A COUNTEREXAMPLE FOR CHARACTERIZING INVARIANT SUBSPACES OF MATRICES BY SINGULARITY SYSTEMS

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Abstract

Let λ_1 be a simple eigenvalue of a matrix A with right and left eigenvectors x_1, y_1, resp., and $x_1^H x_1 = y_1^H x_1 = 1$. Consider the singularity system

$$C(\lambda; u, v) \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} A - \lambda I & v \\ u^H & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

where u, v are sufficiently good approximations to x_1, y_1, resp. Then the bordered matrix $C(\lambda_1; u, v)$ is nonsingular so (1) uniquely defines $x = x(\lambda)$, $\mu = \mu(\lambda)$ for λ close to λ_1, and for $\lambda = \lambda_1$ we have $x(\lambda_1) = x_1/u^H x_1$, $\mu(\lambda_1) = 0$. Hence, $\lambda = \lambda_1$ may be determined by using the scalar singularity condition $\mu(\lambda) = e_{n+1}^T C(\lambda; u, v)^{-1} e_{n+1} = 0$ via, e.g., applying Newton's method as done in the Generalized Rayleigh Quotient Iteration of Schwetlick/Lösche [ZAMM 2000].

Now let $\operatorname{im} X_1$, $X_1^H X_1 = I_p$, be a p-dimensional invariant subspace of A which implies $AX_1 - X_1\Lambda_1 = 0$ with $\Lambda_1 = X_1^H A X_1$. With U, V such that $\operatorname{im} U \approx \operatorname{im} X_1$, $\operatorname{im} V \approx \operatorname{im} Y_1$ sufficiently good where $\operatorname{im} Y_1$ is the corresponding left invariant subspace with $Y_1^H X_1 = I_p$, we try to define $X = X(\Lambda)$, $M = M(\Lambda)$ as functions of Λ by generalizing (1) to the block singularity system

$$C(\Lambda; U, V)[X, M] = \begin{bmatrix} AX - X\Lambda + VM \\ U^H X \end{bmatrix} = \begin{bmatrix} 0 \\ I_p \end{bmatrix} \tag{2}$$

For $\Lambda = \Lambda_* = S^{-1}\Lambda_1 S$ with $S = (U^H X_1)^{-1}$, the linear system (2) is solved by $X = X_* = X_1 S$, $M = M_* = 0$. Hence, when the linear operator $C(\Lambda_*; U, V) : [X, M] \to C(\Lambda_*; U, V)[X, M]$ defined in (2) would be nonsingular, $[X, M]$ would be uniquely defined by (2) for Λ close to Λ_*, and the $(p \times p)$ singularity condition $M(\Lambda) = 0$ could be exploited for computing Λ_* as in case $p = 1$ above.

Since we did not succeed in proving nonsingularity of $C(\Lambda_*; U, V)$ we started looking for a counterexample and, finally, constructed a (5×5)-matrix A with a two-dimensional simple invariant subspace such that even the optimal borderings $U = X_1$ (which leads to $\Lambda_* = \Lambda_1$) and $V = Y_1$ give an operator $C(\Lambda_1, X_1, Y_1)$ that has rank drop 1. Here we used that the Sylvester equation (2) can equivalently be written as standard linear system

$$\mathcal{C}(\Lambda; U, V) \begin{bmatrix} \operatorname{vec}(X) \\ \operatorname{vec}(M) \end{bmatrix} = \begin{bmatrix} I_p \otimes A - \Lambda^T \otimes I_n & I_p \otimes V \\ I_p \otimes U^H & 0 \end{bmatrix} \begin{bmatrix} \operatorname{vec}(X) \\ \operatorname{vec}(M) \end{bmatrix} = \begin{bmatrix} \operatorname{vec}(0) \\ \operatorname{vec}(I_p) \end{bmatrix}.$$