{C}$ is an Archimedean cancellative ICRC.

since $p \vee q = ((p \to q) \to q) \wedge ((q \to p) \to p)$. Similarly $p \vee q$ can be replaced by $(q \to p) \to p$ in (Q_1) and (A_1^2). Thus we can look at (Q_1) and (A_1^2) as weaker forms of Wajsberg axiom. It is known that in the presence of Wajsberg axiom the divisibility axiom is already valid. Thus we have the following lemma showing that $\mathcal{A}_1$ is the variety of Wajsberg hoops.

Lemma 4.1 *Let* $\mathbf{L}$ *be an ICRC. Then* $\mathbf{L}$ *is a Wajsberg hoop iff it satisfies* (A_1).

PROOF: The left-to-right direction is obvious. In order to prove the other direction it is sufficient to show that $\mathbf{L}$ is divisible. This follows from [8, Proposition 3.13 and Lemma 3.5]. $\square$

We start with a couple of useful lemmata describing properties of ICRCs. Then we prove several characterizations of ICRCs satisfying (Q_1).

Lemma 4.2 *Let* $\mathbf{L} \in \mathcal{ICRL}^{\mathcal{C}}$. *Then the following hold:*

1. *If* $p \to q = q$ *then* $q = \max\, [q]_{F(p)}$.
2. *If* F *is a filter in* $\mathbf{L}$ *and* $q = \max\, [q]_F$ *then* $p \to q = q$ *for all* $p \in F$.

PROOF:

1. Let $F(p)$ be the principal filter generated by p. Assume that $p \to q = q$. We will show that $q = \max\, [q]_{F(p)}$. Suppose that $z \in [q]_{F(p)}$. Then $z \to q \in F(p)$. Thus there exists $n \in \mathbb{N}$ such that $p^n \leq z \to q$. By residuation we get $z \leq p^n \to q$. Since we assume that $p \to q = q$, we have $p^n \to q = p^{n-1} \to (p \to q) = p^{n-1} \to q = q$. Thus we obtain that $z \leq q$. Hence $q = \max\, [q]_{F(p)}$.
2. Let $p \in F$. In any ICRL we have $p \to q \geq q$. Since $p \leq (p \to q) \to q$ and $q \to (p \to q) = \mathbf{1}$, we have $(p \to q) \in [q]_F$. As q is maximal, we get $p \to q = q$. $\square$

Lemma 4.3 *Let* $\mathbf{L}$ *be an ICRC. Then* $\mathbf{L}$ *satisfies* (Q_1) *iff each* $[x]_F$ *different from* $[\mathbf{1}]_F$ *has no maximum for all nontrivial filters* F.

PROOF: Firstly, assume that $\mathbf{L}$ satisfies (Q_1). Suppose that there is a nontrivial filter F and $[x]_F \neq [\mathbf{1}]_F$ which has a maximum m. Let $s \in F \setminus \{\mathbf{1}\}$. Then $s \to m = m$ by Lemma 4.2. Consequently, $(s \to m) \to m = m \to m = \mathbf{1}$ and $s \vee m = s < \mathbf{1}$. Hence (Q_1) is not valid in $\mathbf{L}$ (a contradiction).

Secondly, assume that (Q_1) is not valid in $\mathbf{L}$. Then there are $p, q \in L$ such that $(p \to q) \to q = \mathbf{1}$ and $p, q \leq p \vee q < \mathbf{1}$. As $(p \to q) \to q = \mathbf{1}$, we get $p \to q = q$. Thus by Lemma 4.2 it follows that $q = \max\, [q]_{F(p)}$. Since $p < \mathbf{1}$, it follows that $F(p)$ is nontrivial. Moreover, $q \notin F(p)$ otherwise $q = \mathbf{1}$. Thus $[q]_{F(p)} \neq [\mathbf{1}]_F$ and has a maximum. $\square$

Lemma 4.4 *Let* $\mathbf{L}$ *be a ICRC satisfying the quasi-identity* (Q_1) *and* $p, q \in L$ *such that* $p \geq q$. *Then* $q * s \leq p * (p \to q)$ *for all* $s < \mathbf{1}$.

Proof: The cases $p = \mathbf{1}$ and $p = q$ are trivial. Thus suppose $\mathbf{1} > p > q$. Assume that there is $s \in L \setminus \{\mathbf{1}\}$ such that $p * (p \to q) < q * s$. Let $F(s)$ be the filter generated by s and $x \in [p \to q]_{F(s)}$ such that $x > p \to q$. Such x exists. Indeed, either $[p \to q]_{F(s)} \neq [\mathbf{1}]_{F(s)}$ in which case the existence follows from Lemma 4.3 or $[p \to q]_{F(s)} = [\mathbf{1}]_{F(s)}$ and we can take $x = \mathbf{1}$. Then there is $n \in \mathbb{N}$ such that $x * s^{n+1} \leq p \to q < x * s^n$ since $F(s)$ is generated by s (n can be also 0). Let us multiply the first part of the latter inequality by p. Then $p * x * s^{n+1} \leq p * (p \to q) < q * s$. Thus $p * x * s^n < q$ (otherwise $p * x * s^{n+1} \geq q * s$). By residuation $x * s^n \leq p \to q$ which is a contradiction since $x * s^n > p \to q$. □

Lemma 4.5 *Let* $\mathbf{L}$ *be an ICRC satisfying* (Q_1) *and* F *a filter in* $\mathbf{L}$*. Then* (Q_1) *is valid in* $\mathbf{L}/F$ *as well.*

Proof: Suppose not. Then there are $[p]_F, [q]_F \in L/F$ such that $[p]_F > [q]_F$, $[(p \to q) \to q]_F = [\mathbf{1}]_F$ and $[p \vee q]_F < [\mathbf{1}]_F$. The first equality implies $[p \to q]_F = [q]_F$. Thus

$$[p^2 \to q]_F = [p]_F \to [p \to q]_F = [p \to q]_F = [q]_F\,.$$

Consequently, there is $s \in F$ such that $s * (p^2 \to q) \leq q$, i.e. $p^2 \to q \leq s \to q$. Furthermore, $[p \vee q]_F < [\mathbf{1}]_F$ implies $p \notin F$. Hence we have $s > p \geq p^2$. All together we have

$$s \to q \leq p \to q \leq p^2 \to q \leq s \to q\,.$$

Thus $p \to (p \to q) = p^2 \to q = p \to q$ which is a contradiction since $(p \to (p \to q)) \to (p \to q) = \mathbf{1}$ and $p \vee q < \mathbf{1}$. □

Theorem 4.6 *An ICRC* $\mathbf{L}$ *satisfies the quasi-identity* (Q_1) *iff* $\mathbf{L}$ *is either a Wajsberg hoop or there is a minimal nontrivial filter* F *(i.e.,* $\mathbf{L}$ *is subdirectly irreducible) and* $\mathbf{L}/F$ *is a Wajsberg hoop such that each* $[x]_F \neq [\mathbf{1}]_F$ *has no maximum.*

Proof: Firstly, assume that $\mathbf{L}$ satisfies (Q_1) and there is no minimal nontrivial filter. Suppose that $\mathbf{L}$ is not divisible. Then there are $p, q \in L$ such that $\mathbf{1} > p > q$ and $p * (p \to q) < q$. Then $z = q \to p * (p \to q) < \mathbf{1}$ belongs to the filter $F(z)$ generated by z. Let $F(s) \subsetneq F(z)$ be a nontrivial filter strictly smaller than $F(z)$ generated by $s \in L$. Then $p * (p \to q) < q * s$. This leads to a contradiction by Lemma 4.4. Thus $\mathbf{L}$ is a basic hoop. Moreover, as $\mathbf{L}$ satisfies (Q_1), each $[x]_F \neq [\mathbf{1}]_F$ cannot possess a maximum for any nontrivial filter F by Lemma 4.3. Hence $\mathbf{L}$ is a Wajsberg hoop by [1, Theorem 3.6] since $p \to q = q$ implies $p = \mathbf{1}$ or $q = \mathbf{1}$ by Lemma 4.2.

Further, suppose that $\mathbf{L}$ satisfies (Q_1) and is subdirectly irreducible. Thus there is a minimal nontrivial filter F. Assume that $\mathbf{L}/F$ is not divisible. Then there are $[p]_F, [q]_F \in L/F$ such that $[\mathbf{1}]_F > [p]_F > [q]_F$ and $[p]_F * ([p]_F \to [q]_F) < [q]_F$. Since $[p * (p \to q)]_F = [p]_F * ([p]_F \to [q]_F) < [q]_F$, we get that $p * (p \to q) < q * s$ for any $s \in F$ (because $q \to p * (p \to q) \notin F$) but this is a contradiction by Lemma 4.4. Thus $\mathbf{L}/F$ is a basic hoop. Assume that $\mathbf{L}/F$ is not a Wajsberg hoop. Thus [1, Theorem 3.6] gives us $[p]_F, [q]_F$ such that $[p]_F \to [q]_F = [q]_F$ and $[p]_F \neq [\mathbf{1}]_F$ or $[q]_F \neq [\mathbf{1}]_F$. By Lemma 4.2 we get that $[q]_F$ is the maximum of the equivalence class containing $[q]_F$ w.r.t. the filter generated $[p]_F$ which is not trivial. However, this is not possible, as $\mathbf{L}/F$ satisfies (Q_1) by Lemma 4.5, this equivalence class cannot possess a maximum by Lemma 4.3. Hence $\mathbf{L}/F$ is a Wajsberg hoop.

Conversely, if $\mathbf{L}$ is a Wajsberg hoop then (Q_1) is obviously valid in $\mathbf{L}$. Thus assume that $\mathbf{L}$ is not Wajsberg which is subdirectly irreducible with the minimal nontrivial filter F such that $\mathbf{L}/F$ is a Wajsberg hoop and each $[x]_F \neq [\mathbf{1}]_F$ has no maximum. Suppose by the way of contradiction that (Q_1) is not valid in $\mathbf{L}$. Then there are $p, q \in L$ such that $\mathbf{1} > p > q$ and $(p \to q) \to q = \mathbf{1}$. Thus $p \to q = q$. From Lemma 4.2 it follows that $q = \max\ [q]_{F(p)}$. Since $\mathbf{L}/F$ is a Wajsberg hoop, we get that $[\mathbf{1}]_F = [(p \to q) \to q]_F = [p]_F$ by Lemma 4.1. Thus $p \in F$ and $q = \max\ [q]_F$. As $q < \mathbf{1}$, q cannot belong to F. Thus $[q]_F \neq [\mathbf{1}]_F$ and has a maximum which is a contradiction. □

Corollary 4.7 *Let $\mathbf{L}$ be an ICRC. If $\mathbf{L}$ is Archimedean, then $\mathbf{L}$ satisfies (Q_1).*

PROOF: If $\mathbf{L}$ is a Wajsberg hoop then it is obvious. Suppose that $\mathbf{L}$ is not a Wajsberg hoop. Clearly $\mathbf{L}$ is simple, hence subdirectly irreducible and the minimal nontrivial filter is $F = L$. Thus $\mathbf{L}/F$ is the trivial ICRC which is obviously a Wajsberg hoop. Furthermore, it follows that $\mathbf{L}$ contains no equivalence class $[x]_F$ different from $[\mathbf{1}]_F$. Thus (Q_1) is valid in $\mathbf{L}$ by Theorem 4.6. □

It is a natural question to ask whether there is a non-Archimedean ICRC satisfying (Q_1) which is not a Wajsberg hoop. Such algebras exist as it is shown in the following example.

Example 4.8 Let $\mathbf{R}_2$ be the lexicographic product of two copies of the totally ordered additive group of reals and $\mathbb{Z}^-$ the set of non-positive integers. Consider the submonoid $\mathbf{L}$ of $\mathbf{R}_2$ whose universe is the set $F \cup G$ where

$$F = \{\langle 0, y\rangle \in \mathbb{R}^2 \mid y \in \mathbb{Z}^-\}, \quad G = \{\langle x, y\rangle \in \mathbb{R}^2 \mid x < 0\}.$$

Then $\mathbf{L}$ is clearly integral and even residuated. The corresponding residuum is computed as follows:

$$\langle a, b\rangle \to \langle c, d\rangle = \begin{cases} \langle 0, 0\rangle & \text{if } \langle a, b\rangle \leq \langle c, d\rangle\,, \\ \langle c - a, d - b\rangle & \text{if } a > c\,, \\ \langle 0, \lfloor d - b\rfloor\rangle & \text{if } a = c \text{ and } b > d, \end{cases}$$

where $\lfloor x\rfloor$ is the greatest integer less than or equal to x. Thus $\mathbf{L}$ forms a cancellative ICRC which is not divisible. The subset F is its minimum nontrivial filter. The quotient $\mathbf{L}/F$ is isomorphic to the ICRC $\mathbf{R}^-$ defined in Example 2.14 which is a Wajsberg hoop. Moreover, each $[\langle x, y\rangle]_F$ has no maximum for $x < 0$. Hence $\mathbf{L}$ satisfies (Q_1) by Theorem 4.6.

Lemma 4.9 *Let $\mathbf{L}$ be an ICRC satisfying (Q_1) such that $\mathbf{L}$ is not a Wajsberg hoop. Then $\mathbf{L}$ has a co-atom, i.e., the set $L \setminus \{\mathbf{1}\}$ has a maximum.*

PROOF: Suppose that $M = L \setminus \{\mathbf{1}\}$ has no maximum. Clearly $\bigvee M = \mathbf{1}$. Observe that $\mathbf{L}$ cannot be a basic hoop. Since it is not a Wajsberg hoop, it would not be indecomposable by [1, Theorem 3.6], i.e. it would have to be an ordinal sum of at least two nontrivial components $\mathbf{A}_1$ and $\mathbf{A}_2$. Let $p \in A_1$ and $q \in A_2$ such that $\mathbf{1} > p > q$. Then $(p \to q) \to q = q \to q = \mathbf{1}$. Thus by (Q_1) we would get $p = \mathbf{1}$ (a contradiction). Hence $\mathbf{L}$ cannot be divisible and there are $p, q \in L$ such that $\mathbf{1} > p > q$ and $p * (p \to q) < q$. Since $\bigvee_{r \in M}(q * r) = q * \bigvee M = q$, there is $s \in M$ such that $p * (p \to q) < q * s$ which implies that (Q_1) is not valid in $\mathbf{L}$ by Lemma 4.4 (a contradiction). □

Lemma 4.10 *Let* $\mathbf{L}$ *be an ICRC satisfying* (Q_1) *and* $p, q \in L$. *If* $\mathbf{1} > p > q$ *then*

$$((p \to q) \to q)^2 < p\,.$$

PROOF: First, assume that $\mathbf{L}$ is a Wajsberg hoop and $((p \to q) \to q)^2 \geq p$. Since we assume that $\mathbf{L}$ is Wajsberg hoop, we get $(p \to q) \to q = p$ which implies $p^2 \geq p$, i.e., $p^2 = p$. Thus p is idempotent, i.e., either $p = 1$ which is not possible or p is a bottom element which also not possible since $p > q$.

Hence assume that $\mathbf{L}$ is not a Wajsberg hoop. Then $\mathbf{L}$ is subdirectly irreducible and $\mathbf{L}/F$ is a Wajsberg hoop by Theorem 4.6 (F is the minimal nontrivial filter) and there is a co-atom $a \in L$ by Lemma 4.9. Since (Q_1) is valid in $\mathbf{L}$, we have $(p \to q) \to q \leq a < \mathbf{1}$. We claim that there is $x \in [p \to q]_F$ such that $x > p \to q$. Indeed, either $[p \to q]_F \neq [\mathbf{1}]_F$ then the existence follows from Lemma 4.3 or $[p \to q]_F = [\mathbf{1}]_F$ then take $x = \mathbf{1}$. Since F is generated by a, there is $n \in \mathbb{N}$ such that

$$x * a^{n+1} \leq p \to q < x * a^n\,. \tag{1}$$

Since $x * a^n > p \to q$, we get $x * a^n \to q < p$ (if $p \leq x * a^n \to q$ then $x * a^n \leq p \to q$). It follows that

$$(p \to q) \to q \leq x * a^{n+1} \to q = a \to (x * a^n \to q) \leq a \to p\,.$$

Thus $(p \to q) \to q \leq a \to p$. From $(p \to q) \to q \leq a$ we obtain

$$((p \to q) \to q)^2 \leq a * (a \to p) \leq p\,.$$

Now it remains to show that $p \neq ((p \to q) \to q)^2$. Suppose that $p = ((p \to q) \to q)^2$. Then

$$p = ((p \to q) \to q)^2 \leq a * ((p \to q) \to q) \leq a * (a \to p) \leq p\,,$$

showing that all the terms above are equal. It follows $x * a^{n+1} * ((p \to q) \to q) = p * x * a^n$. Moreover, we have $p*x*a^n > q$ because if $p*x*a^n \leq q$ then $x*a^n \leq p \to q$, contradicting the inequality (1). Thus $x*a^{n+1}*((p \to q) \to q) = p*x*a^n > q$ and $(p \to q) \to q > x*a^{n+1} \to q$. But from the first inequality in (1) we also get $(p \to q) \to q \leq x * a^{n+1} \to q$, and we reach a contradiction. □

Theorem 4.11 *Let* $\mathbf{L}$ *be an ICRC. Then* (Q_1) *is valid in* $\mathbf{L}$ *iff* (A_1^2) *is valid in* $\mathbf{L}$.

PROOF: The right-to-left direction is obvious. Assume that $\mathbf{L}$ satisfies (Q_1). The only interesting case, when (A_1^2) could fail, is the case when $\mathbf{1} > p > q$. But in this case we have $((p \to q) \to q)^2 < p$ by Lemma 4.10. □

Finally, it is possible to extend the previous results to all representable ICRLs. In order to prove this we will use results from [25]. First, we have to prove the following lemma.

Lemma 4.12 *Let* $\mathbf{L} \in \mathcal{Q}_1$. *Then the following quasi-identity is valid in* $\mathbf{L}$:

$$((p \to q) \to q) \vee r \approx \mathbf{1} \;\Rightarrow\; p \vee q \vee r \approx \mathbf{1}\,.$$

Proof: Assume that $((p \to q) \to q) \vee r = \mathbf{1}$. First, we will show that

$$((p \to q) \to q) \vee r \leq ((p \vee r) \to q) \to q\,.$$

Clearly, $(p \to q) \to q \leq ((p \vee r) \to q) \to q$ since $\to$ is anti-tone in the first argument. Moreover, $r \leq p \vee r \leq ((p \vee r) \to q) \to q$. Thus we obtain $((p \to q) \to q) \vee r \leq ((p \vee r) \to q) \to q$. Consequently, $((p \vee r) \to q) \to q = \mathbf{1}$. Now, using (Q_1) we get $p \vee q \vee r = \mathbf{1}$. $\square$

Now using the previous lemma, [25, Corollary 9] tells us that the logic corresponding to the quasi-variety $\mathcal{Q}_1$ is complete w.r.t. chains. Since (Q_1) and (A_1^2) are equivalent on chains by Theorem 4.11, we obtain the following theorem.

Theorem 4.13 *Let $\mathbf{L} \in \mathcal{ICRL}^{\mathcal{C}}$. Then (Q_1) is valid in $\mathbf{L}$ iff (A_1^2) is valid in $\mathbf{L}$.*

5 Subvarieties of $\mathcal{ICRL}^{\mathcal{C}}$

In this section we deal with subvarieties of $\mathcal{ICRL}^{\mathcal{C}}$ obtained by a generalization of the identity (A_1^2), namely

$$(A_n^2) \quad \textstyle\bigwedge_{i=1}^{n}((p_{i-1} \to p_i) \to p_i)^2 \leq \bigvee_{i=0}^{n} p_i.$$

The subvariety of $\mathcal{ICRL}^{\mathcal{C}}$ defined by (A_n^2) will be denoted by $\mathcal{A}_2^n$. Analogously, the subvariety of $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$ defined by (A_n^2) will be denoted by $\mathcal{C}an\mathcal{A}_n^2$. Note that if $p_{j-1} \leq p_j$ or $p_{j-1} = \mathbf{1}$ for some $j \in \{1, \ldots, n\}$, then (A_n^2) is valid.

Lemma 5.1 *For all $n \in \mathbb{N} \setminus \{0\}$ we have $\mathcal{A}_n^2 \subseteq \mathcal{A}_{n+1}^2$.*

Proof: Suppose that $\mathbf{L} \in \mathcal{A}_n^2$. Then

$$\bigwedge_{i=1}^{n+1}((p_{i-1} \to p_i) \to p_i)^2 \leq \bigwedge_{i=1}^{n}((p_{i-1} \to p_i) \to p_i)^2 \leq \bigvee_{i=0}^{n} p_i \leq \bigvee_{i=0}^{n+1} p_i\,.$$

Thus $\mathbf{L} \in \mathcal{A}_{n+1}^2$. $\square$

Theorem 5.2 *Let $\mathbf{L}$ be an ICRC with at most n nontrivial Archimedean classes. Then (A_n^2) is valid in $\mathbf{L}$.*

Proof: Clearly the only interesting case is when $\mathbf{1} > p_0 > p_1 \cdots > p_n$. Then at least one pair p_{j-1}, p_j must belong to one nontrivial Archimedean class C. By Proposition 2.10 there is a principal filter F_C corresponding to C and $C = F_C \setminus F$ where F is the predecessor of F_C. If we factorize F_C/F, we get an Archimedean ICRC. Thus (Q_1) is valid in F_C/F by Corollary 4.7. Moreover, by Lemma 4.10 we have $[((p_{j-1} \to p_j) \to p_j)^2]_F < [p_{j-1}]_F$. Thus $((p_{j-1} \to p_j) \to p_j)^2 < p_{j-1}$. Since $\bigwedge_{i=1}^{n}((p_{i-1} \to p_i) \to p_i)^2 \leq ((p_{j-1} \to p_j) \to p_j)^2$ and $p_{j-1} \leq \bigvee_{i=0}^{n} p_i$, we are done. $\square$

Example 5.3 Let us denote the set of nonpositive integers by $\mathbb{Z}^- = \{z \in \mathbb{Z} \mid z \leq 0\}$. Then $\mathbf{Z}^- = (\mathbb{Z}^-, +, \to, \leq, 0)$, where $+$ is the usual addition, $\leq$ is the usual order, and $x \to y = \min\{0, y - x\}$, is an Archimedean cancellative ICRC. Let $\vec{\mathbf{Z}}^n$ be the lexicographic product of n copies of $\mathbf{Z}^-$, i.e. $\vec{\mathbf{Z}}^n = \mathbf{Z}^- \mathbin{\vec{\times}} \cdots \mathbin{\vec{\times}} \mathbf{Z}^-$. Then $\vec{\mathbf{Z}}^n$ is a cancellative ICRC. Observe that $\vec{\mathbf{Z}}^n$ possesses n nontrivial Archimedean classes of the form

$$C_i = \{\langle 0, \ldots, 0, a_i, \ldots, a_n\rangle \in (\mathbb{Z}^-)^n \mid a_i, \ldots, a_n \in \mathbb{Z}^-\}\,.$$

Theorem 5.4 *The chain of varieties $\mathcal{A}^2_n$ is strictly increasing.*

PROOF: Let $\vec{\mathbf{Z}}^n$ be the ICRC from Example 5.3. First, we show that (A^2_{n-1}) is not valid in $\vec{\mathbf{Z}}^n$. Let us evaluate the variables as follows:

$$\begin{aligned} p_0 &= \langle 0, \ldots, 0, -1\rangle\,, \\ p_1 &= \langle 0, \ldots, -1, 0\rangle\,, \\ &\vdots \\ p_{n-1} &= \langle -1, \ldots, 0, 0\rangle\,. \end{aligned}$$

Then $p_{i-1} \to p_i = p_i$ and $p_{n-1} < \cdots < p_1 < p_0 < \langle 0, \ldots, 0\rangle$. Thus $((p_{i-1} \to p_i) \to p_i)^2 = \langle 0, \ldots, 0\rangle$ and $p_0 = \bigvee_{i=0}^{n} p_i < \langle 0, \ldots, 0\rangle$.

Second, $\vec{\mathbf{Z}}^n$ consists of n nontrivial Archimedean classes, therefore (A^2_n) is valid in $\vec{\mathbf{Z}}^n$ by Theorem 5.2. □

Theorem 5.5 *Let $\mathbf{L}$ be an ICRC and $n \geq 2$. Then $\mathbf{L}$ belongs to $\mathcal{A}^2_n$ iff $\mathbf{L}/F_M$ belongs to $\mathcal{A}^2_1$ where F_M is the maximal filter such that F_M is a subalgebra belonging to $\mathcal{A}^2_{n-1}$.*

PROOF: The maximal filter F_M clearly exists and is the union of all filters belonging to $\mathcal{A}^2_{n-1}$. The union is not empty as the trivial filter satisfies this condition.

($\Rightarrow$): Assume that $\mathbf{L} \in \mathcal{A}^2_n$. Let F_M be the maximal filter belonging to $\mathcal{A}^2_{n-1}$. Suppose that $\mathbf{L}/F_M \notin \mathcal{A}^2_1$. Then (Q_1) is not valid in $\mathbf{L}/F_M$ by Theorem 4.11. Thus there must be $p, q \in L$ such that $[\mathbf{1}]_{F_M} > [p]_{F_M} > [q]_{F_M}$, $[(p \to q) \to q]_{F_M} = [\mathbf{1}]_{F_M}$ and $[p]_{F_M} < [\mathbf{1}]_{F_M}$. It follows that $(p \to q) \to q \in F_M$ and $p \notin F_M$. Moreover, we have $[p \to q]_{F_M} = [q]_{F_M}$. Further

$$[p^2 \to q]_{F_M} = [p]_{F_M} \to [p \to q]_{F_M} = [p \to q]_{F_M} = [q]_{F_M}\,.$$

We claim that $p \to q = p^2 \to q$. Since $[p^2 \to q]_{F_M} = [q]_{F_M}$, there is $s \in F_M$ such that $s * (p^2 \to q) \leq q$. By residuation we get $p^2 \to q \leq s \to q$. As $s > p \geq p^2$, we have $s \to q \leq p \to q \leq p^2 \to q$. Consequently, $s \to q = p \to q = p^2 \to q$. Thus $p^2 \to q = p \to (p \to q) = p \to q$ and we obtain that $p \to q = \max\,[p \to q]_{F(p)}$ by Lemma 4.2.

Now, since F_M is maximal such that $F_M \in \mathcal{A}^2_{n-1}$, the subalgebra $F(p) \notin \mathcal{A}^2_{n-1}$. Hence there are $p_0, p_1, \ldots, p_{n-1} \in F(p)$ such that $\mathbf{1} > p_0 > p_1 > \cdots > p_{n-1}$ and

$$\bigwedge_{i=1}^{n-1} ((p_{i-1} \to p_i) \to p_i)^2 > \bigvee_{i=0}^{n-1} p_i\,.$$

Moreover, p_{n-1} cannot belong to F_M otherwise $p_0, \ldots, p_{n-1} \in F_M$ and (A^2_{n-1}) is valid in F_M. Thus $p_{n-1} \in F(p) \setminus F_M$. Since $p \to q = \max\, [p \to q]_{F(p)}$, we get that $p_{n-1} \to (p \to q) = p \to q$ by Lemma 4.2. In particular, $\mathbf{1} > p_{n-1} > p \to q$. Consequently, we obtain

$$\bigwedge_{i=1}^{n-1} ((p_{i-1} \to p_i) \to p_i)^2 \wedge ((p_{n-1} \to (p \to q)) \to (p \to q))^2 =$$

$$= \bigwedge_{i=1}^{n-1} ((p_{i-1} \to p_i) \to p_i)^2 > \bigvee_{i=0}^{n-1} p_i = \bigvee_{i=0}^{n-1} p_i \vee (p \to q)\,.$$

Thus (A^2_n) is not valid in $\mathbf{L}$ (a contradiction).

($\Leftarrow$): Let $\mathbf{L}$ be an ICRC such that $\mathbf{L}/F_M \in \mathcal{A}^2_1$ where F_M is the maximal filter such that $F_M \in \mathcal{A}^2_{n-1}$. Suppose that $\mathbf{L} \notin \mathcal{A}^2_n$. Then there must be $\mathbf{1} > p_0 > p_1, \cdots > p_n$ such that

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 > \bigvee_{i=0}^{n} p_i\,.$$

Suppose that $p_0, \ldots, p_{n-1} \in F_M$. Since (A^2_{n-1}) is valid in F_M, we have

$$\bigwedge_{i=1}^{n-1} ((p_{i-1} \to p_i) \to p_i)^2 \leq \bigvee_{i=0}^{n-1} p_i\,.$$

Thus we get a contradiction:

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \leq \bigwedge_{i=1}^{n-1} ((p_{i-1} \to p_i) \to p_i)^2 \leq \bigvee_{i=0}^{n-1} p_i \leq \bigvee_{i=0}^{n} p_i\,.$$

Hence at least p_{n-1} and p_n cannot belong to F_M. Since $\mathbf{L}/F_M \in \mathcal{A}^2_1$, we obtain $[((p_{n-1} \to p_n) \to p_n)^2]_{F_M} < [p_{n-1}]_{F_M}$ by Lemma 4.10. Consequently, $((p_{n-1} \to p_n) \to p_n)^2 < p_{n-1}$. Finally, we get a contradiction:

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \leq ((p_{n-1} \to p_n) \to p_n)^2 < p_{n-1} \leq \bigvee_{i=0}^{n} p_i\,.$$

□

Theorem 5.6 *The smallest variety containing all algebras from $\mathcal{A}^2_n$ for all $n \in \mathbb{N}$ is the variety $\mathcal{ICRL}^{\mathcal{C}}$.*

Proof: It is known that the variety $\mathcal{ICRL}^{\mathcal{C}}$ has the finite embeddability property (FEP) (see [4, 7, 22]). Thus $\mathcal{ICRL}^{\mathcal{C}}$ is generated by its finite members. Since every finite ICRC $\mathbf{L}$ consists of finitely many Archimedean classes, $\mathbf{L}$ belongs to $\mathcal{A}^2_n$ for some $n \in \mathbb{N}$ by Theorem 5.2.

□

Theorem 5.7 *The chain of subvarieties of $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$ given by identities (A^2_n) is strictly increasing and its limit is $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$.*

PROOF: The fact that the chain is strictly increasing follows already from the proof of Theorem 5.4 if we observe that the used example $\vec{\mathbf{Z}}^n$ in this proof belongs in fact to $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$. The second claim holds since $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$ is generated by members with finitely many Archimedean classes as was shown in [17]. □

6 Applications to MTL-algebras

In this section we present several applications of the results above. First, we will deal with subvarieties of MTL-algebras. Then we will show that the identities (A_n^2) can express the number of Archimedean classes under certain conditions. Finally, the consequences for basic hoops and BL-algebras will be presented.

6.1 Subvarieties of $\mathcal{MTL}$

Now we can employ the lattice isomorphism between $\mathcal{ICRL}^{\mathcal{C}}$ and $\mathcal{SMTL}$ (see Theorem 3.1) together with Theorems 5.6 and 5.4 and show several facts on subvarieties of $\mathcal{MTL}$.

Theorem 6.1 *The chain of subvarieties of $\mathcal{MTL}$ given by identities (A_n^2) is strictly increasing and its limit is $\mathcal{MTL}$.*

PROOF: Let $\mathbf{L} = \bigoplus_{i=0}^n \mathbf{A}_i$ be the ordinal sum of $n+1$ copies of the two-element Boolean algebra $\mathbf{2}$. Then $\mathbf{L}$ has $n+1$ nontrivial Archimedean classes (it is in fact the $n+2$-valued Gödel chain). Thus by Theorem 5.2 (A_{n+1}^2) is valid in $\mathbf{L}$. However, $\mathbf{L}$ does not satisfy (A_n^2). Let $p_{n-i} \in A_i \setminus \{\mathbf{1}\}$ for $0 \le i \le n$. Then $(p_{i-1} \to p_i) \to p_i = p_i \to p_i = \mathbf{1}$ for all i. Thus

$$\bigwedge_{i=1}^n ((p_{i-1} \to p_i) \to p_i)^2 = \mathbf{1} > p_0 \ge \bigvee_{i=0}^n p_i\,.$$

For the second part of the claim, we can argue in the same way as in the proof of Theorem 5.6 since $\mathcal{MTL}$ satisfies FEP. □

Theorem 6.2 *The chains of subvarieties $\mathbf{2} \oplus \mathcal{A}_n^2$ and $\mathbf{2} \oplus \mathcal{C}an\mathcal{A}_n^2$ are strictly increasing and their limits are respectively $\mathcal{SMTL}$ and $\mathcal{PMTL}$.*

PROOF: The claim for $\mathbf{2} \oplus \mathcal{A}_n^2$ is an immediate consequence of Theorems 5.6 and 5.4 together with Theorem 3.1. The same is true for $\mathbf{2} \oplus \mathcal{C}an\mathcal{A}_n^2$ if we use Theorem 5.7 instead of Theorem 5.6. □

We can also easily give an axiomatization of $\mathbf{2} \oplus \mathcal{A}_n^2$ and $\mathbf{2} \oplus \mathcal{C}an\mathcal{A}_n^2$ as subvarieties of $\mathcal{SMTL}$ and $\mathcal{PMTL}$ respectively.

Proposition 6.3 *The variety $\mathbf{2} \oplus \mathcal{A}_n^2$ is a subvariety of $\mathcal{SMTL}$ defined by the identity (A_{n+1}^2) and $\mathbf{2} \oplus \mathcal{C}an\mathcal{A}_n^2$ is a subvariety of $\mathcal{PMTL}$ defined by the same identity.*

Proof: Let $\mathbf{L}$ be a generator of $\mathbf{2} \oplus \mathcal{A}^2_n$, i.e. it is of the form $\mathbf{2} \oplus \mathbf{C}$ for an ICRC $\mathbf{C}$ satisfying (A^2_n). We have to show that (A^2_{n+1}), i.e.

$$\bigwedge_{i=1}^{n+1} ((p_{i-1} \to p_i) \to p_i)^2 \leq \bigvee_{i=0}^{n+1} p_i \,,$$

is valid in $\mathbf{L}$. Clearly, if all p_i's are in C, then the inequality above is trivially satisfied because $\mathbf{C}$ satisfies (A^2_n). Thus the only case, which has to be discussed, is the case when $\mathbf{1} > p_1 > \cdots > p_n > p_{n+1} = \mathbf{0}$. Then

$$\bigwedge_{i=1}^{n+1} ((p_{i-1} \to p_i) \to p_i)^2 = \bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \wedge \neg\neg p_{n+1} =$$

$$= \bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \wedge \mathbf{1} = \bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \leq \mathbf{0} \vee \bigvee_{i=0}^{n} p_i = \bigvee_{i=0}^{n+1} p_i \,.$$

Conversely, assume that $\mathbf{L} \in \mathcal{SMTL}$, where (A^2_{n+1}) is valid. Then by Lemma 2.4 $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ for an ICRC $\mathbf{C}$. It suffices to show that (A^2_n) is valid in $\mathbf{C}$. Suppose not. Then there are $p_0, \ldots, p_n \in C$ such that $\mathbf{1} > p_0 > \cdots > p_n$ and

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 > \bigvee_{i=0}^{n} p_i \,.$$

Then we have

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \wedge \neg\neg p_n = \bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 > \bigvee_{i=0}^{n+1} p_i = \mathbf{0} \vee \bigvee_{i=0}^{n+1} p_i \,,$$

showing that (A^2_{n+1}) is not valid in $\mathbf{L}$ (a contradiction).

The proof for $\mathcal{PMTL}$ can be done analogously, using Lemma 2.5 instead of Lemma 2.4. □

From the previous theorem we can also obtain a negative solution to Hájek's problem mentioned in the introduction.

Corollary 6.4 *Let $\mathcal{K}$ be the class of* ΠMTL*-chains containing at most two nontrivial Archimedean classes. Then the variety $\mathcal{PMTL}$ is not generated by $\mathcal{K}$.*

Proof: Since each ΠMTL-chain in $\mathcal{K}$ satisfies (A^2_2) by Proposition 6.3, the variety generated by $\mathcal{K}$ is strictly smaller than the variety $\mathcal{PMTL}$ by Theorem 6.2. □

6.2 Number of Archimedean classes

The identities (A^2_n) in some cases express the number of Archimedean classes. We start with a general theorem which can be applied to particular cases.

Theorem 6.5 *Let $\mathcal{K}$ be a class of ICRCs satisfying the following conditions:*

1. *$\mathcal{K}$ is closed under homomorphic images.*

2. *Let $\mathbf{L} \in \mathcal{K}$. Then (A_1^2) is valid in $\mathbf{L}$ iff $\mathbf{L}$ is Archimedean.*

Then an algebra $\mathbf{L} \in \mathcal{K}$ satisfies (A_n^2) iff $\mathbf{L}$ contains at most n nontrivial Archimedean classes.

PROOF: The left-to-right direction follows from Theorem 5.2. Let $\mathbf{L} \in \mathcal{K}$ satisfying (A_n^2). We prove the right-to-left direction by induction on n. For $n = 1$ the claim follows from the second condition on $\mathcal{K}$. Now assume that the claim holds for $n-1$. Let F_M be the maximum filter such that F_M is a subalgebra belonging to $\mathcal{A}_{n-1}^2$. Then F_M contains at most $n-1$ nontrivial Archimedean classes by the induction assumption and $\mathbf{L}/F_M$ belongs to $\mathcal{A}_1^2$ by Theorem 5.5. The quotient $\mathbf{L}/F_M$ also belongs to $\mathcal{K}$ by the first condition on $\mathcal{K}$. Thus $\mathbf{L}/F_M$ is Archimedean (hence simple) by the second condition. Consequently, Con $\mathbf{L}/F_M$ is just a two-element chain. It is well known from universal algebra that Con $\mathbf{L}/F_M \cong [\theta_{F_M}, \theta_L]$ (see e.g. [5, Page 54, Theorem 6.20]). Thus $[\theta_{F_M}, \theta_L] = \{\theta_{F_M}, \theta_L\}$. Further, we clearly have Con $F_M \cong [\theta_{\{\mathbf{1}\}}, \theta_{F_M}]$ by Theorem 2.6. Since F_M has at most $n-1$ nontrivial Archimedean classes, Con F_M is a chain containing at most n elements. Altogether, Con $\mathbf{L} = [\theta_{\{\mathbf{1}\}}, \theta_L] = [\theta_{\{\mathbf{1}\}}, \theta_{F_M}] \cup [\theta_{F_M}, \theta_L]$ has at most $n+1$ elements. Thus $\mathbf{L}$ has at most n nontrivial Archimedean classes. □

The first class of algebras, where Theorem 6.5 can be applied, is the case of *complete* ICRCs. Our definition of completeness differs from the usual one since an ICRC need not possess a bottom element. Precisely, we say that an ICRC is complete if each non-empty subset has a supremum. Observe that a subset in this setting has an infimum iff it has a lower bound.

Now it suffices to prove that the class of complete ICRCs satisfies the conditions from Theorem 6.5.

Lemma 6.6 *Let $\mathbf{L}$ be a complete ICRC. Then $\mathbf{L}$ satisfies (A_1^2) iff $\mathbf{L}$ is Archimedean.*

PROOF: The left-to-right direction follows from Corollary 4.7 and Theorem 4.11. For the right-to-left direction we can suppose that $\mathbf{L}$ satisfies (Q_1) by Theorem 4.11. Assume that $\mathbf{L}$ is not Archimedean, i.e. not simple. Then there is a nontrivial filter F such that $F \neq L$, i.e. there is $x \notin F$. Note that x is a lower bound of F. By Lemma 4.3 each $[x]_F \neq [\mathbf{1}]_F$ has no maximum. Hence F cannot have a minimum m (otherwise $m \to x$ would be a maximum of $[x]_F$). Thus the infimum i of F does not belong to F and clearly $i = \max\, [i]_F$ which is a contradiction with the fact that $[i]_F$ has no maximum. Consequently, $\mathbf{L}$ must be simple, hence Archimedean. □

Lemma 6.7 *Let $\mathbf{L}$ be an ICRC and $F \in \mathcal{F}_{\mathbf{L}}$. If $\mathbf{L}$ is complete, then $\mathbf{L}/F$ is complete as well.*

PROOF: Let $\emptyset \neq M \subseteq L/F$. If M has a maximum, then this maximum is a supremum as well. Assume that M has no maximum. Let us define the following subset of L:

$$M' = \bigcup_{[x]_F \in M} [x]_F\,.$$

Observe that $[x]_F \in M$ iff $x \in M'$. Since $\mathbf{L}$ is complete, there is a supremum of M', say m. We claim that $[m]_F$ is a supremum of M. Clearly, $[m]_F \geq [x]_F$ for all $[x]_F \in M$ because $m \geq x$ for any $x \in M'$. Suppose $[y]_F \geq [x]_F$ for all $[x]_F \in M$. Since we assume that M has no maximum, we have in fact $[y]_F > [x]_F$ for all $[x]_F \in M$. Consequently, $y > x$ for all $x \in M'$. Thus $y \geq m$ showing that $[y]_F \geq [m]_F$. □

Using Theorem 6.5 and the lemmata above we get the following theorem.

Theorem 6.8 *Let* $\mathbf{L}$ *be a complete ICRC. Then* (A^2_n) *is valid in* $\mathbf{L}$ *iff* $\mathbf{L}$ *has at most* n *nontrivial Archimedean classes.*

Another interesting class of algebras, where Theorem 6.5 can be applied, is the variety of k-contractive MTL-algebras. These algebras were introduced in [6] as MTL-algebras satisfying a sort of generalized contraction law $x^k \approx x^{k-1}$. Since the class of k-contractive MTL-chains is clearly closed under taking homomorphic images, it suffices to prove the following lemmata.

Lemma 6.9 *Let* $\mathbf{L}$ *be an* k*-contractive MTL-algebra and* $F \in \mathcal{P}_{\mathbf{L}}$*. Then each* $[x]_F$ *has a minimum and a maximum.*

Proof: Since F is a principal filter, it is generated by a single element, say a. Then a^k is idempotent and the minimum of F because $\mathbf{L}$ is k-contractive. We claim that $x * a^k = \min [x]_F$ and $a^k \to x = \max [x]_F$. Let $y \in [x]_F$. Then $x \to y \in F$, i.e. $a^k \leq x \to y$ showing that $x * a^k \leq y$. Further, we have $y \to x \in F$, i.e. $a^k \leq y \to x$. Consequently, $y \leq a^k \to x$. □

Lemma 6.10 *Let* $\mathbf{L}$ *be an* k*-contractive MTL-chain. Then* $\mathbf{L}$ *satisfies* (A^2_1) *iff it is Archimedean.*

Proof: The left-to-right direction follows from Corollary 4.7 and Theorem 4.11. Conversely, assume that (A^2_1) is valid in $\mathbf{L}$. If $\mathbf{L}$ is not Archimedean (i.e. non-simple), then there must be a nontrivial principal filter F such that $F \neq L$. Thus there is $x \in L \setminus F$. By Lemma 6.9 we get that $[x]_F$ has a maximum which is a contradiction by Lemma 4.3. □

Theorem 6.11 *Let* $\mathbf{L}$ *be an* k*-contractive MTL-chain. Then* (A^2_n) *is valid in* $\mathbf{L}$ *iff* $\mathbf{L}$ *has at most* n *nontrivial Archimedean classes.*

Corollary 6.12 *The class of* k*-contractive MTL-algebras generated by chains containing at most* n *Archimedean classes is axiomatized by* (A^2_n)*.*

Let us denote the class of all k-contractive MTL-algebras by $\mathcal{C}_k\mathcal{MTL}$.

Theorem 6.13 *The chain of subvarieties of* $\mathcal{C}_k\mathcal{MTL}$ *given by identities* (A^2_n) *is strictly increasing and its limit is* $\mathcal{C}_k\mathcal{MTL}$*.*

Proof: The strictness can be proved by the same reasoning as in the proof of Theorem 6.1. The fact the the limit of the chain is $\mathcal{C}_k\mathcal{MTL}$ follows since $\mathcal{C}_k\mathcal{MTL}$ satisfies the FEP (see [20]). □

6.3 Basic hoops and BL-algebras

Lemma 6.14 *Let* $\mathbf{L}$ *be a totally ordered basic hoop. Then* (A_1^2) *is valid in* $\mathbf{L}$ *iff* $\mathbf{L}$ *is indecomposable (i.e. Wajsberg).*

PROOF: If $\mathbf{L}$ is indecomposable, then it is a Wajsberg hoop which satisfies (A_1^2). Conversely, assume that (A_1^2) is valid in $\mathbf{L}$. Then by Theorem 4.11 (Q_1) is valid in $\mathbf{L}$ as well. Assume that $\mathbf{L} \cong \mathbf{A} \oplus \mathbf{B}$ for some nontrivial totally ordered basic hoops. Let $x \in A \setminus \{\mathbf{1}\}$. Then B is clearly a nontrivial filter and $[x]_B = \{x\}$ which is a contradiction by Lemma 4.3. □

Theorem 6.15 *A totally ordered basic hoop satisfies* (A_n^2) *iff it is an ordinal sum consisting of at most* n *indecomposable components.*

PROOF: Let $\mathbf{L}$ be a totally ordered basic hoop satisfying (A_n^2). Suppose that $\mathbf{L}$ has more than n indecomposable components, i.e. $\mathbf{L} \cong \bigoplus_{i=0}^{n} \mathbf{A}_i$. Let $p_{n-i} \in A_i \setminus \{\mathbf{1}\}$. Then $\mathbf{1} > p_0 > \cdots > p_n$ and $(p_{i-1} \to p_i) \to p_i = \mathbf{1}$ for all $1 \leq i \leq n$. Consequently, we get

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 = \mathbf{1} > \bigvee_{i=0}^{n} p_i \,,$$

showing that (A_n^2) is not valid (a contradiction).

Conversely, suppose that $\mathbf{L}$ is an ordinal sum of at most n indecomposable components. Again the only case when (A_n^2) could fail is the case for $\mathbf{1} > p_0 > \cdots > p_n$. Since $\mathbf{L}$ has at most n components, there must be $j \in \{1, \ldots, n\}$ such that p_{j-1} and p_j are in the same component, say $\mathbf{A}$. By Lemma 6.14 the identity (A_1^2) is valid in $\mathbf{A}$. Thus $((p_{j-1} \to p_j) \to p_j)^2 \leq p_{j-1} \vee p_j$. Consequently, we obtain

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \leq ((p_{j-1} \to p_j) \to p_j)^2 \leq p_{j-1} \vee p_j \leq \bigvee_{i=0}^{n} p_i \,,$$

showing that (A_n^2) is valid in $\mathbf{L}$. □

Corollary 6.16 *The subvariety of basic hoops (BL-algebras) generated by ordinal sums of at most* n *indecomposable totally ordered basic hoops is axiomatized by* (A_n^2)*.*

Remark 6.17 In fact, in the previous corollary we can replace the identity (A_n^2) by the simpler identity (A_n):

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i) \leq \bigvee_{i=0}^{n} p_i \,,$$

obtaining the axiomatization presented in [1, Lemma 4.2]. This can be done since Lemma 6.14 can be easily improved as follows: a totally ordered basic hoop satisfies (A_1) iff it is indecomposable (i.e. Wajsberg).

Theorem 6.18 *The chain of subvarieties of* $\mathcal{BL}$ *given by identities* (A_n^2) *is strictly increasing and its limit is* $\mathcal{BL}$*.*

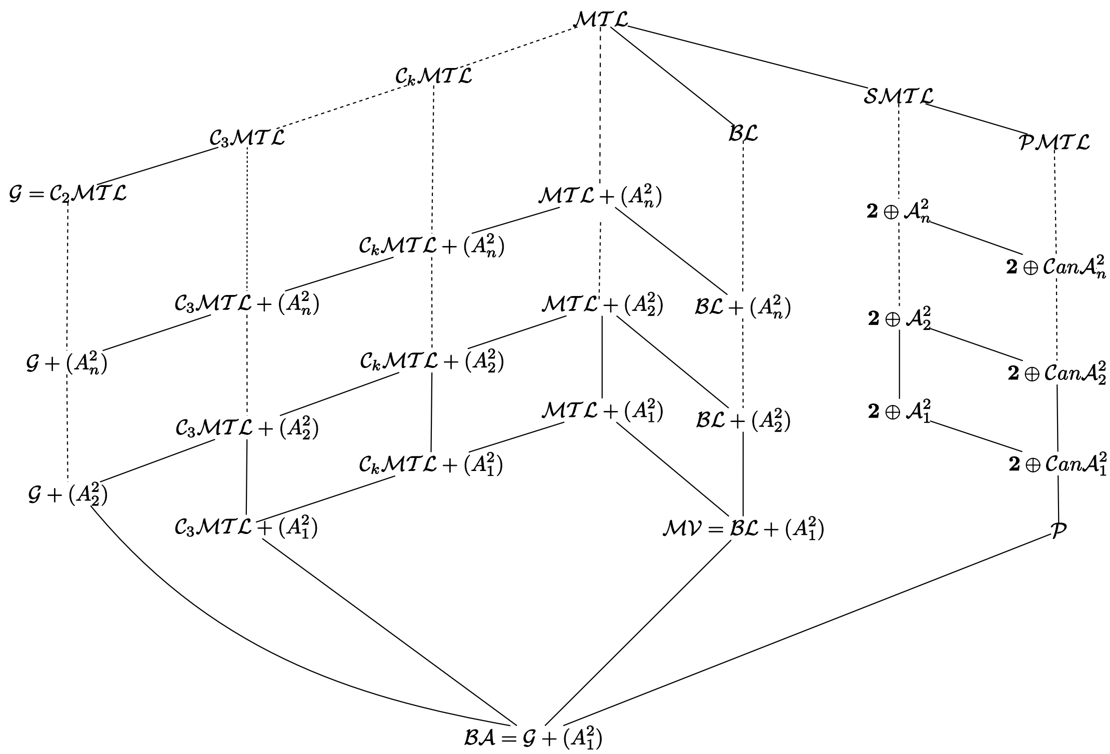

Figure 1: Relations between considered subvarieties of $\mathcal{MTL}$.

Proof: The strictness can be proved as in Theorem 6.13. The second part of the claim holds since $\mathcal{BL}$ is generated by finite ordinal sums of finite Wajsberg hoops (see [1]). $\square$

The relations between investigated varieties are depicted in Figure 6.3. The varieties of Boolean algebras, Gödel algebras, MV-algebras, and product algebras are denoted respectively by $\mathcal{BA}$, $\mathcal{G}$, $\mathcal{MV}$, and $\mathcal{P}$. In order to make the picture simpler, the relations from Proposition 6.3 between $\mathbf{2} \oplus \mathcal{A}_n^2$ and the subvariety of $\mathcal{MTL}$ defined by (A_{n+1}^2) are not included.

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