Numerical Illustration of Theoretical Results for Non-Linear Semi-Coercive Beam Problem

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The semi-coercive problem of a beam on a unilateral elastic subsoil of Winkler's type is considered. The aim of this contribution is numerically illustrate some basic results of the problem, which can be found in the articles [1] and [2]. Concretely, we are interested in dependence of the problem solution on an external load, the error estimates of approximated solutions and the convergence properties of used numerical methods subject to the load. The problem formulations and the results are summarized in Section 1.

1 Semi-coercive problem with beam on unilateral elastic subsoil

Formulation of the problem. We consider a beam of the length l with free ends which is situated in the interval $\Omega = (0, l)$, and assume that the beam is supported by a unilateral elastic subsoil in the interval $\Omega_s := (x_l, x_r)$, $0 \le x_l < x_r \le l$. Such a subsoil is active only if the beam deflects against it. Let E, I and q denote functions that represent, respectively, the Young's modulus of the beam material, the inertia moment of the cross-section of the beam and the stiffness coefficient of the subsoil. The aim is to find the deflection w^* of the axes of the beam caused by the load of the beam.

The variational formulation of the problem has the form

$$(P) \quad \begin{cases} \text{find } w^* \in H^2(\Omega): \quad J(w^*) \le J(v) \quad \forall v \in H^2(\Omega), \\ J(v) := \frac{1}{2}\left(a(v,v) + b(v^-, v^-)\right) - L(v), \quad v \in H^2(\Omega), \end{cases}$$

where

$$a(v_1, v_2) := \int_\Omega EI v_1'' v_2''\, dx, \quad b(v_1, v_2) := \int_{\Omega_s} q v_1 v_2\, dx$$

represent the work of the inner forces and the subsoil, respectively, and $L \in \left(H^2(\Omega)\right)^*$ represents the work of the beam load.

Solvability and dependence on the load. Since the beam does not have fixed ends (it is only laid on the subsoil), the problem solvability depends on the beam load. The existence and uniqueness of the solution w^* of the problem (P) is ensured by the condition

$$F < 0 \quad \text{and} \quad x_l < T < x_r, \tag{1}$$

where $F := L(1)$ is the load resultant and $T := L(x)/L(1)$ is the balance point of the load. The condition (1) means that the load resultant is situated in Ω_s and oriented against the subsoil.

To determine the dependence of the change of the problem (P) solution on the change of the load, we will consider the class $\mathcal{S}_{\delta,\xi,\eta}$ of the loads $L \in \left(H^2(\Omega)\right)^*$ such that $T \in (x_l + \delta, x_r - \delta)$, $F < -\xi < 0$ and $\|L\|_* \le \eta$, with respect to positive parameters δ, ξ, η. Then there exists a positive constant c which depends on the loads from $\mathcal{S}_{\delta,\xi,\eta}$ only through the parameters δ, ξ, η such that

$$\|w_1^* - w_2^*\|_{2,2} \le c\|L_1 - L_2\|_* \quad \forall L_1, L_2 \in \mathcal{S}_{\delta,\xi,\eta}, \tag{2}$$

where $w_i^* = w_i^*(L_i)$ solves the problem (P) with respect to the loads L_i, $i = 1, 2$. Notice that if $\delta \to 0$ or $\xi/\|L\|_* \to 0$ then $c \to +\infty$. Therefore the loads, for which the balance point T is closed to the end points of the subsoil or the size of the load resultant is small with respect to $\left(H^2(\Omega)\right)^*$-norm of the load, will be called *unstable*.

Approximation of the problem. Let us define a partition τ_h, of the interval $\overline{\Omega} = [0, l]$, with the nodal points x_j, $j = 0, 1, \ldots, N$, and with the discretization parameter h. For a partition τ_h, we will define the function space $V_h \subset H^2(\Omega)$ of continuously differentiable and piecewise cubic functions. The form b will be approximated by a numerical quadrature on each subsoiled partition interval. Its approximation has a form

$$b_h(v_1, v_2) := \sum_{i=1}^{m} r_i v_1(z_i) v_2(z_i), \quad v_1, v_2 \in H^2(\Omega),$$

where $z_1 < z_2 < \ldots < z_m$, are the points of the numerical quadratures and the coefficients r_i contain the products of the stiffness coefficients and weights of the numerical quadrature. From a mechanical point of view, the subsoil is substituted by insulated "springs".

The approximated problem corresponding to the partition τ_h will be denoted (P_h) and its solution $w_h^* \in V_h$. The solution w_h^* is unique if the condition (1) holds and the discretization parameter h is sufficiently small. It holds that $w_h^* \to w^*$ in $H^2(\Omega)$ for $h \to 0$. Moreover, if w^* belongs to the sufficiently wide class of the functions from $H^4(\Omega)$ and the numerical quadrature is exact at least for polynomial of the first degree then the convergence order is two.

The algebraic form of the problem (P_h) will be denoted by $(\mathbb{P})$ and has the form

$$(\mathbb{P}) \qquad \begin{cases} \text{find } w^* \in \mathbb{R}^n : \ J(w^*) \le J(w) \quad \forall w \in \mathbb{R}^n, \\ J(w) := \frac{1}{2}(Kw, w)_n + \frac{1}{2}\left(D(Bw)^-, (Bw)^-\right)_m - (f, w)_n, \end{cases}$$

where $n = 2N + 2$, $u_i^- := \min\{0, u_i\}$, $i = 1, 2, \ldots, m$, $u \in \mathbb{R}^m$, and $K \in \mathbb{R}^{n \times n}$, $f \in \mathbb{R}^n$, $D \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times m}$ are, respectively, the stiffness matrix, the load vector, the diagonal matrix containing coefficients r_i and the matrix, which transforms the function values and the values of the first derivatives at the nodal points x_j, $j = 0, 1, \ldots, N$, onto the points z_i, $i = 1, \ldots, m$. The problem $(\mathbb{P})$ can be equivalently rewritten as a non-linear system of the equations:

$$\text{find } w^* \in \mathbb{R}^n : \ \ Kw^* + B^T D(Bw^*)^- = f.$$

Let the matrix $R \in \mathbb{R}^{n \times 2}$ given by the functions 1 and x represent all polynomials from P_1. Notice that $KR = 0$. Let us denote $G := BR \in \mathbb{R}^{m \times 2}$ and $e := R^T f = (F, FT)^T \in \mathbb{R}^2$.

Descent direction method with and without projection. We will also define two families of auxiliary problems. First one is created by the linear problems with bilateral elastic "springs". The concrete problem of this family is specified by a choice of the springs. The choice will be represented by the diagonal matrix $A \in \mathbb{R}^{m \times m}$ containing the numbers 0 and 1:

$$(\mathbb{P}^A) \qquad \text{find } w = w(A) \in \mathbb{R}^n : \ \ (K + B^T DAB)w = f,$$

The second family is created by the non-linear problems specified by a vector $v \in \mathbb{R}^n$:

$$(\mathbb{P}^v) \qquad \text{find } c = c(v) \in \mathbb{R}^2 : \ \ G^T D(Bv + Gc)^- = e.$$

For the vector $v \in \mathbb{R}^n$, we will define the diagonal matrix $A(v) \in \mathbb{R}^{m \times m}$ such that $(A(v))_{ii} = 1$ if $(Bv)_i < 0$, otherwise $(A(v))_{ii} = 0$.

The following algorithms are based on a descent direction method without and with projection.

Algorithm 1

Initialization

$w^{(0)} = 0,$

$A^{(0)} = \{1, \dots, m\}.$

Iteration $k = 0, 1, \dots$

$s^{(k)},\ w^{(k)} + s^{(k)}$ solves $(\mathbb{P}^{A^{(k)}})$,

$\alpha^{(k)} = arg \min\limits_{0 \le \alpha \le 1} J(w^{(k)} + \alpha s^{(k)}),$

$w^{(k+1)} = w^{(k)} + \alpha^{(k)} s^{(k)},$

$A^{(k+1)} = A(w^{(k+1)}).$

Algorithm 2

Initialization

$w^{(0)} = Rc^{(0)}$, $c^{(0)}$ solves $(\mathbb{P}^0)$,

$A^{(0)} = A(w^{(0)}),$

Iteration $k = 0, 1, \dots$

$s^{(k)},\ w^{(k)} + s^{(k)}$ solves $(\mathbb{P}^{A^{(k)}})$,

$\alpha^{(k)} = arg \min\limits_{0 \le \alpha \le 1} J(w^{(k)} + \alpha s^{(k)}),$

$\tilde{w}^{(k)} = w^{(k)} + \alpha^{(k)} s^{(k)},$

$c^{(k)}$, $c^{(k)}$ solves $(\mathbb{P}^{\tilde{w}^{(k)}})$,

$w^{(k+1)} = \tilde{w}^{(k)} + Rc^{(k)},$

$A^{(k+1)} = A(w^{(k+1)}).$

If we denote $w_{h,k} \in V_h$ as a function corresponding to the vector $w^{(k)}$, which is generated by one of these algorithms, then the sequence $\{w_{h,k}\}_k$ converges uniformly (with respect to sufficiently small h) to the solution w_h^* of the problem (P_h) in $H^2(\Omega)$ for both algorithms due to the condition (1).

Algorithm 2 contains the step, where the problem $(\mathbb{P}^v)$ is solved in, contrary to Algorithm 1. The problem $(\mathbb{P}^v)$ represents the projection P of the space $\mathbb{R}^m$ onto the set

$$\Lambda := \{\lambda \in \mathbb{R}^m \mid \lambda \le 0,\ G^T D\lambda = e\}$$

with respect to the scalar product $(D.,.)_m$ in $\mathbb{R}^m$, since $P(Bv) = (Bv + Gc)^-$, where $c = c(v) \in \mathbb{R}^2$ solve the problem $(\mathbb{P}^v)$. Therefore the vectors $(Bw^{(k)})^-$ belong to Λ, where $w^{(k)}$ are generated by Algorithm 2. Since it also holds that $(Bw^*)^- \in \Lambda$, the vectors $w^{(k)}$ have some common properties, given by the properties of Λ, as the unknown vector w^*. For example, the diameter of the set Λ is small for unstable loads. Thus, the vectors $w^{(k)}$ are closed to the vector w^* and we can expect the better convergence property for Algorithm 2 than for Algorithm 1.

2 Numerical Examples

In this section, some theoretical results of the problem and the numerical methods will be illustrated on numerical examples. In all the examples, we will consider the beam of the length $l = 1\ m$ and the equidistant partition with $10 * 2^j$, $j = 1, 2, \dots$, elements. We use the following stopping criterion:

$$\frac{\|r^{(k)}\|_n}{\|f\|_n} \le 10^{-8}, \quad r^{(k)} := f - Kw^{(k)} - B^T DB(w^{(k)})^-$$

and the numerical quadratures, which are respectively denoted NQ_1, NQ_2, NQ_3:

$$\int_{-1}^{1} \phi(\xi)\ d\xi \approx 2\phi(0), \quad \int_{-1}^{1} \phi(\xi)\ d\xi \approx \phi(-1) + \phi(1), \quad \int_{-1}^{1} \phi(\xi)\ d\xi \approx \phi(-\sqrt{3}/3) + \phi(\sqrt{3}/3).$$

Comparison of the algorithms. Let $EI = 5 * 10^5\ Nm^2$, $x_l = 0.1\ m$, $x_r = 0.9\ m$ and $q = 2 * 10^7\ Nm^{-2}$. At the end points $0, l$ of the beam, we will consider the point loads F_0 and F_l. In Example 1, we choose $F_0 = -5000\ N$, $F_l = -5000\ N$ and $F_0 = -5000\ N$, $F_l = -1000\ N$ in

Example 2. The load is stable in Example 1, since the balance point T of the load lies in the central part of the subsoil, contrary to Example 2. Dependence of the number of outer iterations on the number of elements of the partition is shown in Table 1. The quadrature NQ_3 is used.

Ex. 1	40	80	160	320	640	1280
ALG1	4	3	4	4	4	4
ALG2	3	3	3	3	3	3

Ex. 2	40	80	160	320	640	1280
ALG1	6	6	7	8	7	8
ALG2	2	2	2	2	2	2

Table 1: Numbers of outer iterations in dependence on the numbers of elements.

Notice that the number of outer iterations does not depend on refinement of the partition for both algorithms. In Example 1, the numbers of outer iterations are practically the same for both algorithms, contrary to Example 2, which justifies the theoretical result.

Dependence of the solution on the load. We investigate the influence of the small change of the load on the change of the solution of the problem ($\mathbb{P}$) in three examples. The load is stable in Example 1 and unstable in Examples 2 and 3. Concretely, we evaluate the ratio

$$c := \frac{\frac{\|w_1 - w_2\|_A}{\|w_1\|_A}}{\frac{\|f_1 - f_2\|_n}{\|f_1\|_n}}, \quad \|v\|_A^2 := (Av, v)_n, \quad A = K + B^T DB,$$

where the vectors f_1, f_2 represent the loads and the vectors w_1, w_2 the corresponding solutions.

Let $EI = 20\ Nm^2$, $x_l = 0.2\ m$, $x_r = 0.8\ m$ and $q = 1.2 * 10^6\ Nm^{-2}$. We use the equidistant partition with 80 elements and the numerical quadrature NQ_3. We consider three constant loads P_1, P_2, P_3, which are situated respectively in the intervals $(0\ m, 0.1\ m)$, $(0.4\ m, 0.6\ m)$ and $(0.9\ m, 1\ m)$ in all the examples. The concrete values of the load and their small changes (in brackets) are in Table 2.

	$P_1\ [Nm^{-1}]$	$P_2\ [Nm^{-1}]$	$P_3\ [Nm^{-1}]$
Ex. 1	-100 (-1)	0	-100 (-1)
Ex. 2	-100	0	-450 (-1)
Ex. 3	100	-110	100 (+1)

Table 2: Loads and their changes.

The ratios c are respectively 1.00, 25.83 and 44.53 for these examples, which shows that the relative small changes of the unstable loads can cause the relative large changes of the solutions (Examples 2,3), contrary to the stable load (Example 1).

Convergence properties of the numerical quadratures. Let $EI = 1\ Nm^2$, $x_l = 0.2\ m$, $x_r = 0.8\ m$, $q = 1\ Nm^{-2}$ and let the load density have a form

$$f(x) = \begin{cases} -384\left(x - \frac{1}{2}\right)^2 + 32, & x \in (0, \frac{1}{5}) \cup (\frac{2}{5}, \frac{3}{5}) \cup (\frac{4}{5}, 1), \\ -\frac{16}{15}\left(x - \frac{1}{2}\right)^6 + \frac{4}{3}\left(x - \frac{1}{2}\right)^4 - 385\left(x - \frac{1}{2}\right)^2 + \frac{148816}{15000000} + 32, & x \in (\frac{1}{5}, \frac{2}{5}) \cup (\frac{3}{5}, \frac{4}{5}). \end{cases}$$

Then the exact solution of the problem (P) is a function

$$w(x) = -\frac{16}{15}\left(x - \frac{1}{2}\right)^6 + \frac{4}{3}\left(x - \frac{1}{2}\right)^4 - \left(x - \frac{1}{2}\right)^2 + \frac{148816}{15000000}.$$

Notice that $w \in H^4(\Omega)$. For numerical quadratures NQ_1, NQ_2, NQ_3 and different partitions, the terms

$$\|w - w_h\|_{2,2}/h^2, \quad \|v\|_{2,2}^2 := \int_0^1 (v'')^2 \, dx + \int_{1/5}^{4/5} v^2 \, dx \quad \text{and} \quad |w - w_h|_{0,2}/h^2, \quad |v|_{0,2}^2 := \int_0^1 v^2 \, dx$$

are evaluated. The approximated values of the terms are described in Table 3.

$\|w - w_h\|_{2,2}/h^2$	10	20	40	80	160
NQ1	1.0422	1.0606	1.0651	1.0663	1.0666
NQ2	1.0423	1.0606	1.0652	1.0663	1.0664
NQ3	1.0422	1.0606	1.0651	1.0663	1.0666

$\|w - w_h\|_{0,2}/h^2$	10	20	40	80	160
NQ1	0.0575	0.0577	0.0577	0.0577	0.0585
NQ2	0.1559	0.1351	0.1289	0.1302	0.1388
NQ3	0.0005	0.0001	0.0000	0.0000	0.0017

Table 3: Error estimates of the quadratures in dependence on the number of elements.

The results confirm that the order of convergence is two for this example for all the quadratures. The quadratures NQ_1, NQ_2 are exact for polynomials of the first degree, the quadrature NQ_3 is exact for polynomials of the third degree. While the values of the first term are quite similar for all the quadratures, the values of the second term are smaller for NQ_3 than for NQ_1 or NQ_2.

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References

[1] S. Sysala. Unilateral Elastic Subsoil of Winkler's type: Semi-Coercive Beam Problem. Applications of Mathematics. Accepted.

[2] S. Sysala. Numerical Modelling of the Semi-Coercive Beam Problem with Unilateral Elastic Subsoil of Winkler's type. In preparation.