Fast Optimal Topology and Shape Design in Magnetostatics

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1 Introduction

Topology optimization, cf. [1], searches for an optimal distribution of material and void without any restrictions on the structure of the design geometry. Shape optimization, cf. [3], tunes the shape of the geometry, while the topology is fixed. We propose to proceed sequentially with the optimal topology and shape design so that a coarsely optimized topology is the initial guess for the following shape optimization. A necessary non-trivial step in between is an identification and smooth approximation of the fuzzy shapes. A simultaneous topology and shape design was studied, e.g., in [2]. An algorithm that couples topology and shape optimization was presented in [7]. Here we aim at making this algorithm fast by using the adjoint sensitivity analysis to the Newton-method for the governing nonlinear state equation and using a multigrid approach for the shape optimization. Our results are also presented in [6].

2 Topology Optimization for 2-Dimensional Magnetostatics

Let $\Omega \subset \mathbf{R}^2$ be a computational Lipschitz domain that is divided into a Lipschitz subdomain $\Omega_{\mathrm{d}} \subset \Omega$, where the optimal distribution of the ferromagnetics and the air is going to be find, and into a purely air Lipschitz subdomain $\Omega_0 := \Omega \setminus \overline{\Omega_{\mathrm{d}}}$. Let further $Q := \{\rho \in L^2(\Omega_{\mathrm{d}}) : 0 \le \rho \le 1,\ \int_{\Omega_{\mathrm{d}}} \widetilde{\rho}(\rho)\, d\mathbf{x} \le V_{\max}\}$ be a set of admissible material distributions, where $V_{\max} > 0$ is a maximal possible area occupied by the ferromagnetics and where $\widetilde{\rho} \in C^2((0,1))$ penalizes the values of $\rho \in (0, 1/2)$ to be close to 0 and the values of $\rho \in (1/2, 1)$ to 1. Finally, let $J : H^1(\Omega) \mapsto \mathbf{R}$ be a cost functional. We consider the following topology optimization problem:

$$\text{Find } \rho^* \in Q : J(u(\rho^*)) \le J(u(\rho)) \quad \forall \rho \in Q \tag{1}$$

with respect to the 2-dimensional nonlinear magnetostatic state problem

$$\int_\Omega \nu_0\, \mathbf{grad}(u(\rho)) \cdot \mathbf{grad}(v)\, d\mathbf{x} + \int_{\Omega_{\mathrm{d}}} \widetilde{\rho}(\rho)\, (\nu(\|\mathbf{grad}(u(\rho))\|) - \nu_0)\, \mathbf{grad}(u(\rho)) \cdot \mathbf{grad}(v)\, d\mathbf{x} = \int_\Omega J v\, d\mathbf{x} \quad \forall v \in H^1_0(\Omega), \tag{2}$$

where $\nu \in C^2((0,\infty))$ denotes a nonlinear material reluctivity of the ferromagnetics, ν_0 is the vacuum reluctivity constant and $J \in L^2(\Omega)$ is a current density. Note that in general, one has to pose an additional regularization of the topology ρ to avoid the so-called checkerboard effect. However, we are merely interested in a coarsely discretized problem, which is well-posed.

2.1 Nonlinear State Sensitivity Analysis

When solving the problem (1), we use a nested approach, i.e. for a given design we eliminate the nonlinear state equation (2). The latter is discretized by the finite element method using the linear Lagrange nodal elements on triangles, which reads as follows:

$$\mathbf{A}(\mathbf{u}(\boldsymbol{\rho}), \boldsymbol{\rho}) \cdot \mathbf{u}(\boldsymbol{\rho}) = \mathbf{f}, \tag{3}$$

where $\mathbf{A}$ is the assembled reluctivity matrix, $\mathbf{f}$ is the right-hand side vector, $\mathbf{u}$ is the solution vector and $\boldsymbol{\rho}$ is the element-wise constant material function.

The problem (3) is solved by the Newton method. However, the optimization needs to evaluate the gradients of the cost functional with respect to $\boldsymbol{\rho}$. To this goal we derived an adjoint algorithm to the Newton method. Both are depicted below.

Newton method

- Given $\boldsymbol{\rho}$
- $i := 0$
- Solve $\mathbf{A}(\mathbf{0}, \boldsymbol{\rho}) \cdot \mathbf{u}^0 = \mathbf{f}$
- $\mathbf{f}^0 := \mathbf{f} - \mathbf{A}(\mathbf{u}^0, \boldsymbol{\rho}) \cdot \mathbf{u}^0$
- **while** $\|\mathbf{f}^i\|/\|\mathbf{f}\| >$ prec **do**
  - $i := i + 1$
  - Solve $\mathbf{A}'_{\mathbf{u}}(\mathbf{u}^{i-1}, \boldsymbol{\rho}) \cdot \mathbf{w}^i = \mathbf{f}^{i-1}$
  - Find $\tau^i : \|\mathbf{f}^i(\tau^i)\| < \|\mathbf{f}^{i-1}\|$
  - $\mathbf{u}^i := \mathbf{u}^{i-1} + \tau^i \mathbf{w}^i$
  - $\mathbf{f}^i := \mathbf{f} - \mathbf{A}(\mathbf{u}^i, \boldsymbol{\rho}) \cdot \mathbf{u}^i$
  - Store $\mathbf{w}^i$ and τ^i
- **end while**
- Store $\mathbf{u}^i$ and $k := i$
- Calculate objective $J(\mathbf{u}^i, \boldsymbol{\rho})$

Adjoint Newton method

- Given $\boldsymbol{\rho}$, k, $\mathbf{u}^k$, $\{\mathbf{w}^i\}_{i=1}^k$ and $\{\tau^i\}_{i=1}^k$
- $\boldsymbol{\lambda} := J'_{\mathbf{u}}(\mathbf{u}^k, \boldsymbol{\rho})$
- $\omega := \mathbf{0}$
- **for** $i := k, \ldots, 1$ **do**
  - $\mathbf{u}^{i-1} := \mathbf{u}^i - \tau^i \mathbf{w}^i$
  - Solve $\mathbf{A}'_{\mathbf{u}}(\mathbf{u}^{i-1}, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta} = \boldsymbol{\lambda}$
  - $\boldsymbol{\omega} := \boldsymbol{\omega} + \tau^i \mathbf{G}_{\boldsymbol{\rho}}(\mathbf{u}^{i-1}, \mathbf{w}^i, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta}$
  - $\boldsymbol{\lambda} := \boldsymbol{\lambda} + \tau^i \mathbf{G}_{\mathbf{u}}(\mathbf{u}^{i-1}, \mathbf{w}^i, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta}$
- **end for**
- Solve $\mathbf{A}(\mathbf{0}, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta} = \boldsymbol{\lambda}$
- $\frac{dJ(\mathbf{u}^k(\boldsymbol{\rho}), \boldsymbol{\rho})}{d\boldsymbol{\rho}} := \boldsymbol{\omega} + \mathbf{H}_{\boldsymbol{\rho}}(\mathbf{u}^0, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta} + J'_{\boldsymbol{\rho}}(\mathbf{u}^k, \boldsymbol{\rho})$

The sensitivity information of the system matrix is involved in

$$\begin{aligned}
\mathbf{G}_{\boldsymbol{\rho}}(\mathbf{u}, \mathbf{w}, \boldsymbol{\rho}) &:= -\left[\frac{\partial \mathbf{A}'_{\mathbf{u}}(\mathbf{u}, \boldsymbol{\rho})}{\partial \rho_1} \cdot \mathbf{w}, \ldots, \frac{\partial \mathbf{A}'_{\mathbf{u}}(\mathbf{u}, \boldsymbol{\rho})}{\partial \rho_m} \cdot \mathbf{w}\right] - \left[\frac{\partial \mathbf{A}(\mathbf{u}, \boldsymbol{\rho})}{\partial \rho_1} \cdot \mathbf{u}, \ldots, \frac{\partial \mathbf{A}(\mathbf{u}, \boldsymbol{\rho})}{\partial \rho_m} \cdot \mathbf{u}\right], \\
\mathbf{G}_{\mathbf{u}}(\mathbf{u}, \mathbf{w}, \boldsymbol{\rho}) &:= -\left[\frac{\partial \mathbf{A}'_{\mathbf{u}}(\mathbf{u}, \boldsymbol{\rho})}{\partial u_1} \cdot \mathbf{w}, \ldots, \frac{\partial \mathbf{A}'_{\mathbf{u}}(\mathbf{u}, \boldsymbol{\rho})}{\partial u_n} \cdot \mathbf{w}\right] - \mathbf{A}'_{\mathbf{u}}(\mathbf{u}, \boldsymbol{\rho}), \\
\mathbf{H}_{\boldsymbol{\rho}}(\mathbf{u}, \boldsymbol{\rho}) &:= -\left[\frac{\partial \mathbf{A}(\mathbf{0}, \boldsymbol{\rho})}{\partial \rho_1} \cdot \mathbf{u}, \ldots, \frac{\partial \mathbf{A}(\mathbf{0}, \boldsymbol{\rho})}{\partial \rho_m} \cdot \mathbf{u}\right],
\end{aligned}$$

where $\mathbf{A}'_{\mathbf{u}}(\mathbf{u}, \boldsymbol{\rho})$ is the linearization of the nonlinear system matrix.

3 Sequential Coupling of Topology and Shape Optimization

We will use the optimal topology design as the initial guess for the shape optimization. The first step towards a fully automatic procedure is a shape identification, which we are doing by hand for the moment. The second step we are treating now is a piecewise smooth approximation of the shapes by Bézier curves or patches. Let $\rho^{\text{opt}} \in \mathcal{Q}$ be an optimized discretized material distribution. Recall that it is not a strictly 0-1 function. Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ denote vectors of Bézier parameters of the shapes $\alpha_1(\mathbf{p}_1), \ldots, \alpha_n(\mathbf{p}_n)$ which form the interface between the air and ferromagnetic subdomains $\Omega_0(\alpha_1, \ldots, \alpha_n)$ and $\Omega_1(\alpha_1, \ldots, \alpha_n)$, respectively, i.e. $\Omega_1 \subset \Omega_{\text{d}}$, $\overline{\Omega} = \overline{\Omega_0} \cup \overline{\Omega_1}$ and $\Omega_0 \cap \Omega_1 = \emptyset$. Let further $\underline{\mathbf{p}_i}$ and $\overline{\mathbf{p}_i}$ denote the lower and upper bounds, respectively, and let $\mathcal{P} := \left\{(\mathbf{p}_1, \ldots, \mathbf{p}_n) \mid \underline{\mathbf{p}_i} \le \mathbf{p}_i \le \overline{\mathbf{p}_i} \text{ for } i = 1, \ldots, n\right\}$ be the set of admissible Bézier parameters. We solve the following least square fitting problem:

$$\min_{(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathcal{P}} \int_{\Omega_{\text{d}}} \left(\rho^{\text{opt}} - \chi(\Omega_1(\alpha_1(\mathbf{p}_1), \ldots, \alpha_n(\mathbf{p}_n)))\right)^2 d\mathbf{x}, \tag{4}$$

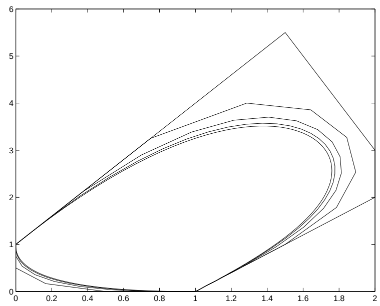


Figure 1: Approximation of Bézier shapes by the refined control polygon

where $\chi(\Omega_1)$ is the characteristic function of Ω_1.

When solving (4) numerically, one encounters the problem of intersection of the Bézier shapes with the mesh on which ρ^{opt} is elementwise constant. In order to avoid it we use the property that the Bézier control polygon converges quite fast, see Fig. 1, to the shape under a refinement procedure. Then the integration in (4) is replaced by a sum over the elements and we deal with intersecting the mesh with a polygon. Note that our least square functional is not twice differentiable whenever a shape touches the grid. This is still acceptable for the quasi-Newton optimization method that we apply.

4 Multilevel Shape Optimization

With the previous notation, the shape optimization problem under consideration is as follows:

$$\text{Find } (\mathbf{p}_1^*, \dots, \mathbf{p}_n^*) \in \mathcal{P} : J(u(\mathbf{p}_1^*, \dots, \mathbf{p}_n^*)) \le J(u(\mathbf{p}_1, \dots, \mathbf{p}_n)) \quad \forall (\mathbf{p}_1, \dots, \mathbf{p}_n) \in \mathcal{P} \tag{5}$$

subject to the 2-dimensional nonlinear magnetostatics

$$\begin{aligned} &\int_{\Omega_0(\alpha_1(\mathbf{p}_1),\dots,\alpha_n(\mathbf{p}_n))} \nu_0 \, \mathbf{grad}(u(\mathbf{p}_1, \dots, \mathbf{p}_n)) \cdot \mathbf{grad}(v) \, d\mathbf{x} \\ &+ \int_{\Omega_1(\alpha_1(\mathbf{p}_1),\dots,\alpha_n(\mathbf{p}_n))} \nu(0) \, \mathbf{grad}(u(\mathbf{p}_1, \dots, \mathbf{p}_n)) \cdot \mathbf{grad}(v) \, d\mathbf{x} = \int_\Omega Jv \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega), \end{aligned} \tag{6}$$

Concerning the finite element discretization throughout the optimization, we use a moving grid approach. The control design nodes interpolate the Bézier shape and the remaining grid nodes displacements are given by solving an artificial discretized linear elasticity problem with the nonzero Dirichlet boundary condition along the design shape. Then, we develop a fairly similar adjoint algorithm for the shape sensitivity analysis as in case of topology optimization.

Perhaps, the main reason for solving the coarse topology optimization as a preprocessing is that we get rid of a large number of design variables in cases of fine discretized topology optimization. Once we have a good initial shape design, we will proceed the shape optimization in a multilevel way in order to speed up the algorithm as much as possible. We propose to couple the outer quasi-Newton method with the nested conjugate gradient method preconditioned by a geometric multigrid (PCG), as depicted in the algorithm below, in which $\mathbf{A}^l(\mathbf{p}_1, \dots, \mathbf{p}_n)$ denotes the reluctivity matrix assembled at the l-th level.

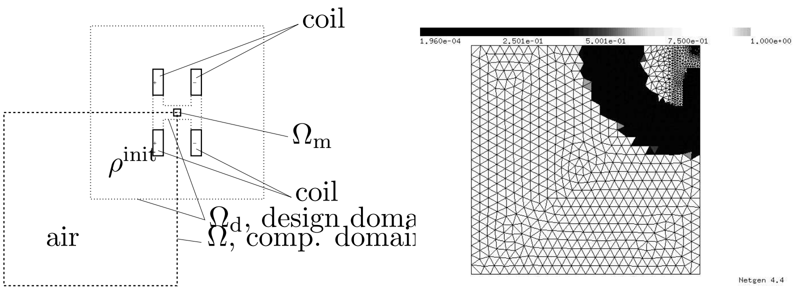


Figure 2: Topology optimization: **(a)** initial design; **(b)** coarsely optimized design ρ^{opt}

Table 1: Multilevel shape optimization

level	design variables	outer Newton iterations	state variables	nested CG iterations	total time (numerical differentiation)
1	19	7	1098		27s
2	40	8	4240	3	3min 9s
3	82	8	16659	4–5	29min 14s
4	166	8	66037	4–5	3h 37min 42s

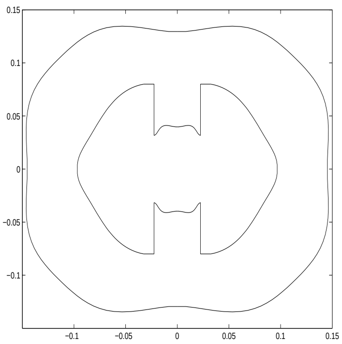
Newton iterations coupled with nested multigrid PCG

Given $\mathbf{p}_1^{\mathrm{init}}, \dots, \mathbf{p}_n^{\mathrm{init}}$
Discretize at the first level $\longrightarrow h^1, \mathbf{A}^1(\mathbf{p}_1^{\mathrm{init}}, \dots, \mathbf{p}_n^{\mathrm{init}})$
Solve by a quasi-Newton method and the nested direct solver $\longrightarrow \mathbf{p}_1^1, \dots, \mathbf{p}_n^1$
Store the first level preconditioner $\mathbf{C}^1 := \left[\mathbf{A}^1(\mathbf{p}_1^1, \dots, \mathbf{p}_n^1)\right]^{-1}$
for $l = 2, \dots$ **do**
  Refine $h^{l-1} \longrightarrow h^l$
  Prolong $\mathbf{p}_1^{l-1}, \dots, \mathbf{p}_n^{l-1} \longrightarrow \mathbf{p}_1^{l,\mathrm{init}}, \dots, \mathbf{p}_n^{l,\mathrm{init}}$
  Solve by a quasi-Newton method and the nested multigrid solver $\longrightarrow \mathbf{p}_1^l, \dots, \mathbf{p}_n^l$
  Store the l–th level preconditioner $\mathbf{C}^l$
end for

5 Numerical Results

We consider a problem depicted in Fig. 2 **(a)**. The aim is to find a distribution of the ferromagnetic core so that the field $\mathbf{grad}(u)$ is homogeneous in the area Ω_{m}. The coarsely optimized topology of the quarter of the geometry is depicted in Fig. 2 **(b)**. Then, we approximated the boundary of the black domain by three Bézier curves and proceeded with the multilevel shape optimization. The performance of the algorithm can be seen from Table 1. The computational times should be divided by the number of design variables, as we have not implemented the adjoint method for the shape sensitivity analysis yet. The final result is depicted in Fig. 3 **(a)** and it is very similar to the existing geometry of the so-called O-Ring electromagnet, see Fig. 3 **(b)**.

So, we have just mathematically approved the well-known fact that round geometries of electromagnets are superior. Next, we will use the machinery e.g. to an optimal design of a shielding of an AC transformer, where the engineers have hardly any knowledge about a proper result. Concerning the mathematics, we will employ an algebraic multigrid preconditioner, an adaptivity



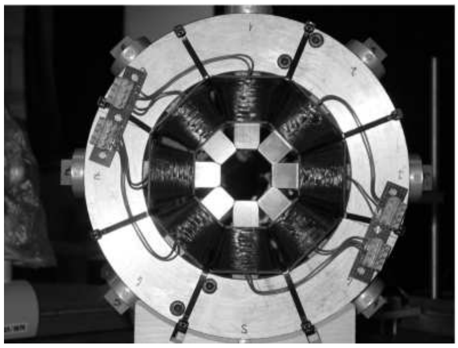


Figure 3: Multilevel shape optimization: **(a)** optimized geometry; **(b)** the O-Ring electromagnet

refinement with respect to the cost functional and 3-dimensional problems.

References

[1] Bendsøe, M.P.: Optimization of Structural Topology, Shape and Material. Springer, Berlin, Heidelberg (1995)

[2] Céa, J., Garreau, S., Guillaume, P., Masmoudi, M.: The shape and topological optimizations connection. Comput. Methods Appl. Mech. Eng. **188**, 713–726 (2000)

[3] Haslinger J., Neittaanmäki P.: Finite Element Approximation for Optimal Shape, Material and Topology Design. Wiley, Chinchester (1997)

[4] Lukáš, D.: Shape optimization of homogeneous electromagnets. In: van Rienen, U., Günther, M., Hecht, D. (eds.) Scientific Computing in Electrical Engineering 2000, Lect. Notes Comp. Sci. Engrg. **18**, pp. 145–152 (2001)

[5] Lukáš, D.: On solution to an optimal shape design problem in 3-dimensional magnetostatics. Appl. Math. **49**:5, 24 pp. (2004)

[6] Lukáš, D.: An integration of optimal topology and shape design for magnetostatics. In: Scientific Computing in Electrical Engineering 2004, to appear

[7] Tang, P.-S., Chang, K.-H.: Integration of topology and shape optimization for design of structural components. Struct. Multidisc. Optim. **22**, 65–82 (2001)