# On the Relation between SUPG Stabilization and GMRES Convergence for a Convection-Diffusion Model Problem 

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## 1 Introduction

In [11, 12] Liesen and Strakoš considered, similarly to $[3,4,5,6,7]$, the following convectiondiffusion model problem

$$
\begin{equation*}
-\nu \nabla^{2} u+w \cdot \nabla u=0 \quad \text { in } \quad \Omega=(0,1) \times(0,1), \quad u=g \text { on } \partial \Omega, \tag{1}
\end{equation*}
$$

where $\nu$ is a scalar diffusion coefficient and $w$ is the velocity field. Using the Streamline Upwind Petrov Galerkin (SUPG) discretization $[8,1,10,6,7]$, the coefficient matrix for the discretized system takes the form

$$
\begin{equation*}
\tilde{A}=\nu A_{d}+A_{c}+\hat{\delta} A_{s}, \tag{2}
\end{equation*}
$$

where $A_{d}=\left\langle\nabla \phi_{j}, \nabla \phi_{i}\right\rangle$ represents the diffusion term, $A_{c}=\left\langle w \cdot \nabla \phi_{j}, \phi_{i}\right\rangle$ represents the convection term, and $A_{s}=\left\langle w \cdot \nabla \phi_{j}, w \cdot \nabla \phi_{i}\right\rangle$ is a stabilization term added to suppress nonphysical oscillations. Here $\phi_{j}, j=1,2, \ldots$, are the bilinear finite element nodal basis functions for an $N$ by $N$ grid with spacing $h=1 /(N+1)$ and $\langle\cdot, \cdot\rangle$ denotes the $L^{2}$ inner product on $\Omega$. To obtain efficient stabilization the parameter $\hat{\delta}$ is usually chosen in what has been shown to be a near optimal way for one dimensional problems and what appears to be a reasonable way for higher dimensional problems as well [3, 4], namely as

$$
\begin{equation*}
\hat{\delta}_{0}=\frac{\delta_{0} h}{\|w\|}, \quad \text { where } \quad \delta_{0}=\frac{1}{2}\left(1-\frac{1}{P_{h}}\right) \tag{3}
\end{equation*}
$$

and $P_{h} \equiv h\|w\| /(2 \nu)$ is the mesh Peclet number. We will refer to $\delta_{0}$ as to the optimal tuning parameter and to discretization with $\hat{\delta}=\hat{\delta}_{0}$ in (2) as to optimal stabilization. Also, any choice of $\hat{\delta}$ in (2) that we consider will be of the form $\hat{\delta}=\frac{\delta h}{\|w\|}$ for some tuning parameter $\delta$.
In [7] it has been observed that the linear systems arising with optimal stabilization are in general more rapidly solved by the GMRES method than systems corresponding to weaker or stronger stabilization. This seems to be true for different choices of the velocity field $w$ and different boundary conditions. In our talk we will point out a way to explain this phenomenon in the special case of a vertical wind $w=[0,1]^{T}$. In this case, it is possible to bring system matrices to a very convenient form with the help of the discrete sine transformation introduced in [12]. This form yields a spectral decomposition that can be exploited in a convergence analysis based on the following equality for GMRES residual norms (see, e.g. [9]),

$$
\begin{equation*}
\left\|r_{k}\right\|=\left\|e_{1}^{T} \cdot\left[r_{0}, A r_{0}, \ldots, A^{k} r_{0}\right]^{+}\right\|^{-1} \tag{4}
\end{equation*}
$$

where $A$ is the transformed matrix, $r_{0}$ the right hand side, $e_{1}$ is the first column of the identity matrix and $X^{+}$denotes the Moore-Penrose pseudoinverse of a matrix $X$.

## 2 The linear systems

In this section we show how we can transform, with a vertical wind $w=[0,1]^{T}$, system matrices to block diagonal form with tridiagonal Toeplitz blocks. We also give formulaes for the entries and the eigenvalues of the system matrix (2) in dependency of the tuning parameter $\delta$. With the help of these formulaes we will be able to explain why the convergence of GMRES with optimal stabilization is faster than with weaker or stronger stabilization.
Using the vertical line ordering for equations and unknowns, the $N^{2}$ by $N^{2}$ system matrix takes the form

$$
\begin{equation*}
A_{V}=A_{V}(h, \nu, \delta)=\nu K \otimes M+M \otimes((\nu+\delta h) K+G), \tag{5}
\end{equation*}
$$

see, e.g., [2, Section 1.1] and [6, pp. 1081 and 1089]. Here

$$
\begin{equation*}
M=\frac{h}{6}\left(S+4 I+S^{T}\right), \quad K=\frac{1}{h}\left(-S+2 I-S^{T}\right), \quad G=\frac{1}{2}\left(-S+S^{T}\right), \tag{6}
\end{equation*}
$$

where $I$ is the identity and $S=\left[e_{2}, \ldots, e_{N}, 0\right]$ is the down shift matrix, are the $N$ by $N$ mass, stiffness and gradient matrices of the one dimensional constant coefficient convection-diffusion equation discretized on a uniform mesh using linear elements. The symmetric tridiagonal Toeplitz matrices $M$ and $K$ can be diagonalized by the same matrix of eigenvectors [12]. Denoting this matrix with $U$ we transform the system matrix $A_{V}$ through

$$
\begin{equation*}
(U \otimes I) A_{V}(U \otimes I)=\nu(U K U) \otimes M+(U M U) \otimes((\nu+\delta h) K+G) \equiv A \tag{7}
\end{equation*}
$$

Elementary algebra shows that $A$ is a block-diagonal matrix consisting of $N$ nonsymmetric tridiagonal Toeplitz blocks $T_{j}$, each of size $N$ by $N$,

$$
\begin{equation*}
A=\operatorname{diag}\left(T_{1}, \ldots, T_{N}\right), \quad T_{j}=\gamma_{j} S+\lambda_{j} I+\mu_{j} S^{T}, \quad j=1, \ldots, N, \tag{8}
\end{equation*}
$$

for more details see [3], [4] and [12]. In [12] we find exact expressions, in dependency of the tuning parameter $\delta$, of the entries of $T_{j}$, namely

$$
\begin{align*}
3 \lambda_{j} & =2 \delta h c_{j}+2 \nu\left(4-\frac{\omega_{j}}{2}\right),  \tag{9}\\
-3 \mu_{j} & =\delta h c_{j}+\nu\left(1+\omega_{j}\right)-\frac{h}{2} c_{j},  \tag{10}\\
-3 \gamma_{j} & =\delta h c_{j}+\nu\left(1+\omega_{j}\right)+\frac{h}{2} c_{j}, \tag{11}
\end{align*}
$$

where $c_{j}=2+\cos j \pi h$ and $\omega_{j}=2 \cos j \pi h$. We consider convection dominated problems and hence assume that the mesh Peclet number satisfies $P_{h} \gg 1$ (with $w=[0,1]^{T}$, this means $\nu \ll h$ ). In addition, we consider tuning parameters $\delta$ of the same order of magnitude as $\delta_{0} \approx 1 / 2$. Hence $\nu \ll \delta h$ is a realistic assumption. Suppose we have scaled the matrix $A$ by $-1 / 3$, then according to (9)-(11) the entries of the scaled matrix are given by

$$
\begin{equation*}
\lambda_{j}=-2 \delta h c_{j}+\mathcal{O}(\nu), \quad \mu_{j}=\delta h c_{j}-\frac{h}{2} c_{j}+\mathcal{O}(\nu), \quad \gamma_{j}=\delta h c_{j}+\frac{h}{2} c_{j}+\mathcal{O}(\nu) . \tag{12}
\end{equation*}
$$

In our analysis we will exploit these simplified expressions of system matrix entries (note that GMRES convergence speed is invariant under scaling of the system matrix).
As for eigenvalues, the eigenvalues of tridiagonal Toeplitz matrices are well-known; for the matrix $T_{j}$ they are given by

$$
\begin{equation*}
\sigma_{j k}=\lambda_{j}+2 \sqrt{\mu_{j} \gamma_{j}} \cos k h \pi, \quad 1 \leq j, k \leq N, \tag{13}
\end{equation*}
$$

see [12]. Now it easy to derive, by substituting (12) in (13), the following convenient expression for the eigenvalues in dependence of $\delta$,

$$
\begin{equation*}
\sigma_{j k}=2 \delta h c_{j}\left(-1+\sqrt{1-\frac{1}{4 \delta^{2}}+\mathcal{O}(\nu)} \cos k h \pi\right), \quad 1 \leq j, k \leq N \tag{14}
\end{equation*}
$$

Using this eigenvalue formula we will show that the eigenvalues of the system matrix are extremely close with optimal stabilization but grow away from each other as soon as the chosen stabilization is weaker or stronger. We will explain, with the help of (4), that these growing eigenvalue distances are the main reason for convergence delay with non-optimal stabilization.

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