THE TECHNIQUE OF HIERARCHICAL MATRICES

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Abstract

The discretisation of partial differential equations (PDEs) leads to large systems of equations. In particular, the boundary element method (BEM) produces fully populated matrices. Several methods try to reduce the costs for storage and matrix-vector multiplication from $\mathcal{O}(n^2)$ to $\mathcal{O}(n \log^* n)$. The technique of hierarchical matrices presented here supports *all* matrix operations, i.e., addition, multiplication, inversion of matrices and LU decompositions up to certain approximation errors. This is of interest also for the sparse matrices from FEM, since the dense inverse or certain Schur complements can be approximated.

Concerning the applications, we mention several topics.

FEM preconditioning: Linear equations with sparse FEM matrices A are usually solved iteratively, provided a good preconditioner is available. The technique of hierarchical matrices allows to approximate the LU-factors (if they exist) which lead to a perfect black-box preconditioner. The most efficient version uses domain decomposition ideas for the construction of the cluster tree.

Matrix equations: The Lyapunov and Riccati equations arise in control theory and define a system of n^2 equations for the n^2 unknown entries of X. Therefore the best possible solve seems to need a work of $O(n^2)$. If the coefficient matrix A arises from an elliptic operator (as in control problems with a state governed by an elliptic boundary value problem), it turns out that the solution X can be well approximated by a hierarchical matrix. The costs add up to $O(nk^2 \log^3 n)$ even in the case of the nonlinear Riccati equation.

Matrix functions: The matrix exponential function $\exp(-tA)$ is of general interest. We are able to compute $\exp(-tA)$ with accuracy ε with a cost of order $O(n \log^p \frac{1}{\varepsilon} \log^q n)$. Similarly, other matrix functions can be computed, in particular, the sign-function $sign(A)$ is a very interesting function.

Beyond the hierarchical matrix technique, one can consider similar constructions involving Kronecker tensor products instead of low-rank matrices. Here we present some problems in high spatial dimensions.