Numerical stability analysis of Krylov subspace methods - an adventure with unexpected consequences

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The contribution is based on a joint work with

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Abstract

Until recently, rounding error analysis for iterative methods for solving large linear algebraic systems, and for Krylov subspace methods in particular, was considered not well developed. [Higham, N.,J., *Accuracy and Stability of Numerical Algorithms*, SIAM, Philadelphia, 2002, Section 17] gives two main reasons. First, in many applications accuracy requirements are modest and can be satisfied without difficulty. Second, rounding error analysis for iterative methods is inherently more difficult than for direct methods, and the results are harder to interpret.

Within the last decade, an interest in rounding errors analysis for Krylov subspace methods has, however, considerably increased. It was observed that rounding errors can delay convergence and significantly increase the cost of computation even *when stopping at a modest accuracy level.* The delay phenomenon in the conjugate gradient method (CG) was linked to the finite precision behaviour of the Lanczos method, and it was understood. Estimates of the energy norm of the error in CG were justified by a mathematically rigorous rounding error analysis. Results on limiting accuracy of various methods gave a guidance for their implementations. On the other hand, rounding error analysis for seemingly attractive implementations used in practice revealed their hidden potentially dangerous numerical instabilities.

In our contribution we illustrate on two recent developments that rounding error analysis can have a surprising impact in other areas of research. In the first part we show that Gauss-Christoffel quadrature for a small number of quadrature nodes can be highly sensitive to small changes in the distribution function, and we describe the relationship of the sensitivity of Gauss-Christoffel quadrature to the convergence properties of the CG and Lanczos methods in finite precision arithmetic. In the second part we explain how rounding error analysis of the generalized minimal residual method (GMRES) has resulted in the core problem decomposition of ortogonally invariant linear algebraic approximation problems. In both cases, rounding error analysis for an iterative method has inspired questions in the areas far from the original problem. Their solution has offered a new insight to mathematical foundations of the classical problems.