Non-Classical Mathematics 2009

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Invited talks

Tutorial on Inconsistent Mathematics

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Inconsistent mathematics aims to study interesting mathematical theories which do not collapse under inconsistency. As such, the background logics for such theories must not contain the classical principle ECQ: from a contradiction to deduce anything. Many such logics are known by now, and it can be seen that, many of the mathematically interesting parts of such theories are invariant under change of logic. In this talk I will do three things. First, I will discuss some results based on algebraic collapse, particularly collapse of groups onto their subgroups. Use will be made of the Routley star operation. Second, I will outline some recent work on inconsistent geometrical theories. I will conclude by discussing some of the philosophical implications of inconsistent mathematics.

Theories, Co-theories and Bi-theories in Non-Classical Mathematics

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A theory is a collection of statements closed under logical consequence. Given classical logical principles, a theory can tell one what to accept (accept all the statements in the theory), and what to reject (reject all the statements whose negations are in the theory), and over what one should – at least as far as the theory is concerned – remain silent (the statements which do not appear in the theory and whose negations also do not appear in the theory). In a theory expressed in something other than classical logic, this nexus is broken. A theory may tell us what to accept, but it no longer gives us guidance on what is to be rejected. In logics admitting truth-value 'gaps', perhaps more is to be rejected than the theory explicitly states by way of negation. In logics admitting truth value 'gluts', perhaps some statements are not to be rejected, even when their negations are explicitly endorsed in the theory.

In this talk, I will introduce co-theories and bi-theories, to keep track both of what is to be accepted and what is to be rejected, in non-classical contexts, and explain what we can do with these notions in theories of arithmetic, classes, properties and truth.

Real and Ideal Entities in a Minimalist Constructive Foundation

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Part 1

Foundations as a choice of the level of abstraction—the minimalist foundation (M.E. Maietti and G.S., 2005)—real and ideal entities—mathematization of existential statements—symmetry and duality in topology.

Part 2

Developing constructive topology over a minimalist foundation—basic topologies and formal topologies—generation by induction and coinduction—formal points and formal spaces—examples: trees and choice sequences, rings and prime filters—constructive version of Hilbert's program.

Semantic Methods in Substructural and Fuzzy Logics

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In this tutorial, I will introduce some semantic techniques that are widely applicable in substructural and fuzzy logics. The properties include:

(1) Local deduction theorem
(2) Disjunction and existence properties
(3) Interpolation property
(4) (strong) consistency of naive comprehension axioms

The first three are based on generalized Dedekind-MacNeille completions, while the last is on Brouwer's fixpoint theorem. Notice that these are often proved by syntactic methods, which are concrete but cumbersome. Semantic methods instead allow for concise and uniform proofs; this is most prominent for (4), where the syntactic consistency proof by (White 1979) for naive set theory over Ł (infinite-valued Łukasiewicz logic) is notoriously complicated. More importantly, the semantic approach highlights a delicate and deep relationship between syntax and semantics; by taking (4) as example again, the syntactic consistency proof relies on a crucial property of logics without contraction that proofs shrink by cut elimination, whereas the semantic proof relies on the fact that logical connectives admit continuous interpretations over $[0, 1]$. Is there any *metalevel* relationship between shrink-by-cut-elimination and continuity? I believe that this kind of metalevel questions is of vital significance for the next generation of substructural and fuzzy logics.

The course is self-contained. Although it focuses on substructural and fuzzy logics, I hope the idea behind will be useful for other branches of nonclassical logics too.

Ternary Semantics, Combinators, K2U

Robert K Meyer (1932–2009)

The stone which the builders rejected
Is become the head of the corner.
(Psalms 118:22)

There is a long and somewhat mysterious connection between the unary functions of λ-calculus and Combinatory Logic (henceforth λ and **CL** respectively) and the ternary relation of relevant semantics. I call this the Key to the Universe (K2U). Our purpose will be to explore K2U. The minimal positive relevant logic is the system **B**+ of [$Sem3]. Basically the semantics of **B**+ (and its various inessential expansions and contractions) is based on the notion of a **3-frame** $\mathbf{K} = \langle K, R\rangle$, where K is a set (of worlds, points) and R is a ternary relation on K. (I. e., $R \subseteq K^3$. Let $\mathbf{K}$ be a 3-frame. Let $\mathbf{2}$ be the set $\{F, T\} = \{0, 1\}$ of truth-values. Let L be the set of formulae A, B, etc., of a propositional logic, built up from countably many atoms p, q, etc. A function $I : L \times K \to 2$ is a *possible interpretation.* Let I be a possible interpretation, fixed in context, let A be a formula and let $w \in K$. I write sometimes

$$[A]w \text{ for } I(A, w) = 1$$
$$\neg[A]w \text{ for } I(A, w) = 0$$

Let $\mathbf{K}$ and w be as just above, and let quantifiers range over K. The possible interpretation I is moreover an *interpretation* provided that certain conditions are met, including in particular the following (classical) *truth-conditions* (when the particles are present in L), associating iterated (material) $\supset$ to the *right.*

($T\wedge$) $[A \wedge B]w = [A]w \wedge [B]w$
($T\vee$) $[A \vee B]w = [A]w \vee [B]w$
($T\to$) $[A \to B]w = \forall a\forall b(Rwab \supset [A]a \supset [B]b)$
($V\to$) $A \to B$ is *verified* $= \forall w([A]w \supset [B]w)$

We shall add to this list of conditions in the presence of other particles in L. We concern ourselves here with **basic positive entailments**. Let A and B be any formulae. Then we say that A *basically entails* B (write $A \leq B$) just in case $A \to B$ is semantically valid. I. e.,

($B\leq$) $A \leq B = A \to B$ is verified on all interpretations in all 3-frames.

Basic entailment is a very weak relation—much weaker than the entailments of standard positive relevant logics like **E**+ of entailment and **R**+ of relevant implication of [$\$ABD$].

Let α and β be sets of formulae. We follow Powers [$\$ALP$] in defining the *modus ponens product* as follows:

($D\circ$) $\alpha \circ \beta = \{C : \exists B((B \to C \in \alpha) \wedge (B \in \beta))\}$

'$\alpha \circ \beta$' (read 'α *mop* β') straightforwardly means the result of detaching all the antecedents B in β from $\to$ statements $B \to C$ in α.

We turn (*at last*) to K2U. Let α be a set of formulas (henceforth *formset*). What K2U does is to *model* the **combinators** of Curry's **CL** in certain formsets. The most conspicuous of these formsets α are moreover *theories* (called *intensional theories* in our [\$$SEM1$]) provided that α is closed under (provable) entailment and conjunction. I. e., to be a theory, α must satisfy, for all $A, B \in \mathbf{L}$, these 2 conditions:

$(\leq E)$ $A \leq B \supset A \in \alpha \supset B \in \alpha$
$(\wedge I)$ $A \in \alpha \wedge B \in \alpha \supset A \wedge B \in \alpha$

In fact, an even more special role is reserved for the prime theories of $\mathbf{B}+$, where a theory α is *prime* iff α satisfies (in addition to $\leq E$ and $\wedge I$), for all formulae A and B,

$(\vee E)$ $A \vee B \in \alpha \supset A \in \alpha \vee B \in \alpha$

Applying $\leq E$ makes the converse $\vee I$ of $\vee E$ trivial from right to left for all theories α.

There is a preliminary modeling of **CL** in arbitrary formsets, the Fools Model of [\$$MBP$]. Where $A \to B$ is an arbitrary $\to$ formula, let $\{A \to B\}$ be the set of all *substitution instances* of that formula.

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Email sent: 28 March 2009

... I apologize that even with the extended deadline today
there is more work to do.  ...  I shall send a more complete
version immediately.

Cordially,
Bob Meyer (Australian National University)
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Email sent: 30 March 2009

... I shall leave you (for now)
with the preliminary version. When cleaned up, it will
contain yet another proof that there is a model of CL in
prime B+ theories (a result that has already been claimed
on my behalf by colleagues in 2 previous joint papers, but
more readily extendible here to richer contexts).

Cordially,
Bob (Meyer)
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Contributed talks

A New Approach to Predicative Set Theory

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The predicativist program for the foundations of mathematics, initiated by Poincaré, and first seriously developed by Weyl, seeks to establish certainty in mathematics without necessarily revolutionizing it (as the intuitionistic program does). The program as is usually conceived nowadays (following Weyl and Feferman) is based on the following two basic principles:

(PRE) Higher order constructs, such as sets or functions, are acceptable only when introduced through definitions. These definitions cannot be circular. Hence in defining a new construct one can only refer to constructs which were introduced by previous definitions.

(NAT) The natural-numbers sequence is a basic well understood mathematical concept, and as a totality it constitutes a set.

The predicativist program was extensively pursued by Feferman, who developed proof systems for predicative mathematics and showed that a very large part of classical analysis can be developed within them. He further conjectured that predicative mathematics in fact suffices for developing all the mathematics that is actually indispensable to present-day natural sciences.

Despite this success, Feferman's systems failed to receive in the mathematical community the interest they deserve. Unlike constructive mathematics, they were also almost totally ignored in the computer science community. The main reason for this seems to be the fact that they are rather complicated in comparison to the impredicative formal set theory ZF, which provides the standard foundations and framework for developing mathematics. In particular: Feferman's systems use complicated systems of types, and both functions and classes are taken in them as independent primitives.

The main goal of this paper is to suggest a new framework for the Weyl-Feferman predicativist program by constructing an absolutely (at least in our opinion) reliable predicative *pure set theory* PZF whose language is type-free, and from a platonic point of view, the universe V of ZF (whatever this universe is) is a model of it.

Our basic idea is that principle (PRE) means that the predicatively acceptable instances of the comprehension schema are those which determine the collections they define in an absolute way, independent of the extension of the "surrounding universe". This idea is implemented using a syntactic safety relation between formulas and sets of variables. This safety relation is obtained as a common generalization of syntactic approximations of the notion of domain-independence used in database theory, and syntactic approximations of Gödel's notion of absoluteness used in set theory.

One important feature of our framework is that it requires us to make an extensive use of abstraction terms. In fact the main axiom of PZF is the comprehension schema $\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$, where φ is syntactically safe with respect to $\{x\}$. Unlike the official language of ZF, this well reflects the real mathematical practice of working with sets. Still, this does not involve an essential departure from first-order language. in contrast, in order to implement also Principle (NAT) within our framework, we find it necessary (and natural) to really go beyond first-order languages. This is done by using *ancestral* logic, which is strictly stronger than first-order logic, but much weaker then full second-order logic.

Another important feature of our framework is that it is not committed to any particular underlying logic. It is possible (and makes sense) to use it together with classical logic, but it equally makes sense to use it in combination with some non-classical logic, especially (but not only) intuitionistic logic.

Infinitesimal Calculus Based on a Fuzzy Notion of Infinitesimal

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The original idea of infinitesimals, upon which the infinitesimal calculus was based till the XIX century, turned out to be rather problematic from the point of view of rigorous mathematics. The modern treatment of mathematical analysis circumvented the problems by introducing the ε–δ definitions, thus eliminating the notion of infinitesimal from standard mathematics. Nevertheless, several alternative approaches have been developed that try to save the notion of infinitesimal and reconstruct it in a rigorous manner, while preserving the original appealing and intuitive motivations that have been obscured in standard mathematics. For example, Robinson's non-standard analysis [5] reconstructs infinitesimals by means of an ultraproduct construction, which can be axiomatized in specific non-standard set theories [6, 3]. Mortensen's non-classical approach [4] admits that the original notion of infinitesimal (a positive real smaller than all positive reals) is inconsistent, and builds its theory in a paraconsistent logic, which is sufficiently weak to avoid destruction of the theory by explosive inconsistencies, but turns out to be sufficiently strong to reconstruct the infinitesimal calculus.

This paper offers another non-classical approach to infinitesimals, based on formal fuzzy logic. A straightforward idea that suggests itself is to take infinitesimality as a gradual ("fuzzy") property, with infinitesimals approximated (to a larger or smaller degree) by standard reals: the smaller a positive real, the closer it is to being an infinitesimal. Even though there are no true infinitesimals among reals, very small reals are very close to being ones. (Compare the informal reasoning about $\mathrm{d}x$ in limit considerations, e.g., in physics.) In other words, the smaller a positive real x, the larger the truth degree of the predication 'x is infinitesimal'. Infinitesimals thus form a fuzzy subset C of standard reals, whose membership values $\mu_C(x)$ increase as the absolute value of x gets closer to 0, though they never reach the value 1 (as there are no true infinitesimals). The value of $\mu_C(x)$ expresses the degree to which x can play the role of an infinitesimal (which is obviously fairly large for very small x).

By means of such a notion of fuzzy infinitesimal, which can be formalized, e.g., in Henkin-style higher-order fuzzy logic ŁΠ [1] or MTL_Δ, definitions of limits, derivatives, etc., may be formulated as usual in the calculus of infinitesimals: e.g., the fact that f is continuous in x_0 can be defined as

$$(\forall x)(|x - x_0| \in C \rightarrow |f(x) - f(x_0)| \in C). \tag{1}$$

The properties of so defined notions will have to be derived by the rules of (higher-order) fuzzy logic, as formulae, in which the fuzzy predicate C occurs, are in general fuzzy, i.e., can have truth values between 0 and 1. Particular values of $\mu_C(x)$ (and the resulting truth values of more complex formulae, such as the degree of being a derivative of f in x_0) can vary in particular models chosen by the user; the theory only assumes that C satisfies certain conditions (viz, to increase towards 0 and have the full height, but no prototypes), which are expressible in higher-order fuzzy logic. The fuzzy equivalence relation $x \doteq_C y \equiv_{\mathrm{df}} |x - y| \in C$, which appears in the definitions, can be interpreted as the (degree of) indistinguishability of x and y. (Notice that the general non-idempotence of conjunction in fuzzy logic prevents Poincaré's paradox and allows fuzzy indistinguishability to be fully transitive in the sense of fuzzy logic.) The notions defined by means of C (or $\doteq_C$) then have a very natural meaning, e.g., the definition (1) just says that f is continuous iff indistinguishable arguments get indistinguishable values. Thus apparently, the infinitesimal calculus can be reconstructed by means of a notion of fuzzy infinitesimal, formalized in higher-order fuzzy logic.

However, this straightforward idea does not quite work: the notion expressed by (1) is actually not that of a limit, but rather one akin to being 1-Lipschitz in x_0 (the condition corresponds exactly to the 1-Lipschitz property for a particular C, namely $\mu_C(x) = 1 - x$, in standard models of Łukasiewicz higher-order logic). A finer analysis is therefore needed.

A correct account is obtained by considering not one, but a *system* $\mathscr{C}$ of fuzzy sets C representing the 'fuzzy neighborhoods' of 0 (or measures of being infinitesimal). Quantification over C in suitably defined $\mathscr{C}$ then leads to a correct definition of the limit (and similarly for other notions of the infinitesimal calculus):

$$\mathrm{Lim}_{\mathscr{C}}(f, x_0, y_0) \equiv_{\mathrm{df}} (\exists C \in \mathscr{C})(\forall x)(x \doteq_C x_0 \to f(x) \doteq_{C_0} y_0).$$

A representation theorem is available for such a notion of limit, showing that in standard models of Łukasiewicz or product fuzzy logic, if $\mathscr{C}$ is the crisp system of all C satisfying the above-mentioned conditions of antitony and full height and C_0 is an element of $\mathscr{C}$, then $\mathrm{Lim}_{\mathscr{C}}(f, x_0, y_0)$ is fully true in the model iff $\lim_{x\to x_0} f(x) = y_0$. (Observe that under our approach, limits are defined by a Σ_2-formula of fuzzy logic; compare this with the Π_1-formula in nonstandard analysis and the Π_3-formula in classical analysis, both in classical logic.)

The fuzzy infinitesimal calculus follows the method of 'number-free fuzzy mathematics' described in [2], which uses the fact that various properties of real-valued functions are expressible as properties of (fuzzy) sets in standard models of fuzzy logic. Under the number-free account, the relation $\doteq_C$ is a natural generalization of the classical notion of metric. With a suitably chosen system $\mathscr{C}$ (as hinted above), all basic notions of the infinitesimal calculus can be reconstructed, their usual properties proved in higher-order fuzzy logic ŁΠ, and representation theorems obtained that ensure that in certain standard models our notions coincide with the classical ones. Moreover, the fuzzy infinitesimal calculus naturally accommodates such genuinely fuzzy notions as closeness or largeness, which are both natural and important in applications (cf., e.g., the predicate $\gg$ 'much larger' in physics), which in classical analysis have to be neglected or modelled in a rather complicated manner. Technical details of the fuzzy infinitesimal calculus, though omitted here for space reasons, will be presented at the talk and given in detail in the full paper.

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Paraconsistency with Classical Logic beyond STT

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Instead of weakening classical logic, one way to deal with paradoxes while approximating naive comprehension consists in weakening the comprehension schema. In the system I propose for consideration, the latter happens in a somewhat intricate manner so that it is nonetheless a theorem that a set s is a member of the set $\{x : F(x)\}$ if, and only if it is a theorem that $F(s)$. Classical logic is respected in the full sense that all theorems of classical predicate logic remain theorems, and no theorem of the system contradicts classical logic. Importantly, this is so even in the presence of the invoked liberal comprehehension principles. Instead, what has to go is modus ponens in one of its interpretations. This failure of modus ponens, however, is compensated for by the semantical validity of a row of other inference rules, some of which are mentioned below, and other features. The language presupposed is the language of set theory extended with a truth operator T. The semantical justification of the system follows in the tradition of Herzberger and Gupta (who build upon Kripke and Moschovakis and others, of course), though with the important difference that we first of all focus upon sets and set membership (some would, in view of the inherent failure of extensionality, prefer the term "property theory"). It is throughout the semantical process on (given) ordinals presupposed that at an ordinal o the set s is a member of the set $\{x : F(x)\}$ iff $TF(s)$ holds at the ordinal o. Furthermore, $TF(s)$ holds at an ordinal o iff for some ordinal $m < o$, $F(s)$ holds at all ordinals n such that $m = n$ or $m < n < o$. (Notice that this gives the empty extension of T at the ordinal zero.) Given some choice as to how to assign values (here: sets, or closed set-terms) to free variables, this semantical process reaches a closure ordinal (the first stabilizing ordinal) c such that any formula Tp holds at c iff p holds at all ordinals larger than or equal to c. (We require that all variables are assigned to a set-term and that all set terms will be assigned.) At the closure ordinal c, we will, given the assignment of free variables, have delimited a unique maximal consistent set of formulas, i.e. a unique model for a logic of the truth operator T together with the comprehension principle that a set s is a member of the set $\{x : F(x)\}$ iff $TF(s)$ holds (at c). In much of what is stated in the foregoing, we largely follow (though at some important points extend) the considerations by Andrea Cantini in "Logical Frameworks for Truth and Abstraction", Elsevier 1996, §68 and §69 (consider especially Cantini's system ELST).

The next metalogical steps are crucial: (1) Let V be the (maximal consistent) set of formulas that are valid at the closure ordinal. (2) Let I be the set of formulas that are derivable in some (preferably non-trivial) initial attempt to capture as many sentences as possible from V by means of some recursive axiomatization. (3) Let D be the least set of formulas such that: i) I is a subset of D, ii) D is a subset of V, iii) D, as I and V, respects the assignment of set-terms to free variables; iv) D is, as I and V, closed under modus ponens (we presuppose an axiomatization of I which, going back to a strategy suggested by Tarski, does not presuppose generalization as a primitive inference rule), and v) D is, as V, closed under the "rule" that if a sentence $-T-A$ is a member of D then either TA is a member of D or $-TA$ is a member of D (we call this rule "the Bifurcation Rule"). (4) Define L to be the set of formulas A such that $-T-A$ is a member of D.

Write $D(A)(L(A))$ for the statement that A is a member of $D(L)$. Since for any formula A, $D(TA \Rightarrow -T-A)$, and since $D(A)$ only if not $D(-A)$, we will have that $D(TA)$ only if both $L(A)$ and not $L(-A)$. Furthermore, given the Bifurcation Rule, we have that both $L(A)$ and not $L(-A)$ only if $D(TA)$. So: $L(A)$ iff $D(-T-A)$, and both $L(A)$ and not $L(-A)$ iff $D(TA)$. The system L is the centre of our attention. We say that A is a maxim of L iff $L(A)$ and not $L(-A)$. A formula A is a minor of L iff both $L(A)$ and $L(-A)$. As emphasized above, L respects classical logic even though modus ponens fails. A plethora of new and unfamiliar inference rules are induced, however. E.g., if A is a maxim (minor), then so is TA; if TA is a maxim (minor) then so is A; if A is a maxim and $(A \Rightarrow B)$ is a maxim, then so is B; if A is a maxim and $A \Rightarrow B$ is a minor, then B is a minor; If A is a minor and $A \Rightarrow B$ is a maxim, then B is a theorem; if $T(Ex)F(x)$ is a theorem then $(Ex)TF(x)$ is a theorem; if $(Ax)TF(x)$ is a maxim then $T(Ax)F(x)$ is a maxim. The system is of course highly non-constructive and impredicative, so that any given recursive list of axioms and inference rules will be incomplete.

Even though L respects classical logic, paradoxical sentences will be minors. So if e.g. R is Russell's infamous set, it will be a theorem that R is a member of R, and it will be a theorem that R is not a theorem of R. But it will not be a theorem that R is a member of itself and R is not a member of itself. So, L is, if you will, non-adjunctive. As all informed people know, some intuitions will have to be adjusted in order to deal coherently with paradoxes. In L, a large part of the burden is shifted upon how we intuitively are to interpret the connectives.

The system proposed is very flexible. In the talk I want to focus upon how we may use a fixed-point construction to obtain a model (set) M which interprets Bounded ("predicative") Zermelo set theory + Transitive Closure + full Induction (in fact, additional principles may be added to construct analogous sets M'), and in consequence its strength goes beyond that of the Simple Theory of Types (with infinity). Also, if time permits, I would like to relate certain considerations concerning the set HNP of hereditarily non-paradoxical sets (also obtained via a fixed-point construction), which show that although this set is not closed under ordinary power or replacement, it is so closed under analogous rules (here "analogous" is used in two different meanings); this has the consequence that HNP, unlike M, will contain an indefinite amount of (countable) von Neumann ordinals.

Nonstandard Analysis from a Philosophical Point of View

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1. Nonstandard Analysis is develop on the same grounds that standard, real analysis, e.g. classical logic, axioms for the real numbers field, axiom of choice, and it is in this sense a part of classical mathematics. It is its basic structure, i.e. non-Archimedean field of hyperreals, and **techniques** such as transfer principle, S-continuity, hyperfinite sets and Loeb measures that make it non-standard. **Problems** it deals with, those it shares with standard analysis, make it to be still analysis rather than a new branch of mathematics.

In my talk I sketch a project in philosophy of mathematics (see [2], [3],[14]) that is designed to investigate the notions of a mathematical problem and a mathematical technique. It is well-known that *the same* theorem, eg. triangle proportionality theorem, Pythagorean theorem (see [10], VI. 2, VI.8), can be proved with different techniques, eg. theory of proportion (see [10], Book V) or the arithmetic of real numbers (see [4]). However, different techniques usually refer to different mathematical structures. Since there is some common ground between standard and nonstandard analysis, I choose mathematical analysis to develop a notion of a mathematical problem and a mathematical technique. I discuss [8] to present a problem that is not comprehended in any axiomatic reconstruction of mathematics. Next, I present a brand new technique of nonstandard analysis, namely that of hyperfinite sets. Finally, I address the question of a mathematical technique itself.

In the talk basic knowledge of nonstandard analysis is assumed (see point 5 below).

2. *Mathematics over Metaphysics.* In [8] Dedekind introduces "a real definition of the **essence of continuity**" that could form a sufficient basis for "a rigorous exposition of differential calculus". It is also believed that opposed to continuity is discreteness (see [1],[13]). Nowadays Dedekind's continuity is just a characterization of a totally ordered set. In this context, discrete, as opposed to continuous, means discrete order. Next to continuous order, there are other notions of continuity in use in mathematics, e.g. Dedekind complete ordered field or topological field (to mention only those that characterize an algebraic field). Mathematics also provides a more general meaning of discrete: in topology, discrete, as opposed to connected, could be rendered as totally disconnected space.

The field of hyperreals is not Dedekind continuous, and it is also a totally disconnected topological space, so, in a sense, it is a discrete space. In spite of this, within the framework of Nonstandard Analysis basic theorems of standard analysis can be proved. Comparing standard and nonstandard analysis I show that beyond mathematical rules for defining numbers such as Dedekind cut, Cauchy completeness, standard part theorem or hyperfinite sum there is nothing like *the linear continuum.*

3. *Finite-Infinite-Hyperfinite.* In classical mathematics the set of natural numbers $\mathbb{N}$ forms a standard measure of infinity: a set A is finite iff there is a bijection between it and a some natural number n, otherwise it is infinite. This Cantorian approach focuses on the cardinality of a set. However, one can take into account the well-known properties of finite sets, namely: (1) a subset of a finite set is finite, (2) a finite and totally ordered set has a greatest and a least element, (3) if A, B are finite then $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}$. **Hyperfinite sets**, being either finite or denumerable in Cantor's sense, share with standard finite sets (in a sense clarified below) these properties. I present some arguments of Nonstandard Analysis that make use of these properties and their standard analysis counterparts (eg. Riemann integral, Lebesgue measure) that refer to the notion of limit and, in consequence, to the axiom of continuity.

4. *Mathematical techniques over logic.* Philosophically motivated programs to reconstruct analysis on different grounds than those provided by real analysis are based on a tacit assumption that there is some ground structure of analysis, usually called *real numbers* (see [5],[11]). As a result they mimic basic real analysis concepts (eg. ordered field, sequence and limit, continuity of a function) to develop but a new branch of mathematics. Since the field of rationals is a common ground between standard, constructivist (see [5], p. 42) and intuitionist (see [11], p. 16) analysis I present the ordered field of rational numbers just as a mathematical technique rather than a construction.

5. Basic facts and definitions (see [6],[7],[9],[12]). Let $(\mathbb{R}, +, \cdot, 0, 1, <)$ be the field of real numbers, $\mathcal{F}$ – a nonprincipal ultrafilter on $\mathbb{N}$. The relation defined by

$$(r_n) \equiv (s_n) \leftrightarrow_{df} \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$$

is an equivalence relation on the set $\mathbb{R}^{\mathbb{N}}$. The set of hyperreals $\mathbb{R}^*$ is the quotient set $\mathbb{R}^* =_{df} \mathbb{R}^{\mathbb{N}}/_{\equiv}$.

Addition, multiplication and order of hyperreals are defined by

$$[(r_n)] \oplus [(s_n)] =_{df} [(r_n + s_n)], \qquad [(r_n)] \otimes [(s_n)] =_{df} [(r_n \cdot s_n)],$$

$$[(r_n)] \prec [(s_n)] \leftrightarrow_{df} \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{F}.$$

The standard real number r is identified with equivalence class r^* of the constant sequence $(r, r, ...)$, i.e. $r^* =_{df} [(r, r, ...)]$.

Theorem $(\mathbb{R}^*, \oplus, \otimes, 0^*, 1^*, \prec)$ is a non-Archimedean, real closed field.

The set of infinitisimal hyperreals Ω is defined by

$$x \in \Omega \leftrightarrow_{df} \forall \theta \in \mathbb{R}_+[\ |x| \prec \theta^*\ ].$$

We say that x is infinitely close to y, $x \approx y$, iff $x - y \in \Omega$.

The set of limited hyperreals $\mathbb{L}$ is defined by

$$x \in \mathbb{L} \leftrightarrow_{df} \exists \theta \in \mathbb{R}_+[\ |x| \prec \theta^*\ ].$$

Standard Part Theorem: $\forall x \in \mathbb{L} \exists! r \in \mathbb{R}[\ r^* \approx x\ ]$.

The standard part of a limited hyperreal x is denoted by ${}^o x$, i.e. ${}^o x = r$.

The set of hypernaturals $\mathbb{N}^*$ is defined by

$$[(n_j)] \in \mathbb{N}^* \leftrightarrow_{df} \{j \in \mathbb{N} : n_j \in \mathbb{N}\} \in \mathcal{F}.$$

The set of infinite hypernaturals $\mathbb{N}_\infty$ is defined by $\mathbb{N}_\infty =_{df} \mathbb{N}^* \setminus \{n^* : n \in \mathbb{N}\}$.

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of reals. Then an extension of $(s_n)_{n \in \mathbb{N}}$ to a hypersequence $(s^*_K)_{K \in \mathbb{N}^*}$ is defined by

$$s^*_K =_{df} [(s_{k_j})] = [(s_{k_1}, s_{k_2}, ...)], \quad \text{where} \quad K = [(k_j)] = [(k_1, k_2, ...)].$$

Basic Theorem Let (s_n) be a sequence of real numbers, let $a \in \mathbb{R}$. Then

$$\lim_{n \to \infty} s_n = a \leftrightarrow \forall K \in \mathbb{N}_\infty [s^*_K \approx a^*].$$

Let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}$. An internal set $[H_n]$ is a subset of $\mathbb{R}^*$ defined by

$$[(r_n)] \in [H_n] \leftrightarrow_{df} \{n \in \mathbb{N} : r_n \in H_n\} \in \mathcal{F}.$$

When $\{n \in \mathbb{N} :\ H_n \text{ is finite}\} \in \mathcal{F}$, then $[H_n]$ is called hyperfinite. When $H_n = A$, for all n, then the set $[H_n] = [A, A, \dots]$ is denoted by A^*, thus $\mathbb{N}^* = [\mathbb{N}, \mathbb{N}, \dots]$, $\mathbb{Z}^* = [\mathbb{Z}, \mathbb{Z}, \dots]$, and $(a, b)^* = [(a, b), (a, b), \dots]$, for $a, b \in \mathbb{R}$.

The internal cardinality $\big|[H_n]\big|$ of a hyperfinite set $[H_n]$ is a defined by

$$\big|[H_n]\big| =_{df} [\overline{\overline{H_n}}],$$

where $\overline{\overline{H_n}}$ stands for the standard cardinality of H_n.

Theorem: (1) Any internal set is finite or uncountable.
(2) An internal subset of a hyperfinite set is hyperfinite.
(3) Any hyperfinite set has a greatest and a least element.
(4) The union and intersection of any two hyperfinite sets F and G are hyperfinite, with internal cardinality $|F \cup G| = |F| \oplus |G| - |F \cap G|$.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real functions such that $f_n : A_n \mapsto \mathbb{R}$. An internal function $[f_n] : [A_n] \mapsto \mathbb{R}^*$ is defined by

$$[f_n]([(r_n)]) =_{df} [(f_n(r_n))].$$

The **hyperfinite sum** of an internal function $[f_n]$ over a hyperfinite set $[H_n]$ is a hyperreal number defined by

$$\sum_{a\in[H_n]} [f_n](a) =_{df} [(\sum_{a\in H_n} f_n(a))].$$

Let $N \in \mathbb{N}_\infty$, the hyperfinite time line is the hyperfinite set

$$T = \{\frac{k}{N} :\ k \in \mathbb{Z}^*,\ -N^2 \preceq k \preceq N^2\}.$$

Let $\mathcal{A}$ be the set of all internal subsets of T, i.e. $\mathcal{A} = \{A \subset T :\ A \text{ is internal}\}$. $\mathcal{A}$ is an algebra of sets. Let μ be the *counting* measure on $\mathcal{A}$ defined by

$$\mu(A) = \frac{|A|}{N}.$$

A real valued map ${}^o\mu : \mathcal{A} \mapsto [0, \infty]$, defined by

$${}^o\mu(A) = \begin{cases} {}^0(\mu(A)), & \text{if } \mu(A) \text{ is limited} \\ \infty, & \text{otherwise} \end{cases}$$

is additive and for any sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ holds

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A} \to {}^o\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}}{}^o \mu(A_n).$$

Theorem There is a unique extension of ${}^o\mu$ to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. The completion of this measure is the Loab measure μ_L and the completion of $\sigma(\mathcal{A})$ is the Loab σ-algebra $L(\mathcal{A})$.

Theorem Let $\mathcal{B}$ be the set $\mathcal{B} = \{B \subset \mathbb{R} : \ st^{-1}(B) \in L(\mathcal{A})\}$, where $st^{-1}(B) = \{t \in T : \ {}^o t \in B\}$. Then a measure λ on $\mathcal{B}$, defined by

$$\lambda(B) = \mu_L(st^{-1}(B)),$$

is the Lebesgue measure.
Corollary For any $a, b \in \mathbb{R}$, with $a < b$,

$$\mu_L(\{t \in T : \ a^* \prec t \prec b^*\}) =^o \mu\Big(\frac{|T \cap (a,b)^*|}{N}\Big) = b - a.$$

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Fuzzy Class Theory: A State of the Art

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It is indisputable that mathematical structures arising around vague/fuzzy/non-bivalent concepts have a broad range of applications; therefore they have been intensively investigated during the last four decades. The discipline studying these structures is, maybe unfortunately, called *Fuzzy Mathematics*.

There is an ongoing project of the Prague research group in fuzzy logic, directed towards developing the *logic-based* fuzzy mathematics, i.e., an 'alternative' mathematics built in a formal analogy with classical mathematics, but using a suitable formal fuzzy logic instead of the classical logic. First steps in the development thereof were enabled by recent results in Mathematical Fuzzy Logic, especially by the emergence of Henkin-style higher-order fuzzy logics, studied by Libor Běhounek and the present author, see [6]. This approach leads not only to an axiomatization, but also to a systematic study utilizing proof-theoretic and model-theoretic methods. Moreover, the unified formalism allows an interconnection of particular disciplines of fuzzy mathematics and provides the formal foundations of (part of) fuzzy mathematics.

The core of the project is a formulation of certain formalistic methodology (see [7]), proposing the foundational theory (see [6]), and studying the particular disciplines (see the list below) of fuzzy mathematics within this theory using our methodology. The proposed foundational theory is called Fuzzy Class Theory (FCT) and it is a first-order theory over multi-sorted predicate fuzzy logic, with a very natural axiomatic system which approximates nicely Zadeh's original notion of fuzzy set [19].

The papers written within the project so far can be divided into several groups (for more comprehensive list of papers together with their preprints and more details about the project in general see its webpage `www.cs.cas.cz/hp`):

- Methodological issues: [3, 7, 9]
- Formalism of FCT: [6, 14] and freely available primer [8]
- Fuzzy relations: [4, 10]
- Fuzzy topology: [11, 12, 13]
- Fuzzy filters and measures: [17, 18]
- Fuzzy algebra and (interval) analysis: [1, 2, 5, 16]

In this talk we survey the basic logical prerequisites, formulate the methodological standpoint, put it in the context of other nonclassical-logic-based mathematics (intuitionistic, relevant, substructural, etc.), sketch the formalism of FCT and illustrate it using simple examples from the theory of fuzzy relations.

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Towards Metamathematics of Weak Arithmetics over Fuzzy Logic

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This abstract describes continuing investigation of a very weak arithmetic FQ^- that results from the well-known Robinson arithmetic Q by not assuming that addition and multiplication are total functions (the axiom system Q^-) and, secondly, weakening the classical logic to the basic mathematical fuzzy logic $BL\forall$ (or to the monoidal t-norm logic $MTL\forall$). This investigation was started in my paper Mathematical fuzzy logic and natural numbers (Fundamenta Informaticae 81 (2007) 155-163) where the first Gödel incompleteness of FQ^- (i.e. essential incompleteness) is proved. Here we first discuss Q^- over the Gödel fuzzy logic $G\forall$, or alternatively over the intuitionistic predicate logic, showing essential incompleteness and essential undecidability; then we prove essential undecidability of FQ^- (correcting an error in the mentioned paper), show a variant of the second Gödel incompleteness theorem for an extension of FQ^- with the Δ-connective and present a model of the last theory which is fuzzy (non-crisp), has commutative addition and multiplication and non-associative addition. The abstract is an excerption from a paper of mine named "Towards metamathematics of weak arithmetics over fuzzy logic" and submitted for publication.

The theory FQ^- over BL$\forall$ (with crisp equality) has the following language: the unary function symbol S, the constant constant $\bar{0}$ and ternary predicates A, M s (addition, multiplication). The axioms are:

(Q1) $S(x) \neq \bar{0}$
(Q2) $S(x) = S(y) \to x = y$
(Q3) $x \neq \bar{0} \to (\exists y)(x = S(y))$
(Q4) $A(x, \bar{0}, y) \equiv x = y$
(Q5) $A(x, S(y), z) \equiv (\exists u)(A(x, y, u) \,\&\, z = S(u))$
(Q6) $M(x, \bar{0}, y) \equiv y = \bar{0}$
(Q7) $M(x, S(y), z) \equiv (\exists u)(M(x, y, u) \,\&\, A(u, x, z))$
(Q8) $x \leq y \equiv (\exists z)A(z, x, y)$.

Numerals are defined as usual: $\bar{m} = \underbrace{S \ldots S(\bar{0})}_{m}$.

Alternatively we may study the theory FQ^- over other fuzzy predicate logics as Łukasiewicz logic Ł$\forall$, Gödel logic G$\forall$, product logic $\Pi\forall$ etc.

The theory FQ^-_Δ over the logic $BL\forall_\Delta$ is the extension of FQ^- by the unary connective Δ (Baaz's Delta) The theory FQ^-_Δ extends FQ^- by the usual axoms for Δ plus the following two:

(A) $(\Delta A(x, y, z_1) \,\&\, \Delta A(x, y, z_2)) \to z_1 = z_2$,
(M) $(\Delta M(x, y, z_1) \,\&\, \Delta M(x, y, z_2)) \to z_1 = z_2$.

Let us distinguish two notions of a complete theory (over some logic): T is *c-complete* (classically complete) if for each sentence φ, T proves φ or T proves $\neg\varphi$. And T is l-complete (or linear) if for each pair φ, ψ of sentences T proves $\varphi \to \psi$ or T proves $\psi \to \varphi$.

Fact. Let T be a theory over $G\forall$ (or over $Intc\forall$). If T is consistent then there is a $T' \supseteq T$ which is consistent and c-complete; moreover, if T is decidable then T' is decidable. Consequently, if

T is a consistent theory over $G\forall$ (or over $Intc\forall$) then it is consistent as a theory over Boolean logic.

Theorem. Q^- over Gödel logic (or over intuitionistic logic) is essentially c-incomplete and essentially undecidable: each consistent axiomatized extension of it is both c-incomplete and undecidable.

The classical deduction theorem implies that an extension of a decidable theory by a new axiom is also decidable. But I have shown that the last statement is not true for theories over several fuzzy logics, among them $BL\forall$. We can prove the essential undecidability of FQ^- (i.e. Q^- over the fuzzy logic $BL\forall$) by analyzing the proof of Theorem 9 in the famous monograph by Tarski, Mostowski and Robinson. Needless to say, that book deals with theories over classical logic; thus our analysis just checks if or how an analogical proof for theories over fuzzy logic can be made.

A set $X \subseteq N$ is definable in a theory T (over $BL\forall$, as FQ^-) by a formula $\Psi(x)$ if for each $n \in N$, $n \in X$ implies $T \vdash \Psi(\bar{n})$ and $n \notin X$ implies $T \vdash \neg\Psi(\bar{n})$. A function $F : N \to N$ is definable in T by a formula $\Phi(u, v)$ if for each $n \in N$, $T \vdash \Phi(\bar{n}, v) \equiv v = \overline{F(n)}$.

Lemma.

(1) Every recursive function (of one argument) is definable in FQ^- and in each its extension.

(2) Let T be a theory over $BL\forall$ extending FQ^-. Let D be the function satisfying $\varphi_n(\bar{n}) = \varphi_{D(n)}$ for each n, let V be the set of all formulas provable in T. If both D and V are definable then T is contradictory.

Theorem 1. (Cf. Tarski-Mostowski-Robinson Corollary 2.) FQ^- is essentially undecidable.

Remarks.

(1) There is a third notion of completeness of a theory: a consistent theory (over $BL\forall$) is maximal (or m-complete, if you prefer) if for each closed φ, $T \not\vdash \varphi$ implies that $T \cup \{\varphi\}$ is inconsistent, thus for some m, T proves $\neg(\varphi^m)$. (In classical logic this is clearly equivalent to completeness of T.) Over $BL\forall$ maximality proves l-completeness and clearly each axiomatizable maximal theory is decidable, thus from essential undecidability we get a "non-maximality theorem": No axiomatizable extension of FQ^- is maximal. But we do have the first incompleteness theorem of FQ^-, as proven in my paper mentioned above, which also implies the non-maximality theorem.

(2) The monoidal t-norm based logic $MTL\forall$ introduced by Esteva and Godo is weaker (more general) than $BL\forall$. The reader may check that all the results of the present paper hold true for Q^- and Q^-_Δ as theories over $MTL\forall$.

Lemma. The classical Q^- is interpretable in the $BL\forall_\Delta$-theory FQ^-_Δ.

Theorem 2. The classical theory $I\Sigma_0 + exp$ does not prove the consistency of our fuzzy theory FQ^-_Δ. (Second Gödel incompleteness.)

Finally let us mention that there is a model of FQ^-_Δ with commutative (but non-associative) operations which is fuzzy, i.e. the predicates A and M are interpreted fuzzily. Details are in my submitted paper.

A ZF-like Theory in Fuzzy Logic

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This contribution is based on [HH03]. We develop a formal axiomatic theory FST ('fuzzy set theory') in the language and style of ZF within the logic BL$\forall$ (introduced by Hájek in [Háj98]) expanded with the Δ connective (introduced by Baaz in [Baa96]). We learn from, and take the approach of, the papers [Kla67], [Got84], [Pow75], [Gra79], [TT84], [TT92], [Shi99]. Our theory is governed by a non-classical logic and generates a cumulative universe of sets, unlike the more usual theory of fuzzy sets, introduced by Zadeh in [Zad65], where fuzzy sets are identified with real-valued functions on a fixed domain, hence the universe of sets is flat, and the theory is based on classical logic.

In constructing the theory, we strive to make its axioms as strong as possible, but to avoid strengthening the underlying logic. Here we are compelled to make several important choices, such as limiting ourselves to a crisp (two-valued) equality, or choosing carefully between several classically equivalent versions of ZF-axioms.

We show that FST is distinct from ZF or its classical fragments by constructing a BLΔ-valued universe over an arbitrary complete linearly ordered BLΔ-algebra in which all axioms of FST are valid.

Finally, we show relative consistency of ZF w. r. t. FST by exhibiting its inner model, consisting of hereditarily crisp sets.

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Functional Algebraic Models for Intuitionistic Arithmetic: Two Examples

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In one of his works (see [1] or [2]) A. Dragalin suggested a new general approach for description of very different kinds of models of intuitionistic theories. The main A. Dragalin's idea is using of FUNCTIONAL pseudobulean algebra (FPBA) for evaluations of not only closed formulas but also for any formulas. Such FPBA is not full but only must be closed under upper and lower bounds. There are many models for intuitionistic theories (firstly for intuitionistic arithmetic HA) which can be presented as a functional algebraic model (FAM) (for different languages, not only HA-language) and use in its definition FPBA-construction. We notice that algebraic model for some language Ω with complete pseudobulean algebra (PBA) as its algebra of truth values can be thought as a special case of FPBA (it is all functions).

Theorem (A. Dragalin)
If A is FAM for Ω and formula φ is derivable in HPC, then $\| \varphi \|=1$.

In [2] A. Dragalin gave many examples of arithmetic models of realizability type as FAM. The first original Kleene's realizability of 1945 (as formal analog and non-formal analog). Then as noted above any model with complete PBA can be presented as suitable FAM. The Lifschitz's realizability for proving HA+CT! $\nvdash$ CT. The realizability which was used by M. Beeson for proving nonderivability of theorems on the continuity of effective operators (fp-realizability). Every formal theory (for example, HA) can be considered as FAM. This is well known Lindenbaum-Tarski algebra. Formal version of stroke(slash)-realizability of Kleene (A. Dragalin, in [2]).

In my topic I would like to speak about:
1. the special realizability (see [3] for definition and [5] for proving), which use for proof of interpretation HA+CT+P in HA, can be presented as FAM.
2. a semantical version (original version of Kleene, see [4]) of stroke (slash)-realizability of Kleene cannot be presented as FAM (see [6]).

And finally, I would like to present the "lifting" of A. Dragalin's construction to intuitionistic set theories.

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Fundamental Results for Pointfree Convex Geometry

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Inspired by locale theory or pointfree topology (see [2, 3]), we propose "pointfree convex geometry". Pointfree convex geometry is a theory of convexity structures which does not presuppose the notion of "point" and is primarily based on that of "region". It studies the lattice structures of convex sets in a purely algebraic way.

The following are fundamental results in locale theory (see [2]): (i) there is a dual adjunction between the category of frames and the category of topological spaces; (ii) there is a dual equivalence between the category of spatial frames and the category of sober topological spaces. In this talk, results corresponding to (i) and (ii) are shown for pointfree convex geometry.

Along with topology, convex geometry is an important branch of mathematics (see [1, 4]). The notion of topological space in topology corresponds to that of convexity space in convex geometry, which is defined as follows (see [4]). For a set S and a subset $\mathcal{C}$ of the power set of S, $(S, \mathcal{C})$ is called a convexity space iff it satisfies the following: (a) $\emptyset, S \in \mathcal{C}$; (b) $\mathcal{C}$ is closed under arbitrary intersections; (c) if $\{X_i \in \mathcal{C} \,;\, i \in I\}$ is totally ordered, then $\bigcup\{X_i \,;\, i \in I\} \in \mathcal{C}$. An element of $\mathcal{C}$ is called a convex set in a convexity space $(S, \mathcal{C})$.

As a pointfree version of convexity space, we introduce the notion of convexity algebra, which corresponds to that of frame in locale theory and is defined as follows. A poset L is called a convexity algebra iff it satisfies the following: (a) L has the top element 1 and the bottom element 0; (b) L has arbitrary meets; (c) if $\{x_i \in L \,;\, i \in I\}$ is totally ordered, then $\{x_i \,;\, i \in I\}$ has a join in L; (d) arbitrary joins of totally ordered subsets of L distribute over arbitrary meets.

By introducing the concepts of spatiality and sobriety, we show the following theorems corresponding to (i) and (ii) above: (1) there is a dual adjunction between the category of convexity algebras and the category of convexity spaces; (2) there is a dual equivalence between the category of spatial convexity algebras and the category of sober convexity spaces. In particular, the duality in (2) provides: (2.1) a representation of a spatial convexity algebra by the algebra of convex sets in the "spectrum" of the convexity algebra; (2.2) a representation of a sober convexity space by the "spectrum" of the algebra of convex sets in the convexity space. As (i) and (ii) are for locale theory, (1) and (2) are considered as fundamental results for pointfree convex geometry.

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Games for First Order Kripke Models

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The possible-world interpretation on one hand, and strong connections with classical model theory on the other, make Kripke semantics an attractive and powerful tool in semantical investigations of constructive first order theories. Many important problems, such as independence or non-provability, can be successfully tackled with use of Kripke models, and there are many results of this kind concerning intuitionistic first order arithmetic and intuitionistic set theory, to mention only these two prominent examples. However, in contrast to classical model theory, the general theory of Kripke models is still not well developed. This fact motivates research in the theory of Kripke models viewed as a counterpart of classical model theory. The aim of my talk is to present some recent general results in this field.

The question whether given two structures validate the same formulae plays a fundamental role in semantical investigations of logical systems. So, the problem of finding a suitable condition for logical equivalence in terms of the algebraic or set-theoretic properties of the structures in question appears in a natural way. We tackle this problem in case of Kripke semantics for first order intuitionistic theories and ask: when a first-order formula of a particular complexity is valid in a given Kripke model? The concept that we introduce to answer the above question originates in the well-known back-and-forth technique. This idea is being widely applied in modern model theory, e.g., it is a foundation of Ehrenfeucht-Fraïssé games that find many applications in finite model theory and beyond. The back-and-forth technique can also be viewed as a foundation of the notion of bisimulation, the idea which turned out to be a very powerful tool in investigations of process algebras, context-free grammars, and Kripke semantics of modal and intermediate logics. We present our main result in terms of games which are played on Kripke models. The corresponding notion of bounded bisimulation was introduced in [1].

In order to sketch our main result, let us fix some notation and terminology. Let $\mathcal{K}$ and $\mathcal{M}$ be Kripke models whose nodes and the corresponding worlds are denoted as α, β and $\mathcal{K}(\alpha)$, $\mathcal{M}(\beta)$ respectively. A pointed model $(\mathcal{K},\alpha)$ is a model $\mathcal{K}$ with the distinguished node α. We say that a world $\mathcal{K}(\alpha')$ is accessible for $\mathcal{K}(\alpha)$ if the node α' is accessible for α.

Let $(\mathcal{K},\alpha)$ and $(\mathcal{M},\beta)$ be two pointed Kripke models and let π be an injective map from $\mathcal{K}(\alpha)$ to $\mathcal{M}(\beta)$. Moreover, let p, q, and r be non-negative integers. Below we describe the notion of game of length (p,q,r) on the pointed Kripke models $(\mathcal{K},\alpha)$ and $(\mathcal{M},\beta)$ in the initial position π. As usual, the game is played by two players, called Spoiler and Duplicator, that play in turns. There are three kinds of moves that can be played by the players; the three parameters p, q and r are related to these three kinds of moves. In each of his moves Spoiler chooses a Kripke model and: either a world, or an element in a world, or a world and an element in it. Duplicator, in turn, must respond with an appropriate choice in the other Kripke model. The choice of worlds is limited to those that are accessible to the worlds chosen at the previous step. The actual configuration of the game is determined by the triple of the parameters (i,j,k) describing the number of moves of particular kinds that have been already played, a pair of nodes α' and β' of the models $\mathcal{K}$ and $\mathcal{M}$ respectively and a map σ from $\mathcal{K}(\alpha')$ to $\mathcal{M}(\beta')$. The map σ can be viewed as an extension of the initial map π. The game ends if $i=p$, $j=q$ and $k=r$. So, in the course of the play each player has to take $p+q+r$ moves, where each of the numbers p, q and r corresponds to a particular kind of rules that were played during the game. Now let α', β' and σ determine the final configuration of the game. We say that Spoiler wins the game if σ is a partial isomorphism between the classical structures $\mathcal{K}(\alpha')$ and $\mathcal{M}(\beta')$.

In the simplest case, when the initial position π is the empty map, our main result can be stated as follows. If there is a winning strategy for Duplicator in any game on the pointed models $(\mathcal{K}, \alpha)$ and $(\mathcal{M}, \beta)$ of length (p, q, r), then the models $(\mathcal{K}, \alpha)$ and $(\mathcal{M}, \beta)$ validate the same sentences whose number of nested implications, universal quantifiers and existential quantifiers is p, q and r respectively.

Obviously, the notion of Kripke model can be viewed as a generalization of that of classical first order structure, since every classical structure can be presented as a Kripke model over the one-node frame. In this particular case of one-node Kripke models, the notion of game presented above coincides with that of Ehrenfeucht-Fraïssé game. So, our result can be viewed as a Kripke model analogue of the well-known Ehrenfeucht-Fraïssé Theorem concerning logical equivalence of classical first-order structures.

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Powers of Positive Elements in Constructive C^*-algebras

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1 Introduction

In [9], Ogasawara has shown that the following theorems; Let $\mathfrak{A}$ be a C^*-algebra. Then for any positive elements x, y,

1. if $0 \leq x \leq y$ then $0 \leq x^{1/2} \leq y^{1/2}$
2. if $0 \leq x \leq y$ implies always $0 \leq x^2 \leq y^2$ then $\mathfrak{A}$ is commutative.

In [10], these statements are generalised, however both proofs proceeds based on the term of the spectrum of an element. The spectrum of an element is difficult to treat in Bishop style constructive mathmatics, indeed [5], there are Brouwerian examples that we cannot establish elementary properties of spectra which are obtained in classical mathematics. Hence, these proofs are not acceptable in **BISH**. In this note, we will show that these statement has constructive proofs in **BISH**.

Constructive C^*-algebras An *involution* on an algebra $\mathfrak{A}$ over a field $\mathbb{K}$ is a map $^* : \mathfrak{A} \to \mathfrak{A}$ such that for all $x, y \in \mathfrak{A}$ and $a, b \in \mathbb{K}$,

$$(xy)^* = y^*x^*, (x^*)^* = x, (ax + by)^* = \overline{a}x^* + \overline{b}y^*,$$

where $\overline{a}, \overline{b}$ are conjugates of a and b.

A $*$-*algebra* over a field $\mathbb{K}$ is an algebra with an involution. Let $\mathfrak{A}$ be an algebra over a complex field $\mathbb{C}$ with unit $\mathbf{e}$. An algebra $\mathfrak{A}$ is called a *Banach algebra* if $\mathfrak{A}$ is a Banach space with $||\mathbf{e}|| = 1$, and $||xy|| \leq ||x||||y||$ for all x, y in $\mathfrak{A}$.

Note that by definition a Banach space in constructive mathematics is separable, hence a Banach algebra is separable.

Definition 1 (Constructive C^*-algebra)**.** A Banach $*$-algebra $\mathfrak{A}$ is called a C^*-*algebra* if $||x^*x|| = ||x||^2$ for all x in $\mathfrak{A}$.

If $\mathcal{H}$ is a Hilbert space, we denote by $\mathfrak{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be *compact* if $\{T(x) : ||x|| \leq 1\}$ is totally bounded. Every compact operator is normable and has its adjoint. The set of all compact operators on a Hilbert space is an example of constructive C^*-algebra.

Definition 2 (Concrete C^*-algebra)**.** A self-adjoint $*$-subalgebra $\mathfrak{R}$ of normable elements of $\mathfrak{B}(\mathcal{H})$ is called *concrete* C^*-*algebra* if it is complete and separable with respect to the norm.

Classically, the set $\mathfrak{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ is an example of C^*-algebra. However, $\mathfrak{B}(\mathcal{H})$ is not an example of constructive C^*-algebra, there is a Brouwerian example [8] shows that every bounded linear operator in $\mathfrak{B}(\mathcal{H})$ is not normable.

Positive elements An element x in a C^*-algebra $\mathfrak{A}$ is *normal* if $x^*x = xx^*$ and is *self-adjoint* if $x = x^*$ holds. Trivially, any self-adjoint element is a normal element: for any x in $\mathfrak{A}$ there exist self-adjoint elements x_1, x_2 such that $x = x_1 + \imath x_2$, where $x_1 = (x + x^*)/2, x_2 = (x^* - x)/2\imath$. Let $\mathfrak{A}$ be a C^*-algebra. For each $x \in \mathfrak{A}$, the set $Sp(x) = \{\lambda \in \mathbb{C} : (x - \lambda\mathbf{e})^{-1}$ does not exist $\}$ is called the *spectrum of* x. Classically, an element x in a C^*-algebra $\mathfrak{A}$ is *positive* if it satisfies x is normal and $Sp(x) \subset [0, \infty)$. However, the spectrum of an element is not easy to treat constructively. In [5], there are Brouwerian examples that we cannot establish elementary properties of spectra which are obtained in classical mathematics. Hence, we need to modify the definition of a positive element in constructive C^*-algebra.

The spectrum Σ of a commutative Banach algebra $\mathfrak{A}$ consists of all nonzero bounded multiplicative linear functionals. Each element in the spectrum Σ of $\mathfrak{A}$ is called a *character*. With regard to commutative constructive C^*-algebras, the Gelfand representation theorem states that every commutative C^*-algebra is isomorphic to the space of continuous functions on its spectrum.

Note that the spectrum of a Banach algebra is compact classically. Constructively, this is not true for Banach algebras [4], but it is true for C^*-algebras.

Theorem 1 (Gelfand representation theorem [11])**.** *Let $\mathfrak{A}$ be a commutative C^*-algebra and let $\mathsf{C}(\Sigma)$ be the set of all complex valued continuous functions on the spectrum Σ of $\mathfrak{A}$. Then there exists a norm preserving $*$-isomorphism from $\mathfrak{A}$ onto $\mathsf{C}(\Sigma)$.*

Let $\mathfrak{A}$ be a C^*-algebra and x a normal element in $\mathfrak{A}$. Then $[\mathbf{e}, x]$ denotes the commutative C^*-subalgebra of $\mathfrak{A}$ generated by $\mathbf{e}$ and x and $\mathsf{C}(\Sigma)$ denotes the set of all complex valued continuous functions on the spectrum Σ of $[\mathbf{e}, x]$. For a norm preserving $*$-isomorphism ϕ from $[\mathbf{e}, x]$ onto $\mathsf{C}(\Sigma)$ (the existence of such ϕ is assured by Theorem 1), we say that $\phi(x)$ is a *name* of x. A name $\phi(x)$ of x is nonnegative if it satisfies $\phi(x)u \geq 0$ for all $u \in \Sigma$.

Definition 3 (Positive element)**.** An element x in a C^*-algebra $\mathfrak{A}$ is called *positive* if $\phi(x)$ is nonnegative.

Since $[\mathbf{e}, x]$ is defined for a normal element x, and the name $\phi(x)$ of x is nonnegative, it is easy to see that a positive element is self-adjoint.

Let x, y be self-adjoint elements, then we write $x \geq y$ for $x - y$ is a positive.

It is easy to see that the following results holds constructively and the proofs can be found of [12].

Lemma 2. *Let x be a self-adjoint element in a C^*-algebra $\mathfrak{A}$. Then there exist unique positive elements x_+ and x_- such that $x = x_+ - x_-$, $x_+x_- = 0$ and $||x|| = \max\{||x_+||, ||x_-||\}$.*

Theorem 3. *The following conditions are equivalent constructively.*
(1) x is a positive element in a C^*-algebra $\mathfrak{A}$.
(2) There is a self-adjoint element y in $\mathfrak{A}$ such that $x = y^2$.
(3) There exists an element y in $\mathfrak{A}$ such that $x = y^*y$.
(4) x is self-adjoint and $||a\mathbf{e} - x|| \leq a$ for any $a \geq ||x||$.
(5) x is self-adjoint and $||a\mathbf{e} - x|| \leq a$ for some $a \geq ||x||$.

2 Powers of positive elements in constructive C*-algebras

First we need modify the property operator monotonicity, which is classically defined the term of spectrum [10]. So we define operator monotonicity without using the term of the spectrum of an element.

Definition 4 (operator monotone)**.** An operator f on an intervel in $[a, b] \subseteq \mathbb{R}$ is called operator monotone (increasing) if changing the domain $[a, b]$ of f to $\mathfrak{A}$, then f can also be considered an operator on $\mathfrak{A}$ with $x \leq y \Longrightarrow f(x) \leq f(y)$, whenever $[-||y||, ||y||] \subseteq [a, b]$ in $\mathbb{R}$

For any $\alpha > 0$, we define the function f_α on $(-1/\alpha, \infty)$ by $f_\alpha(t) = (1 + \alpha t)^{-1}t$. It is easy to see that f_α is an example of an operator monotone increasing on $(-1/\alpha, \infty)$.

Proposition 4. *If $0 < \beta \leq 1$ the function $t \to t^\beta$ is operator monoton increasing on $\mathbb{R}^+$.*

This Proposition follows the theorem of the first case.

Theorem 5. *Let $\mathfrak{A}$ be a C^*-algebra. If $0 \leq x \leq y$ then $0 \leq x^\alpha \leq y^\alpha$ for any α with $0 < \alpha \leq 1$.*

Next, we will consider the second case. Remark that if α preserves order then so does α^n. Then using the previous theorem, wecan take arbitrary $r \in \mathbb{R}$ as the exponents. Therefore it suffices to prove the theorem with $\alpha = 2$. Combine the facts that for any elements in $\mathfrak{A}$ can be expressed by self-adjoint elements and the Lemma 2, we obtain the second result.

Theorem 6. *Let $\mathfrak{A}$ be a C^*-algebra. If $0 \leq x \leq y$ implies always $0 \leq x^\alpha \leq y^\alpha$, for some $\alpha > 1$ then $\mathfrak{A}$ is commutative.*

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Plurals and the Cumulative Set Hierarchy

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This paper is a step towards providing an interpretation of the cumulative hierarchy of sets which makes no reference to abstract objects, and instead, roughly speaking, uses phrases like 'it is possible to introduce a name token such that..'. I start with a modal interpretation of plural quantification and extend the strategy to another theory, a theory of the so-called cumulative naming structures. This gets us closer to ZF but the axiom of the power set and the axiom of extensionality don't hold. I hint at a modification of this theory that does validate all the axioms of ZF in a fairly intuitive interpretation of the language of ZF, but is essentially a theory of ways name tokens could be.

Let us start with the question of ontological commitment of plural quantification. First, I introduce the language of Quantified Name Logic (QNL) and provide it with a set-theoretic semantics. The **language of QNL** is generated by the alphabet containing brackets, **name variables**: a, b, c, d, …, the copula ε, the existential quantifier $(\exists a)$ (the universal quantifier (a) has its usual definition), and two Boolean connectives: negation $\backsim$, and conjunction $\&$. The set of **well-formed formulas** of QNL is the least set satisfying the following conditions (I freely move to infix notation):

(i) If a_1 and a_2 are name variables, $\varepsilon(a_1,a_2)$ is a well-formed formula,

(ii) If A_1 and A_2 are well-formed formulas and a is a name variable, also $\backsim(A_1)$, $\&(A_1,A_2)$ and $(\exists a)(A_1)$ are well-formed formulas.

Quite an **imparsimonious** but a fairly **standard semantics** for QNL is given as follows (QNL is pretty much a variant of Boolos' logic of plurals - the expressive power of both languages, *modulo* set-theoretic semantics, is the same). Take the domain to be a set of objects and take the range of name variables to be the power set of the domain. An **S-model** of QNL is a pair $<D, I>$ such that D is an arbitrary set and I is a total function which maps name variables into the power set of D (i.e. to each name variable it assigns a subset of the domain). Neither D nor $I(a)$ for any a has to be non-empty. **Truth in an S-model** is defined by the following conditions:

- $<D, I>$ models $\varepsilon(a,b)$ iff $I(a)$ is a singleton and $I(a)$ is a subset of $I(b)$.
- Phrases for negation and conjunction are standard: a model models a negation iff it doesn't model the negated formula, a model models a conjunction iff it models both conjuncts.
- $<D, I>$ models $(\exists a)A$ iff $<D, I^a>$ models A for some I^a which differs from I at most at a.

A sentence is **S**-valid iff it is true in any **S**-model.

One of the standard objections against nominalistic acceptability of the logic of plurals is that it needs a formal semantics, the set-theoretic semantics commits the pluralist to sets, and the substitutional interpretation of plural quantification does not provide the language with the required expressive power (we "run out of tokens", if they're supposed to be finite strings over a finite alphabet). In order to provide an answer to that objection, I give a semi-substitutional semantics which avoids the objections usually raised against the substitutional interpretation of plural quantifiers.

I develop a **Kripke semantics for QNL**. It is a modal interpretation, where the plural quantifier '$(\exists a)$' (suppose A does not contain free variables other than a) is intuitively read as 'it is possible to introduce a name a, which would make A substitutionally true' (the semantics is different from that of Chihara). QNL with Kripke semantics has the same expressive power as QNL with set-theoretic semantics.

A naming structure is a tuple $<I, W>$ where I is a set (of bare individuals) and W is a set of possible worlds. A **possible world** is a tuple $<N, d>$ where I and N are disjoint sets and d is a subset of the Cartesian product of N and I. A **bare world** is the possible world where N is the empty set. The following conditions all have to be satisfied:

- $B=<\emptyset, \emptyset>$ belongs to W (i.e. the naming structure contains the bare world).

- For any w in <N,d> different from B, N is non-empty and countable.
- The accessibility relation on possible worlds is defined by the following condition. Let w=<N, d>, w'=<N', d'>. Rww' if and only if both: (i) N is a proper subset of N', (ii) the restriction of d' to N (i.e. the set of those d'-related pairs whose first elements belong to N) is d.

Let <N, d>=w belong to W. A naming structure **M**=<I, W> is **w-complete** if and only if for any subset A of N there exists a w'=<N',d'> in M such that Rww' and there is an x in N' such that for any y in I, d'(x,y) if and only if y belongs to A. **M** is **complete** iff for any w in W, **M** is w-complete.

An **M**-interpretation is a triple <M, w, v>, where **M** is a naming structure, w=<N, d> is a possible world in **M** and v either assigns to every variable in QNL an element of N, if N is non-empty, or is the empty function on the set of variables of QNL otherwise. If **M** is a complete naming structure, then we say that this **M**-interpretation is complete.

Let <M, w, v> be an M-interpretation, w=<N, d>. Also, let a and b be QNL-variables and A and B be QNL-formulas.

- <M, w, v> models aεb iff v(a) and v(b) are defined and there exists a unique x in I such that <v(a),x> is in d and there is a y in I such that both <v(a),y> and <v(b),y> are in d.
- The clauses for negation and conjunction are fairly standard. The model models a negation of a formula iff v isn't the empty function and it doesn't model the negated formula; and it models a conjunction iff it models both conjuncts.
- <M, w, v> models (∃a)A iff for some w' in M, Rww' and <M, w', v'> models A, where v' differs from v at most in what it assigns to a.

A sentence is **true in a naming structure M** if and only if it is satisfied in its bare world under any valuation. A sentence is valid if and only if it is true in any naming structure. A sentence is complete-valid if it is true in any complete naming structure. It turns out that this semantics is in a sense equivalent to set-theoretic semantics: for any QNL sentence A, A is **S**-valid if and only if A is complete-valid.

Next, let's take a look at the modal factor involved in this semantics by comparing it to a certain two-sorted first-order **modal logic of naming (MLN)**. The **language** of MLN contains two sorts of variables: individual variables $x, y, z, x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots$ and name variables $n, m, o, n_1, n_2, \ldots, m_1, m_2, \ldots, o_1, o_2, \ldots$ Besides, it contains quantifiers ranging over objects of those two sorts, the classical propositional connectives, two modal operators (say, M for possibility and L for necessity), no predicate variables, the identity symbol and one two-place predicate constant D. Formation rules are standard (the only new thing is that *D* takes name variables as first arguments and individual variables as second arguments). **Models of MLN** are just naming structures. An **MLN-interpretation** is a tuple <M, w, i>, where M=<I, W> is a naming structure, <N, d>=w is in M and i is (a) undefined if N is empty, and (b) maps all individual variables into I and all name variables into N otherwise. **Satisfaction of MLN-formulas** in interpretations is defined as follows:

- <M, w, i> models D (n,x) iff i is defined and <i(n),i(x)> is in d.
- <M, w, i> models $t_1=t_2$ iff $i(t_1)=i(t_2)$, where each t_i is one of the variables (arguments of the identity symbol don't have to be of the same sort).
- The clauses for negation and conjunction are standard.
- <M, w, i> models (∃t)A iff there is an interpretation i' (mapping individual variables into I and name variables into N) that differs from i only in what it assigns to t and <M, w, i'> models A.
- <M, w, i> models M(A) iff there is a w' such that Rww' and <M, w', i> models A.

An MLN-sentence is true in a naming structure if it is satisfied in the bare world that underlies it.

Intuitively, we read 'M(A)' as 'there is a way names could be such that A' and 'D(n,x)' as 'x is one of the objects denoted by n' or '*n* refers to x' (where it is not assumed that names do not have to refer uniquely).

Clearly, there is a translation from QNL into MLN, and QNL with Kripke semantics can be embedded in the language of MLN. Since this embedding preserves models (i.e. models for QNL and

MLN are the same, what changes is just the interpretation of symbols), it seems that the ontological commitment of QNL with Kripke semantics does not go beyond the ontological commitment of first-order (two-sorted) modal logic (with one relation constant). Since it is much less plausible that first-order modal logic commits one to abstract objects than that plural quantification does, this strengthen the case for the ontological innocence of QNL.

The strategy can be extended to provide an account of a cumulative hierarchy of names. A **cumulative naming structure** is a tuple <I, W>, where I (also denoted by 'N_0') is a set of bare individuals and W is a set of cumulative possible worlds. A **cumulative possible world** (c.p.w., for short) is a tuple <d, (N_{i+})>, where (N_{i+}) is a denumerable family of sets of names indexed with positive natural numbers, d is the subset of the Cartesian product of the union of all N_i's and the union of I with the union of all N_i's), and the following conditions are satisfied (let $i \in N_+$):

$$(\forall i)[\{y|(\exists x \in N_i)\langle x,y\rangle \in d\} \subseteq I \cup \bigcup_{k<i} N_k]$$
$$(\forall i > 1)[x \in N_i \rightarrow (\exists y \in N_{i-1})\langle x,y\rangle \in d]$$
$$B = \langle d,(N_i)\rangle \in W, \text{where } (\forall i)N_i = \emptyset$$
$$\text{For any } i, N_i \text{ is countable.}$$
$$(\forall w)[w = \langle d,(N_i)\rangle \rightarrow (\exists i)(\forall k > i)N_k = \emptyset$$

These conditions, roughly speaking, say that (respectively), the denotation relation d is "downward-looking" (a name can name only objects below it in the hierarchy), any name of lever higher than level one is non-empty (or: empty names are of level one by default), the empty world doesn't contain names at any level, there are at most countably many names at any level, and for any world, its sets of names are non-empty only up to a certain level. If w=<d, (N_i)> and x is in N_i, we say that x is a name of level i in w.

The notions of accessibility and of completeness of a naming structure are obvious generalizations of the notions that we have already introduced. Suppose $i \in N_+$. Let <I, W> be a cumulative naming structure and let w=<d, (N_i)> and w'=<d', (N'_i)> belong to W. Then, Rww' if and only if:

$$(\forall i)N_i \subseteq N_i^{'}$$
$$(\exists i)N_i \subset N_i^{'}$$
$$(\forall i)\{\langle x,y\rangle|x \in N_i \wedge \langle x,y\rangle \in d\} = \{\langle x,y\rangle|x \in N_i \wedge \langle x,y\rangle \in d\}$$

That is, a world is accessible if it extends the names that already exist in the world that accesses it. Let M=<I, W> be a cumulative naming structure and let w=<d, (N_i)> (i>0) belong to W. Clearly, there exists the least natural number k (0 is treated as a natural number but not as a positive natural number) such that for any $i>k$, N_i is empty. M is said to be **w-cumulatively complete** (w-complete, for short) if and only if for any:

$$A \subseteq \bigcup_{n\geq 0, n<k+1} N_n$$

There is a possible world <d', (N'_i)>=w' in W such that w' is at most of level k+1 (that is, $(\forall i > k)N_i^{'} = \emptyset$), both Rww' and:

$$(\exists x \in N_{k+1}^{'})(\forall y)\left(y \in \bigcup_{n\geq 0, n\leq k} N_n \rightarrow (\langle x,y\rangle \in d' \leftrightarrow y \in A)\right)$$

M is said to be ***cumulatively complete*** iff for any w in W, M is w-cumulatively complete.

If w is of level k (N_k is the highest non-empty element of w), then the **domain of names** of w (denoted by $D_{N(w)}$) is the union of of all N_i for $1 \leq i \leq k$, and the **domain of objects** of w (denoted by $D_{O(w)}$) is the union of $D_{N(w)}$ and I.

Now, I will define a language that resembles the language of set theory, and the satisfaction relation for this language. The **language of cumulative naming logic** (CNL) contains the standard (first-order) logical symbols (including identity), variables x_i that (under an interpretation) will take pure individuals as values, variables a_i that (under an interpretation) will take either names or pure individuals as values, quantifiers that can bind variables of both sorts. Besides, the language contains one primitive symbol D - a two-place predicate (which can take variables of both sorts as arguments in arbitrary combinations) that in the intended reading means `denotes'. A CNL term is either an individual variable, or an a_i variable (I will use standard simplifications when dropping subscripts).

Complete cumulative name structures are intended **models of the language of CNL**. A CNL interpretation is a tuple $<M, w, v>$ such that M is a complete cumulative naming structure, w is a c.p.w. which belongs to it, and v (i) maps individual variables into I if I is not empty, and is undefined on individual variables otherwise, and (ii) maps the variables a_i into $D_{O(w)}$ if this set is non-empty and is undefined on a_i variables otherwise. Let A, B be CNL formulas and let a, b be CNL terms. The **satisfaction under an interpretation** is defined by:

- $<M, w, v>$ models $D(a,b)$ iff $<v(a), v(b)>$ is in d.
- $<M, w, v>$ models $a=b$ iff $v(a) = v(b)$.
- The clauses for Boolean connectives are standard.
- $<M, w, v>$ models $(\exists a)A$ iff $<M, w', v'>$ models A, for some w' such that Rww' and for some v' which differs from v at most at a.

Instead of $(\exists x)x=a$ I will write $U(a)$. Instead of $D(a,b)$ I will just write $b \in a$ (so '∈' here has a slightly different meaning than it has in set theory).

Certain (translations of) principles that hold for sets in ZF (with urelements) hold also for possible names. Some of them are:

$$(\forall a)(U(a) \rightarrow \backsim (\exists b) b \in a)$$
$$(\forall a)\big[\sim U(a) \rightarrow (\exists b)\big(\sim U(b) \,\&\, (\forall c)(c \in b \leftrightarrow \varphi(c))\big)\big]$$
$$(\exists a)(\sim U(a) \,\&\, (\forall b)(b \in a \leftrightarrow U(b)))$$
$$(\exists a)(\sim U(a) \,\&\, \sim(\exists b) b \in a)$$
$$(\forall a, b)(\exists c)(a \in c \,\&\, b \in c)$$

If we write $a \in \bigcup b$ for $(\exists c)(c \in b \,\&\, a \in c)$ the following also holds:

$$(\forall a)(\exists b)(\forall c, d)(c \in d \,\&\, d \in a \rightarrow c \in b)$$

Let's abbreviate $(\sim U(a) \,\&\, \sim(\exists b) b \in a)$ by $\emptyset(a)$. Then the following is valid:

$$(\forall a)[\sim(U(a) \,\&\, \sim\emptyset(a) \rightarrow (\exists b)(b \in a \,\&\, (\forall c)(c \in b \rightarrow \sim c \in a))]$$

There are, however certain axioms of ZF whose renderings fail miserably. The CNL rendering of the **axiom of extensionality**:

$$(\forall a, b)(\sim U(a) \,\&\, \sim U(b) \rightarrow ((\forall c)(c \in a \leftrightarrow c \in b) \rightarrow a = b))$$

and the **axiom of power set** in its name-theoretic translation is:

$$(\forall a)(\exists b)(\forall c)((\forall d)(d \in c \rightarrow d \in a) \rightarrow c \in b)$$

The axiom of extensionality fails because it is possible that there are coextensive and yet different name tokens. The axiom of power set fails because in the case of an infinite domain it would require that a possible world contains non-denumerably many name tokens. The first problem can be fixed easily: we just *define* identity symbol in a non-standard way so that coextensive possible names are identical *ex definitione*. The second problem requires a more elaborate move that lies beyond the scope of this paper. Let me, however, just indicate what this strategy would look like.

First, we start off with a cumulative naming structure. Then we stratify the possible worlds according to how high in the semantic ascent the tokens that exist in them are. For instance, if a possible world contains only names that name individuals, it is a world of level 1. If it also contains names that name names in a world of level 1 but no names of "higher" type, it is of level 2, etc. (formal definitions are easily available). Then, the crucial move is that we allow the reference relation of a name in a possible world *w* "reach" outside of that world, that is, a name *x* in *w* is now allowed to "refer" to objects that don't exist in *w*. What *x* can refer to instead are all those objects that exist in worlds of lower level than *w*. That way we still have a cumulative hierarchy and don't run into any paradoxes, but also we validate the axiom of power set because now there is no problem with a name referring to non-denumerably many name tokens, as long as those tokens don't exist in a single possible world.

The Revenge of the Modest Liar

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In this talk, we discuss a few ways out from the problem whether the truth predicate commutes with connectives (i.e. conjunction is true iff both conjunctions are true etc.) [HPS00] in Łukasiewicz infinite-values predicate logic $\forall$**Ł** - call it the problem of commutativity.

The liar sentence dose not imply a contradiction in $\forall$**Ł**, therefore we can assume the existence of a total truth predicate in arithmetic consistently. The typical example is $\mathbf{PA{\L}Tr}_2$ [HPS00] which is the theory over Łukasiewicz logic whose axioms are all axioms of classical **PA**, the induction scheme for formulae possibly containing the truth predicate **Tr** and T-schemata,

$$\varphi \equiv \mathbf{Tr}(\ulcorner \varphi \urcorner)$$

for a total truth predicate $\mathbf{Tr}(x)$ and any formula φ of the language of $\mathbf{PA{\L}Tr}_2$ where $\ulcorner \varphi \urcorner$ is the Gödel code of φ. We note that the language of $\mathbf{PA{\L}Tr}_2$ is semantically closed. Since the total truth predicate exists, the truth conception represented in $\mathbf{PA{\L}Tr}_2$ seems to be an example of the transparent view of truth [BG08] (like deflationist's one).

However, the truth conception in $\mathbf{PA{\L}Tr}_2$ is not as transparent as we hoped. *The liar's revenge* phenomenon, any *solution* of the liar arises another problem [B07], arises: $\mathbf{PA{\L}Tr}_2$ is ω-inconsistent [R93]. This is due to the possibility of defining *the modest liar* sentence: $\lambda \equiv (\exists n)\mathbf{Tr}(\ulcorner n \times \neg\lambda \urcorner)$, *I am at least a little false.* This involves that $\mathbf{PA{\L}Tr}_2$ has only non-standard models though **PA** has a standard model, therefore Hartry Field concluded that the truth conception in the fuzzy logic is not enough conservative [Fl08]. Moreover, as proved in [HPS00], the theory $\mathbf{PA{\L}Tr}_3$ extending $\mathbf{PA{\L}Tr}_2$ by the formalized commutation scheme

$$(\forall x, y)(Form(x)\&Form(y) \rightarrow [\mathbf{Tr}(x \dot{\rightarrow} y) \equiv (\mathbf{Tr}(x) \rightarrow \mathbf{Tr}(y))])$$

(and similarly for negation) is contradictory.

Since the commutativity has been regarded as one of the desired properties of **Tr**, this result seems to cause a feeling of anxiety. How can we solve this problem? It follows directly from the T-schema that the axiom schema $\mathbf{Tr}(\ulcorner \varphi \rightarrow \psi \urcorner) \equiv (\mathbf{Tr}(\ulcorner \varphi \urcorner) \rightarrow \mathbf{Tr}(\ulcorner \psi \urcorner))$ is provable in $\mathbf{PA{\L}Tr}_2$ for each pair of sentences φ, ψ. This can be called the axiom *schema* of commutativity of truth. Hence this form of commutativity of truth predicate is consistent. However, the revenge of the modest liar arises: the *overspill* phenomenon, any infinite set of natural numbers (e.g. the set of Gödel codes of formulae) contains non-standard natural numbers [Y05]. The unprovability of formalized commutativity is due to the non-standardness: the formalized commutativity fails when y is a non-standard natural number in the modest liar case.

The lesson is that many concepts, as the commutativity or the standardness, need not to be involved by the transparent view of truth. Since we abandon the classical logic, we should abandon the classical details of the total truth predicate. It is easy to justify that the non-standardness does not matter. Since infinite process (like non-terminated programs) are common in the context in co-inductive definition in computer science [MT91], circularly defined sentences as λ seem to have a natural interpretation. Roughly speaking, λ is saying that $\underbrace{\lambda \bar{\wedge} \cdots \bar{\wedge} \lambda}_{\text{finite length}}$ is of truth value 1 though the formula, which is equivalent to the formula whose Gödel code is the fixed point of $f(x) = \ulcorner \mathbf{Tr}(x) \bar{\wedge} \mathbf{Tr}(x) \urcorner$, is not of truth value 1.

We should take the ω-inconsistency seriously and positively. Therefore, Field's conclusion that ω-inconsistency is too expensive to save the truth theory in Łukasiewicz predicate logic also seem to be flavored with a classical stereotype: $\mathbf{PAŁTr}_2$ should not be thought as an extension of classical theory, but be thought as a formal theory of circular phenomenons from an essentially different viewpoint.

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