On the relation between SUPG stabilization and GMRES convergence for a convection-diffusion model problem

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joint work with

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SNA08, Liberec, January 29, 2008.

1. The convection-diffusion model problem

We consider a model problem that has been studied by many authors, e.g. [Fischer, Ramage, Sylvester, Wathen - 1999], [Ernst - 2000], [Elman, Ramage - 2001, 2002], [Liesen, Strakoš - 2005]:

$$\begin{aligned} -\nu\nabla^2 u + w\cdot\nabla u &= 0 \quad \text{in} \quad \Omega = (0,1)\times(0,1), \\ u &= g \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where

- ν: scalar diffusion coefficient
- $w = [0,1]^T$: velocity field (wind)
- g : given by discontinuous inflow (Raithby) boundary conditions

$$\begin{aligned} u(x,0) = u(1,y) &= 1 \quad \text{for } \frac{1}{2} \le x \le 1 \text{ and } 0 \le y < 1, \\ u(x,y) &= 0 \quad \text{elsewhere on } \partial\Omega \end{aligned}$$

1. The convection-diffusion model problem

We use Streamline Upwind Petrov Galerkin (SUPG) discretization ([Brooks, Hughes - 1979]) to suppress non-physical oscillations. The idea is to add an artificial diffusion term in the local flow direction. The modified coefficient matrix C has the form

$$C_{i,j} = \nu \langle \nabla \phi_j, \nabla \phi_i \rangle + \langle w \cdot \nabla \phi_j, \phi_i \rangle + \hat{\delta} \, \langle w \cdot \nabla \phi_j, w \cdot \nabla \phi_i \rangle,$$

where ϕ_j, $j = 1, 2, \ldots$ are the bilinear finite element nodal basis functions for an N by N grid with spacing $h = 1/(N+1)$ and $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product on Ω.

Here, $\hat{\delta}$ is the stabilization parameter. Stabilization parameters are chosen as

$$\hat{\delta} = \frac{\delta \, h}{\|w\|},$$

for some positive tuning parameter $0 < \delta < 1$.

1. The convection-diffusion model problem

For linear finite element discretization in a one-dimensional constant coefficient problem it can be shown that the choice

$$\delta_0 = \frac{1}{2}\left(\coth(P_h) - \frac{1}{P_h}\right), \quad \text{i.e.} \quad \hat{\delta}_0 = \frac{h}{2\|w\|}\left(\coth(P_h) - \frac{1}{P_h}\right)$$

where $P_h \equiv h/(2\nu)$ is the mesh Peclet number, yields the exact solution at the node points [Brooks, Hughes - 1982].

In two dimensions one frequently uses the simplified value [Elman, Ramage - 2001, 2002]

$$\delta_0 = \frac{1}{2}\left(1 - \frac{1}{P_h}\right), \quad \text{i.e.} \quad \hat{\delta}_0 = \frac{h}{2\|w\|}\left(1 - \frac{1}{P_h}\right)$$

In the following we call this choice of stabilization optimal stabilization. Note that with large mesh Peclet numbers we have $\delta_0 \approx 1/2$.

1. The convection-diffusion model problem

Consider an example where we use

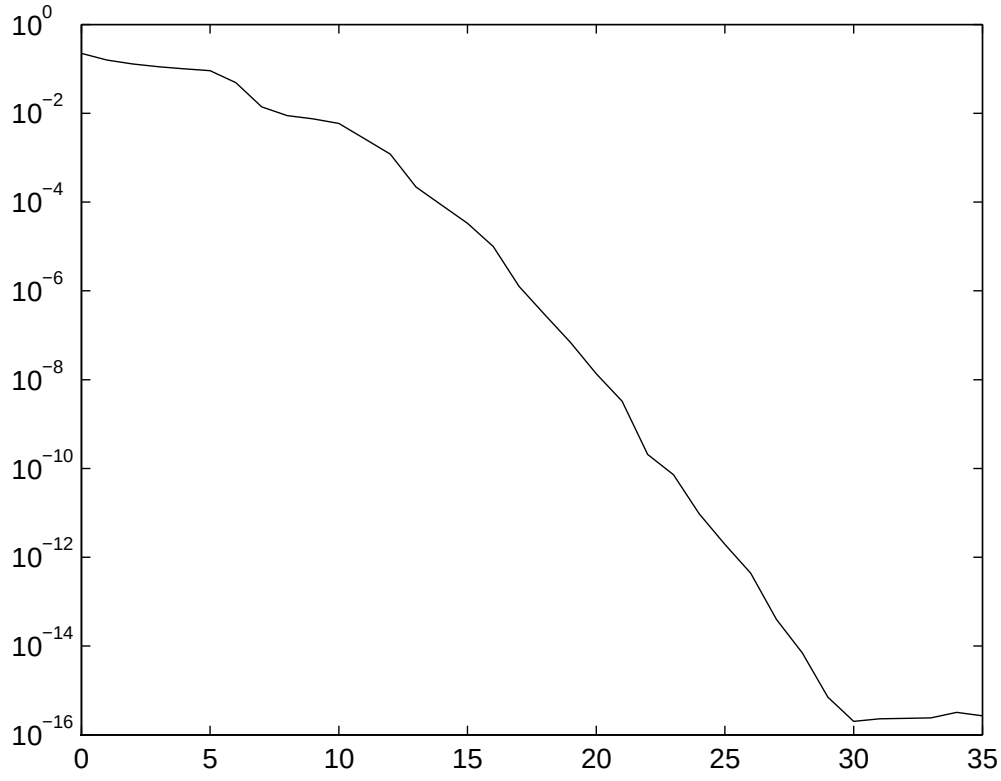
- A 6×6 grid (linear system dimension 36)
- The small diffusion coefficient $\nu = 0.00025$
- Hence large Peclet number $P_h = 285$ and optimal tuning parameter $\delta_0 = 0.498$ close to 0.5
- GMRES to solve the resulting linear systems
- The zero initial guess

In addition to the optimal tuning parameter $\delta_0 = 0.498$ we consider also

$$\delta_1 = 0.398, \quad \delta_2 = 0.298, \quad \delta_2^+ = 0.698, \quad \delta_3^+ = 0.798.$$

1. The convection-diffusion model problem

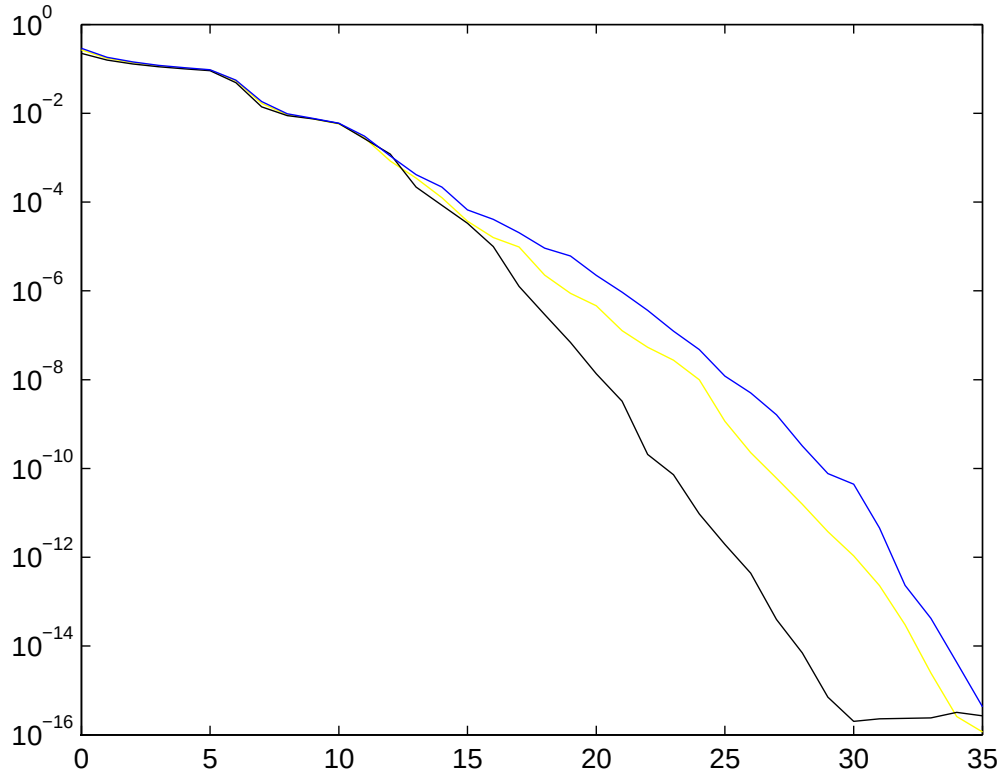
GMRES for the convection-diffusion model problem



Black: δ_0.

1. The convection-diffusion model problem

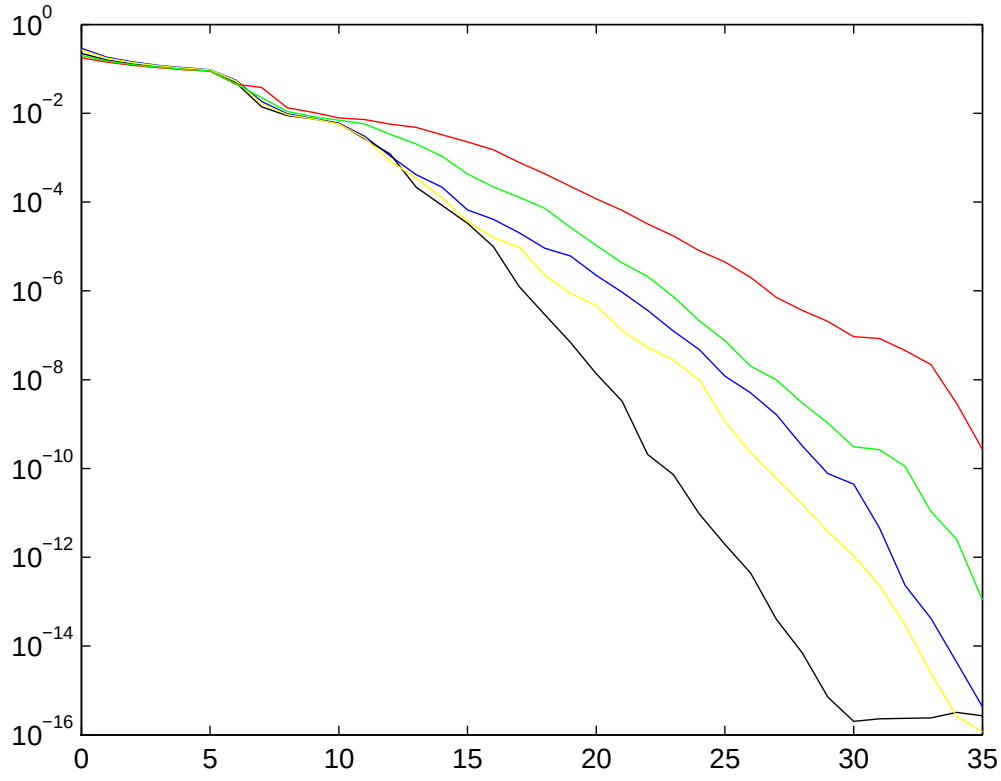
GMRES for the convection-diffusion model problem



Black: δ_0, yellow: δ_2^+, blue: δ_3^+.

1. The convection-diffusion model problem

GMRES for the convection-diffusion model problem



Black: δ_0, red: δ_2, green: δ_1, yellow: δ_2^+, blue: δ_3^+.

1. The convection-diffusion model problem

The fact that GMRES converges worse for worse stabilization shows a fascinating and rather lucky relation between discretization and behavior of an iterative solver.

This fact has been observed in [Fischer, Ramage, Silvester, Wathen - 1999], but has not been fully explained. (Explanation of the behavior of GMRES for δ_0 has been more or less completed [Liesen, Strakoš - 2005], [DT - 2005].)

In this talk we offer an explanation. In particular, we will present lower bounds for $\delta \neq \delta_0$.

2. Properties of the system matrices

With the discrete sine transformation from [Liesen, Strakoš - 2005], system matrices $\mathbf{A}$ are block diagonal with tridiagonal Toeplitz blocks of dimension $N \times N$:

$$\mathbf{A}\,x = \begin{pmatrix} \mathbf{T}_1 & & & \\ & \mathbf{T}_2 & & \\ & & \ddots & \\ & & & \mathbf{T}_N \end{pmatrix} x = \begin{pmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(N)} \end{pmatrix} = b,$$

where

$$\mathbf{T}_j = \begin{pmatrix} \lambda_j & \mu_j & & & \\ \gamma_j & \lambda_j & \mu_j & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_j & \lambda_j & \mu_j \\ & & & \gamma_j & \lambda_j \end{pmatrix}.$$

2. Properties of the system matrices

Assuming $P_h = h/(2\nu) \gg 1$ we can write the entries in simplified form as

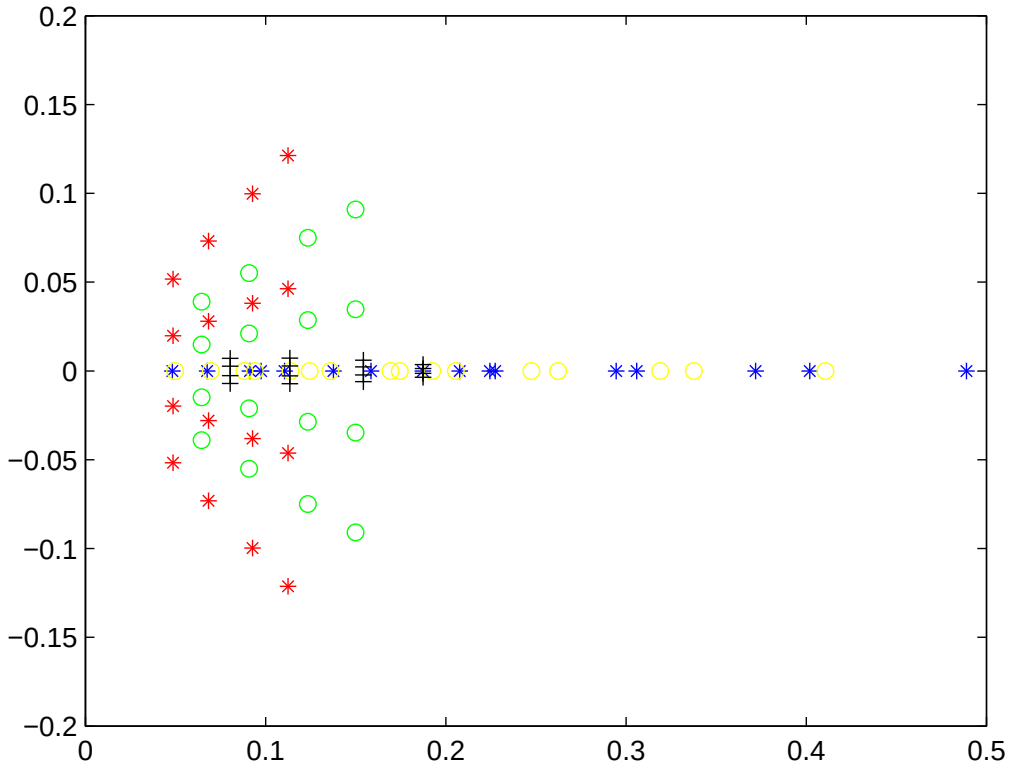
$$\begin{aligned}
\lambda_j &= 2\delta h c_j + \mathcal{O}(\nu), \\
\mu_j &= -\delta h c_j + \frac{h}{2} c_j + \mathcal{O}(\nu), \\
\gamma_j &= -\delta h c_j - \frac{h}{2} c_j + \mathcal{O}(\nu),
\end{aligned}$$

where $c_j = 2 + \cos jh\pi$. Note that for $\delta_0 \approx 1/2$ we have $\mu_j \approx 0$.

The eigenvalues of the whole matrix are given by

$$\sigma_{jk} = \lambda_j + 2\sqrt{\mu_j \gamma_j} \cos kh\pi, \quad 1 \le j, k \le N.$$

2. Properties of the system matrices



Spectra for δ_0(black cross), δ_2(red star), δ_1(green circle), δ_2^+(yellow circle) and δ_3^+(blue star) with $N = 4$.

2. Properties of the system matrices

The eigenvectors of T_j are given by

$$w_{k,j} = \text{diag}\,(\zeta_\delta^{-\frac{1}{2}}, \zeta_\delta^{-1}, \ldots, \zeta_\delta^{-\frac{N}{2}}) \begin{pmatrix} \sin kh\pi \\ \vdots \\ \sin Nkh\pi \end{pmatrix}, \qquad \zeta_\delta = \frac{\mu_j}{\gamma_j} = \frac{1-\frac{1}{2\delta}}{1+\frac{1}{2\delta}} + \mathcal{O}(\nu),$$

with $1 \leq k \leq N$.

Lemma 1 The condition number κ of the eigenvector basis formed of eigenvectors $w_{k,j}$, $1 \leq k, j \leq N$, equals

$$\kappa = \left| \frac{1-\frac{1}{2\delta}}{1+\frac{1}{2\delta}} \right|^{\frac{1-N}{2}}.$$

For $\delta_0 \approx 1/2$ we obtain a huge κ. The matrix with the *worst* conditioned eigenvector basis gives the *best* GMRES convergence !

3. Analysis for the $(n-1)$st step

Let A_δ be the system matrix with tuning parameter δ. We will use the well-known equality (e.g. [Ipsen - 2000])

$$\|r_k\| = \left\| e_1^T \cdot [b_\delta, A_\delta b_\delta, \ldots, A_\delta^k b_\delta]^+ \right\|^{-1}$$

and an analysis similar to [Liesen, Tichy - 2004]. For $k = n-1$ the Moore-Penrose pseudo-inverse translates to

$$[b_\delta, A_\delta b_\delta, \ldots, A_\delta^{n-1} b_\delta]^+ = [b_\delta, A_\delta b_\delta, \ldots, A_\delta^{n-1} b_\delta]^{-1}.$$

Let the spectral decomposition of A_δ be

$$A_\delta = W_\delta \Sigma_\delta W_\delta^{-1},$$

and write the right hand side in the eigenvector basis as

$$b_\delta = \sum_{j=1}^{n} \theta_i w_i.$$

3. Analysis for the $(n-1)$st step

We have

$$A_\delta^k\, b_\delta = W_\delta \Theta_\delta (\sigma_1^k, \ldots, \sigma_n^k)^T,$$

where $\Theta_\delta =$ diag $(\theta_1, \ldots, \theta_n)$. Hence

$$[b_\delta, A_\delta b_\delta, \ldots, A_\delta^{n-1} b_\delta] = W_\delta \Theta_\delta M_\delta,$$

where M_δ is the Vandermonde matrix who's ith row is $(1, \sigma_i, \sigma_i^2, \ldots, \sigma_i^n)$.

For the one but last residual norm we obtain

$$\|r_{n-1}\| = \left\| e_1^T \cdot [b_\delta, A_\delta b_\delta, \ldots, A_\delta^k b_\delta]^{-1} \right\|^{-1} = \left\| e_1^T \cdot M_\delta^{-1} \Theta_\delta^{-1} W_\delta^{-1} \right\|^{-1}.$$

First we take a closer look at the matrix $\Theta_\delta^{-1} W_\delta^{-1}$. This matrix is also block diagonal, $\Theta_\delta^{-1} W_\delta^{-1} =$ diag$(\Theta_1^{-1} W_1^{-1}, \ldots, \Theta_N^{-1} W_N^{-1})$.

3. Analysis for the $(n-1)$st step

The matrix W_δ^{-1} has block diagonal structure; every single block contains an eigenvector matrix of the form

$$W_j = \text{diag}\,(\zeta_\delta^{-\frac{1}{2}}, \zeta_\delta^{-1}, \ldots, \zeta_\delta^{-\frac{N}{2}})V, \quad V_{i,j} = \sin(jih\pi), \quad \zeta_\delta = \frac{1-\frac{1}{2\delta}}{1+\frac{1}{2\delta}} + \mathcal{O}(\nu).$$

Let the part of the right hand side corresponding to block j have the entries $\rho_1, \ldots, \rho_N$. We have, exploiting $V^{-1} = \frac{2}{N+1}V$,

$$W_j^{-1}\begin{pmatrix}\rho_1\\ \ldots \\ \rho_N\end{pmatrix} = \frac{2}{N+1}V\,\text{diag}\,(\zeta_\delta^{\frac{1}{2}}, \zeta_\delta, \ldots, \zeta_\delta^{\frac{N}{2}})\begin{pmatrix}\rho_1\\ \ldots \\ \rho_N\end{pmatrix} = \frac{2}{N+1}\sum_{i=1}^{N}\rho_i\zeta_\delta^{\frac{i}{2}}\begin{pmatrix}\sin ih\pi\\ \ldots \\ \sin iNh\pi\end{pmatrix}.$$

Hence if Θ_j is the diagonal matrix whose entries are the components of the right hand side in the eigenvector basis W_j, then

3. Analysis for the $(n-1)$st step

$$\begin{aligned}\Theta_j^{-1}W_j^{-1} &= \text{diag}\left(\sum_{i=1}^{N}\rho_i\zeta_\delta^{\frac{i}{2}}\sin ih\pi,\ldots,\sum_{i=1}^{N}\rho_i\zeta_\delta^{\frac{i}{2}}\sin iNh\pi\right)^{-1} V\,\text{diag}\,(\zeta_\delta^{\frac{1}{2}},\zeta_\delta,\ldots,\zeta_\delta^{\frac{N}{2}})\\ &= \frac{1}{\rho_1}\hat{V}_\delta^{(j)}\text{diag}\,(1,\zeta_\delta^{\frac{1}{2}},\zeta_\delta,\ldots,\zeta_\delta^{\frac{N-1}{2}}),\end{aligned}$$

where

$$(\hat{V}_\delta^{(j)})_{i,k} \equiv \frac{\rho_1\sin(ikh\pi)}{\sum_{m=1}^{N}\zeta_\delta^{\frac{m-1}{2}}\rho_m\sin(imh\pi)}.$$

Note that $|\zeta_\delta| = |\frac{1-\frac{1}{2\delta}}{1+\frac{1}{2\delta}}| \ll 1$ and hence the denominators in $(\hat{V}_\delta^{(j)})_{i,j}$ are restricted to the first few summands. With Raithby boundary conditions the effect is even emphasized because $\rho_1 \gg \rho_i,\ i \neq 1,$ in many blocks $\Rightarrow$

$$(\hat{V}_\delta^{(j)})_{i,k} \approx \frac{\rho_1\sin(ikh\pi)}{\rho_1\sin(ih\pi)}.$$

3. Analysis for the $(n-1)$st step

$$\Theta_\delta^{-1} W_\delta^{-1} \approx \begin{pmatrix} \frac{1}{\rho_1^{(1)}}\mathbf{E} & & & \\ & \frac{1}{\rho_1^{(2)}}\mathbf{E} & & \\ & & \ddots & \\ & & & \frac{1}{\rho_1^{(N)}}\mathbf{E} \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

where $\frac{1}{\rho_1^{(j)}}$ is the first component of the right hand side corresponding to block j.

From this form it can be seen that the matrix $\Theta_\delta^{-1} W_\delta^{-1}$ is only slightly dependent on the choice of δ. In particular, it seems the influence of the conditioning of the eigenvector matrix W_δ is eliminated by taking into account the right hand side through the matrix Θ_δ.

3. Analysis for the $(n-1)$st step

Having analyzed $\Theta_\delta^{-1} W_\delta^{-1}$ in

$$\|r_{n-1}\| = \left\| e_1^T \cdot M_\delta^{-1} \Theta_\delta^{-1} W_\delta^{-1} \right\|^{-1},$$

we proceed with $e_1^T \cdot M_\delta^{-1}$. It is well known there exist exact formulaes for the entries of inverses of Vandermonde matrices (e.g. [Muir - 1960]),

$$e_1^T \cdot M_\delta^{-1} = (-1)^{n-1} \left(\frac{\prod_{k\neq 1} \sigma_k}{\prod_{k\neq 1}(\sigma_1 - \sigma_k)}, \frac{\prod_{k\neq 2} \sigma_k}{\prod_{k\neq 2}(\sigma_2 - \sigma_k)}, \ldots, \frac{\prod_{k\neq n} \sigma_k}{\prod_{k\neq n}(\sigma_n - \sigma_k)} \right).$$

The behavior of the expression on the right can be explained with the following two lemma's.

Lemma 2. The difference between two eigenvalues of block T_j is

$$\sigma_{jk} - \sigma_{jl} = c_1 \left(2 + \cos jh\pi\right) \sqrt{1 - \frac{1}{4\delta^2}} \left(\cos kh\pi - \cos lh\pi\right).$$

$\Rightarrow$ for $\delta_0 \approx 1/2$ we have $\sigma_{jk} - \sigma_{jl} \approx 0$ and $\left| \frac{\prod_{k\neq i} \sigma_k}{\prod_{k\neq i}(\sigma_i - \sigma_k)} \right| \to \infty$.

3. Analysis for the $(n-1)$st step

Lemma 3. The product of a pair of eigenvalues σ_{jk} and σ_{jl} satisfying $k+l=N+1$ equals

$$\sigma_{jk}\cdot\sigma_{jl} = c_1^2\,(2+\cos jh\pi)^2\left(1-\left(1-\frac{1}{4\delta^2}\right)\cos^2 kh\pi\right).$$

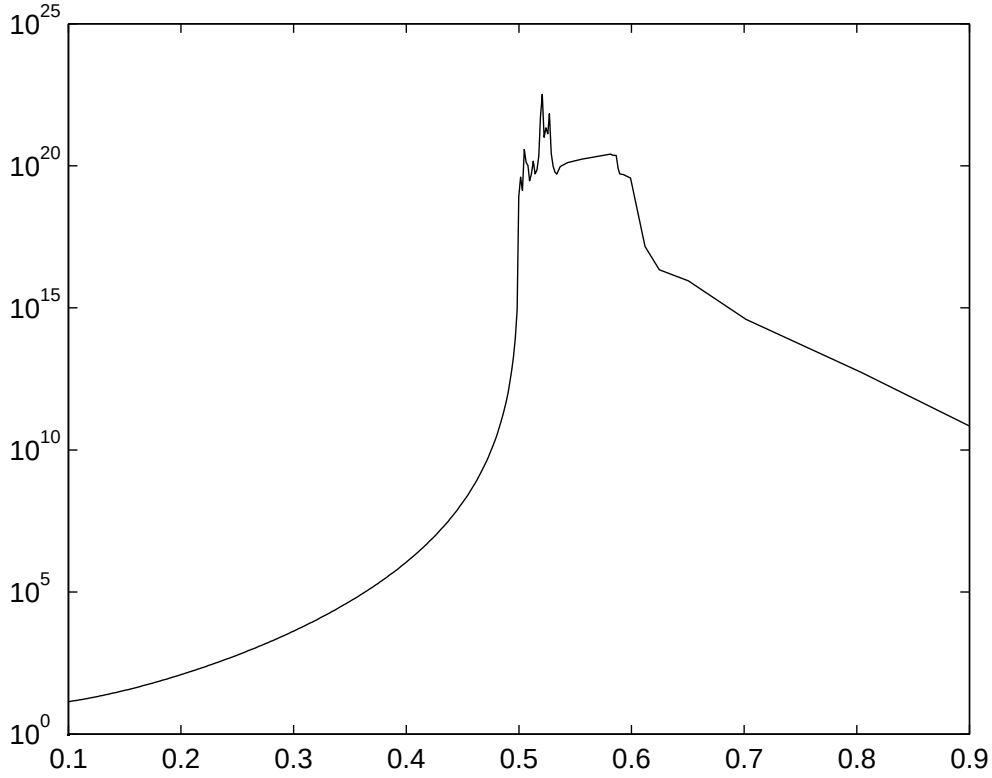
The function $|\sigma_{jk}(\delta)\cdot\sigma_{jl}(\delta)|$ is slightly decreasing in δ.

We obtain for the function

$$m(\delta)\equiv\|e_1^T\cdot M_\delta^{-1}\| = \left\|\left(\frac{\prod_{k\neq 1}\sigma_k}{\prod_{k\neq 1}(\sigma_1-\sigma_k)},\ldots,\frac{\prod_{k\neq n}\sigma_k}{\prod_{k\neq n}(\sigma_n-\sigma_k)}\right)\right\|.$$

the following behavior:

3. Analysis for the $(n-1)$st step



The function $m(\delta) \equiv \|e_1^T \cdot M_\delta^{-1}\|$ with $N = 6$.

3. Analysis for the $(n-1)$st step

All together, we have

$$\|r_{n-1}\|^{-1} = \| \, e_1^T \cdot M_\delta^{-1} \Theta_\delta^{-1} W_\delta^{-1} \|$$

$$\approx \left\| e_1^T \cdot M_\delta^{-1} \begin{pmatrix} \frac{1}{\rho_1^{(1)}}\mathbf{E} & & & \\ & \frac{1}{\rho_1^{(2)}}\mathbf{E} & & \\ & & \ddots & \\ & & & \frac{1}{\rho_1^{(N)}}\mathbf{E} \end{pmatrix} \right\|, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

We can show that convergence is indeed governed by $\|e_1^T \cdot M_\delta^{-1}\|$. Hence the *main* reason for convergence delay by step $n-1$ with non-optimal stabilization are the relatively large distances between eigenvalues.

An example where eigenvalues do govern convergence behavior even when eigenvectors are extremely ill-conditioned, i.e. matrices are highly non-normal!

4. Bounds for the general step

In practice GMRES processes will often have converged before step $n-1$. For a step $k-1 < n-1$ we must deal with

$$\|r_k\|^{-1} = \| \, e_1^T \cdot [b_\delta, A_\delta b_\delta, \ldots, A_\delta^{k-1} b_\delta]^+ \| = \left\| e_1^T \cdot \left(W_\delta \Theta_\delta M_\delta \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\|, \quad (1)$$

and explicit formulae for the entries of the Vandermonde inverse M_δ^{-1} cannot be exploited.

As a tool to explain the behavior of (1) we will use the UL-decomposition of $M^{-1} = UL$ [Golub, van Loan - p. 186]:

$$L_{i,j} = \prod_{l=1, l \neq j}^{i} (\sigma_j - \sigma_l)^{-1} \quad \Rightarrow$$

4. Bounds for the general step

$$L = \begin{pmatrix} 1 & & & \\ \frac{1}{\sigma_1-\sigma_2} & \frac{1}{\sigma_2-\sigma_1} & & \\ \frac{1}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)} & \frac{1}{(\sigma_2-\sigma_1)(\sigma_2-\sigma_3)} & \frac{1}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$U_{i,j} = \epsilon_{j-i}(\sigma_1, \ldots, \sigma_{j-1}), \quad \text{where} \quad \epsilon_k(\sigma_1, \ldots, \sigma_l) = \sum \sigma_{\nu_1} \cdot \sigma_{\nu_2} \ldots \sigma_{\nu_k} \quad \Rightarrow$$

$$U = \begin{pmatrix} 1 & -\sigma_1 & \sigma_1\sigma_2 & -\sigma_1\sigma_2\sigma_3 & \ldots \\ & 1 & -\sigma_2-\sigma_1 & \sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3 & \ldots \\ & & 1 & -\sigma_1-\sigma_2-\sigma_3 & \ldots \\ & & & 1 & \\ & & & & \ddots \end{pmatrix},$$

4. Bounds for the general step

Then

$$\|r_k\|^{-1} = \left\| e_1^T \cdot \left(W_\delta \Theta_\delta M_\delta \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\| = \left\| e_1^T \cdot \left(W_\delta \Theta_\delta L_\delta^{-1} U_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\|$$

$$= \left\| e_1^T \cdot \left(W_\delta \Theta_\delta L_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix} (U_\delta^{(k)})^{-1} \right)^+ \right\| = \left\| e_1^T \cdot U_\delta^{(k)} \left(W_\delta \Theta_\delta L_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\|,$$

where $U_\delta^{(k)}$ denotes the left upper $k \times k$ block of U_δ.

With

$$e_1^T \cdot U_\delta^{(k)} = [1, -\sigma_1, \sigma_1 \cdot \sigma_2, \ldots, (-1)^{k-1} \prod_{j=1}^{k-1} \sigma_j],$$

we have isolated eigenvalue products. To isolate important eigenvalue differences in L_δ we use a scaling matrix $\Pi_\delta^{(k)}$:

4. Bounds for the general step

With an ordering of eigenvalues such that the largest entries of L_δ are on the diagonal, let

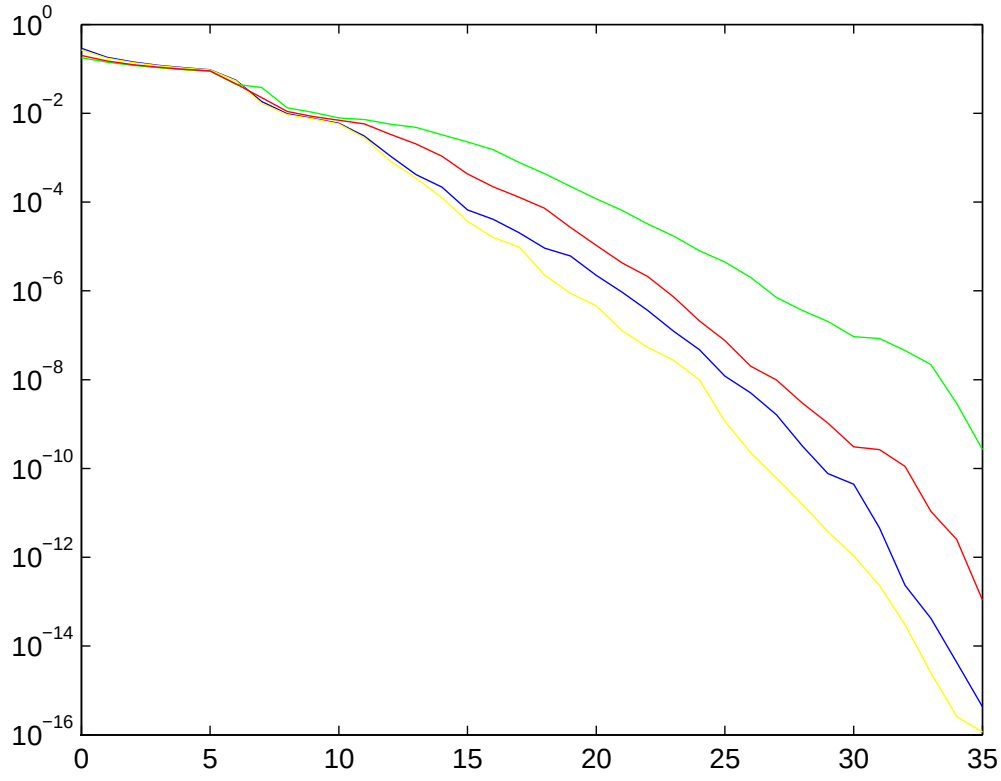
$$\Pi_\delta^{(k)} = \text{diag}(L_\delta) = \left(\text{diag}(1, \prod_{j\neq 1}^{2} |\sigma_1 - \sigma_j|, \prod_{j\neq 2}^{3} |\sigma_2 - \sigma_j|, \ldots, \prod_{j\neq k-1}^{k} |\sigma_{k-1} - \sigma_j|)\right)^{-1}$$

Now, with s_{min} denoting the minimal singular value,

$$\begin{aligned}
\|r_k\| &= \left\| e_1^T \cdot \left(W_\delta \Theta_\delta M_\delta \begin{pmatrix} I_k \\ 0 \end{pmatrix}\right)^+ \right\|^{-1} = \left\| e_1^T \cdot U_\delta^{(k)} \left(W_\delta \Theta_\delta L_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix}\right)^+ \right\|^{-1} \\
&= \left\| e_1^T \cdot U_\delta^{(k)} \Pi_\delta^{(k)} \left(W_\delta \Theta_\delta L_\delta^{-1} \Pi_\delta^{(n)} \begin{pmatrix} I_k \\ 0 \end{pmatrix}\right)^+ \right\|^{-1} \\
&\geq s_{min}\left(W_\delta \Theta_\delta L_\delta^{-1} \Pi_\delta^{(n)} \begin{pmatrix} I_k \\ 0 \end{pmatrix}\right) \left\| e_1^T \cdot U_\delta^{(k)} \Pi_\delta^{(k)} \right\|^{-1} .
\end{aligned}$$

4. Bounds for the general step

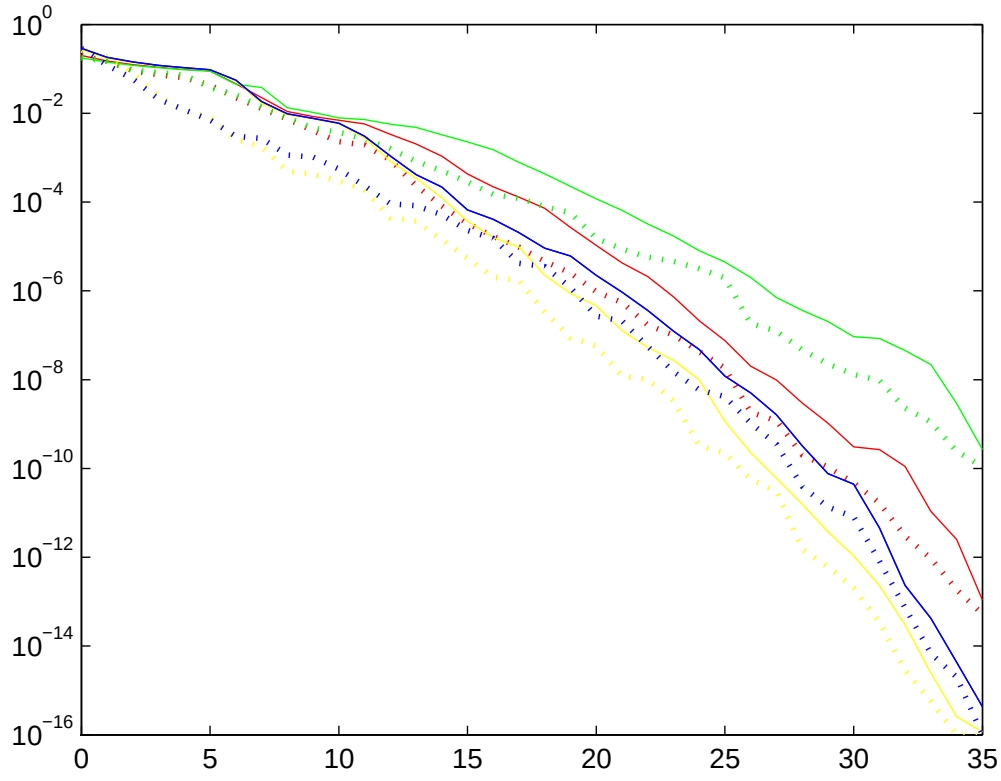
GMRES for the convection-diffusion model problem



Red: δ_2, green: δ_1, yellow: δ_2^+, blue: δ_3^+.

4. Bounds for the general step

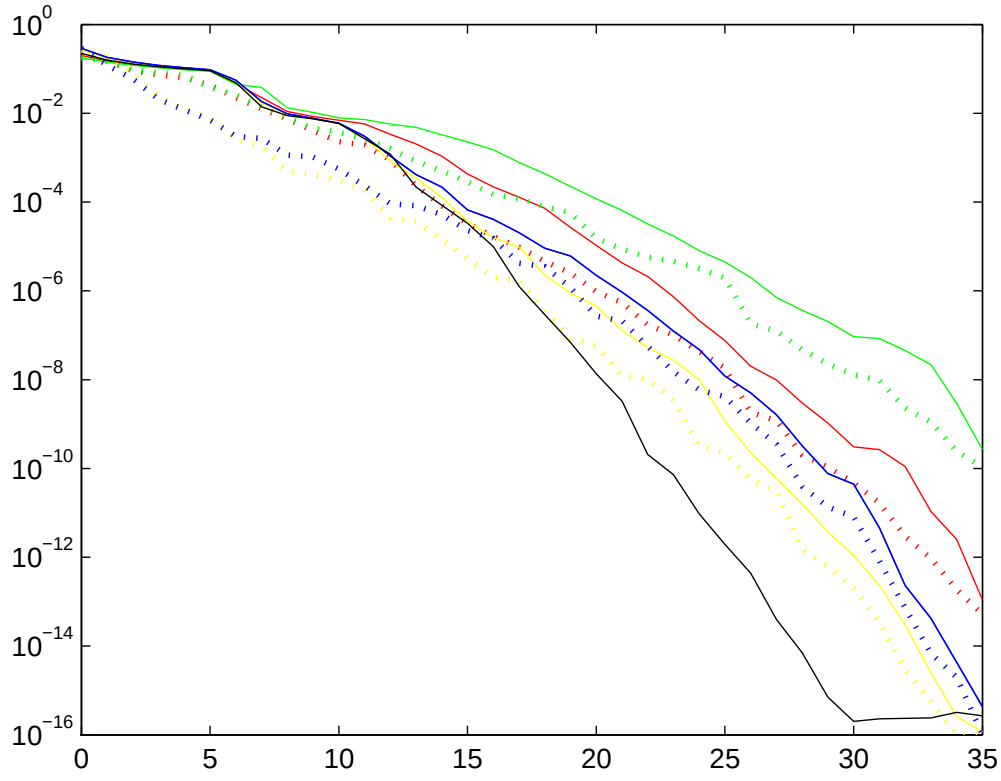
GMRES for the convection-diffusion model problem



Red: δ_2, green: δ_1, yellow: δ_2^+, blue: δ_3^+.

4. Bounds for the general step

GMRES for the convection-diffusion model problem



Black: δ_0, red: δ_2, green: δ_1, yellow: δ_2^+, blue: δ_3^+.

4. Bounds for the general step

We can extend to the less accurate but more descriptive bound

$$\begin{aligned} \|r_k\| &\geq s_{min}\left(W_\delta \Theta_\delta L_\delta^{-1} \Pi_\delta^{(n)} \begin{pmatrix} I_k \\ 0 \end{pmatrix}\right) \left\| e_1^T \cdot U_\delta^{(k)} \, \Pi_\delta^{(k)} \right\|^{-1} \\ &\geq \left\| (\Pi_\delta^{(n)})^{-1} L_\delta \right\|^{-1} \left\| \Theta_\delta^{-1} W_\delta^{-1} \right\|^{-1} \left\| e_1^T \cdot U_\delta^{(k)} \, \Pi_\delta^{(k)} \right\|^{-1} \quad . \end{aligned}$$

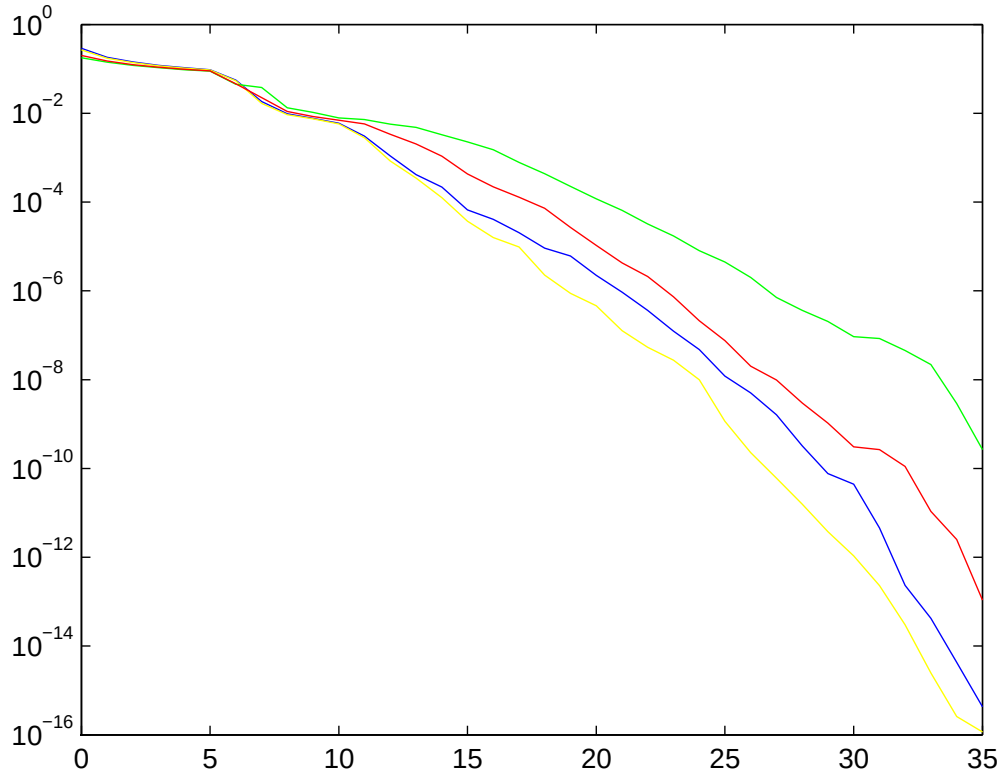
We have seen before that $\Theta_\delta^{-1} W_\delta^{-1}$ is nearly independent from the choice of δ. The matrix $(\Pi_\delta^{(n)})^{-1} L_\delta$ is just a lower triangular matrix with ones on its diagonal and entries with a slow decay when moving away from the diagonal. Essential is that $\left\| e_1^T \cdot U_\delta^{(k)} \, \Pi_\delta^{(k)} \right\|$ where

$$\Pi_\delta^{(k)} = \text{diag}\left(1, \frac{1}{\prod_{j\neq 1}^{2} |\sigma_1 - \sigma_j|}, \frac{1}{\prod_{j\neq 2}^{3} |\sigma_2 - \sigma_j|}, \ldots\ldots, \frac{1}{\prod_{j\neq k-1}^{k} |\sigma_{k-1} - \sigma_j|}\right)$$

is heavily decreasing when growing away from δ_0; hence the main reason for convergence delay are again growing distances between eigenvalues.

4. Bounds for the general step

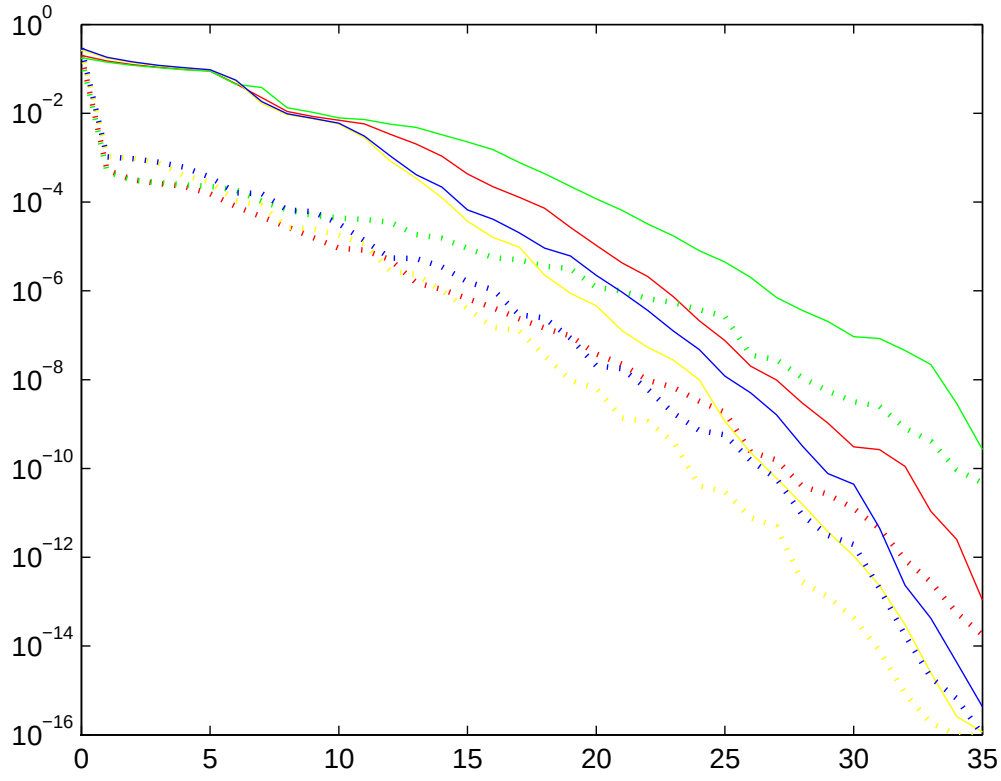
GMRES for the convection-diffusion model problem



Red: δ_2, green: δ_1, yellow: δ_2^+, blue: δ_3^+.

4. Bounds for the general step

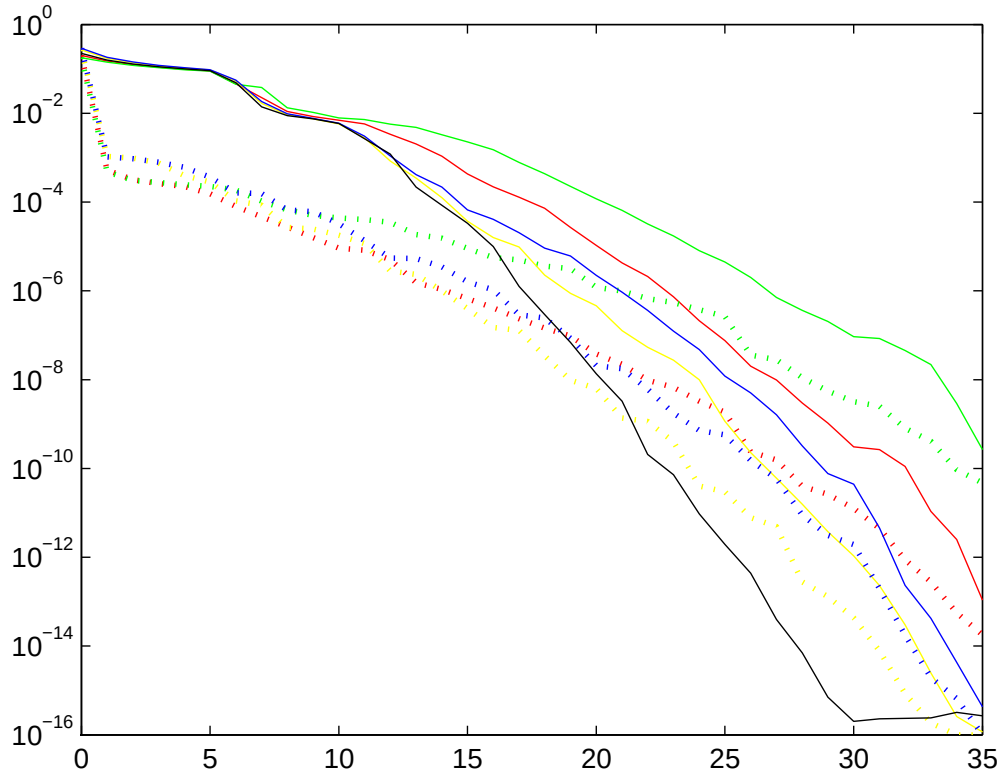
GMRES for the convection-diffusion model problem



Red: δ_2, green: δ_1, yellow: δ_2^+, blue: δ_3^+.

4. Bounds for the general step

GMRES for the convection-diffusion model problem



Black: δ_0, red: δ_2, green: δ_1, yellow: δ_2^+, blue: δ_3^+.

5. Conclusions

Thank you for your attention.

Supported by the project “Information Society” of the Grant Agency of the Academy of Sciences of the Czech Republic under No. 1ET400300415 and by project number KJB100300703 of the Grant Agency of the Academy of Sciences of the Czech Republic