Hierarchical Bases on Rectangles

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1 Hierarchy

We assume a weak formulation of an elliptic partial differential equation,

$$a(u,v) = (f,v), \tag{1}$$

for $u, v \in H_0^1(\Omega)$, Ω polygonal domain in R^2,

$$a(u,v) = \int_\Omega A\nabla u.\nabla v \,\mathrm{d}x \qquad \text{and} \qquad (f,v) = \int_\Omega f v \,\mathrm{d}x, \tag{2}$$

A symmetric positive definite in Ω, $f \in L^2(\Omega)$.

We suppose a finite element (FE) approximation V of $H_0^1(\Omega)$. The algebraic multilevel iterative (AMLI) method in its two-level form exploits a splitting of the FE space into two hierarchical spaces of functions, let us denote (reminding slightly the wavelet notation) the coarse grid space by U and the space of functions corresponding to the added nodes of the fine grid by W, and let $V = U \oplus W$. Then the equation (1) can be transformed into the 2×2 block form

$$\begin{pmatrix} B_W & B_{WU} \\ B_{UW} & B_U \end{pmatrix} \begin{pmatrix} \tilde{u}_W \\ \tilde{u}_U \end{pmatrix} = \begin{pmatrix} F_W \\ F_U \end{pmatrix}, \tag{3}$$

where $\tilde{u}_W$ and $\tilde{u}_U$ are the coefficients of the solution with respect to the basis of U and of W, respectively.

We can choose the block diagonal of the matrix B of the system (3) as for the preconditioning matrix M_{add} (additive form) or we can use the block Gauss-Seidel preconditioning M_{mult} (multiplicative form). Thus the preconditioned system (3) has then the condition numbers bounded by

$$\kappa(M_{add}^{-1}B) \le \frac{1+\gamma}{1-\gamma}$$

and

$$\kappa(M_{mult}^{-1}B) \le \frac{1}{1-\gamma^2},$$

respectively, where γ is the constant in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality

$$|a(u,w)| \le \gamma\sqrt{a(u,u)a(w,w)},$$

$u \in U$, $w \in W$. See [2] and the references therein for a more detailed theory.

2 CBS constants

We consider two different refinements of the space of bilinear FEs on rectangles. The coarse space U contains piecewise bilinear functions, i.e. on one macroelement, there are four bilinear basis functions. After refining the mesh, five new nodes are given in each macroelement. Four of them are in the centers of the edges and the fifth is in the center of the element. These are connected to functions of W. In the first case, W is composed from piecewise bilinear functions with smaller supports and we call it W_1. In the second case, the nodes are connected to polynomials of the second order and the space of them is denoted by W_2. These two types of the refinement can be called h- and p-hierarchy, respectively.

Several hierarchical FE spaces have been studied and the uniform CBS constant estimates were found, see e.g. [2, 3, 4, 5]. We now show that the CBS constant is uniformly bounded in case of p-hierarchy of bilinear FEs when matrix A in equation (2) is diagonal. The upper bound is slightly greater than that for h-refinement. In comparison, the p-refinement of the linear FEs on triangles yields the uniform CBS constant estimate equal to one, thus this splitting is unusable for AMLI methods [4].

Theorem 1. *Let A be a diagonal and constant matrix on each macroelement. Then the CBS constants for the hierarchical h-refinement of bilinear FEs and for the equation (1) are not greater than $\sqrt{\frac{3}{8}}$ and $\sqrt{\frac{3}{4}}$, respectively, for isotropic operator and regular elements and for anisotropic equation or elements, respectively. The CBS constants for the hierarchical p-refinement of bilinear FEs are not greater than $\sqrt{\frac{5}{11}}$ and $\sqrt{\frac{9}{11}}$, respectively, for isotropic operator and regular elements and for anisotropic equation or elements, respectively.*

The proof of the theorem will be available in [6]. In the case when matrix A in (2) is constant and positive definite but not diagonal, the uniform CBS constatnt estimate for the p-hierarchy is equal to one. Thus for such problems the AMLI methods don't yield better convergence than one-level iterative solvers.

3 Preconditioning of B_W

Diagonal block B_W is well conditioned (the condition number is $O(1)$), still the condition number may grow due to the changes of variables in A on macroelements. In papers [1, 2] the new idea of constructing a preconditioning matrix C_W for the block B_W which leads to the uniformly bounded condition number has been introduced. The hierarchical linear FEs on triangles are considered there. The preconditioning matrix C_W is assembled from the macroelement stiffness matrices corresponding to the space W. From these individual matrices, some of the off-diagonal entries are deleted in such manner that after a proper reordering the basis functions of W (reordering rows and columns in C_W), the preconditioning matrix C_W becomes tri-diagonal. The off-diagonal elements which are not deleted from the macroelement stiffness matrices may be called "strong connections".

We show that this idea can be adopted for the hierarchical h- and p-refinement of bilinear FEs as well. The strong connections for five basis functions of W on a macroelement for h- and for p-refinements of bilinear FEs are chosen based on the validity of the relation

$$\frac{A_{11}}{A_{22}} \geq \frac{d_1^2}{d_2^2}.$$

where A_{11} and A_{22} are the diagonal elements of matrix A in (2) and $d_1 \times d_2$ is the size of the particular coarse element. The examples of numbering the basis functions of the refining space W for case of h- and p-refinements are presented in [6].

Theorem 2. *There exist generalized tri-diagonal preconditioning matrices such that the preconditioned diagonal blocks B_W have the condition numbers not greater that* 4.2 *and* 8.2 *for the hierarchical h- and p-refinement of bilinear FEs, respectively.*

We also provide [6] a numerical estimate for the upper bound of the condition number of the preconditioned diagonal block B_W when a deeper hierarchy is used. The existence of a reordering the elements in order to get a tri-diagonal matrix is also shown.

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References

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