Hierarchical bases of finite elements on rectangles

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The Problem.
We search for a function u in a Hilbert space H such that

$$a(u, v) = f(v)$$

for all $v \in H$.

Bilinear form $a(.,.)$ is elliptic H, $f(.)$ is linear in H. Energetic norm is given $|||v||| = \sqrt{a(v, v)}$.

Discretization of the problem. U_h is a finite-dimensional subspace of H generated by a set of finite element basis functions corresponding to the mesh size h. Find $u_h \in U_h$ such that

$$a(u_h, v) = f(v)$$

for all $v \in U_h$.

A bigger space V_h, $U_h \subset V_h \subset H$. Approximate solution v_h in V_h is given by

$$a(v_h, v) = f(v), \quad v \in V_h.$$

Hierarchy

Hierarchical decomposition of the space V_h

$$V_h = U_h \oplus W_h.$$

We assume that there exists a constant γ ($\gamma < 1$) not dependent on h such that the strengthened Cauchy - Bunyakowski - Schwarz (CBS) inequality

$$|a(u, w)| \leq \gamma |||u||| \, |||w|||$$

holds for all $u \in U_h, w \in W_h$.

In this case we solve a system of equations in a 2×2 block form

$$B \begin{pmatrix} \tilde{u}_W \\ \tilde{u}_U \end{pmatrix} = \begin{pmatrix} B_W & B_{WU} \\ B_{UW} & B_U \end{pmatrix} \begin{pmatrix} \tilde{u}_W \\ \tilde{u}_U \end{pmatrix} = \begin{pmatrix} F_W \\ F_U \end{pmatrix}.$$

If we choose for M the block diagonal of B, the preconditioned system of equations has the condition number

$$\kappa(M^{-1}B) \leq \frac{1+\gamma}{1-\gamma}.$$

CBS constant

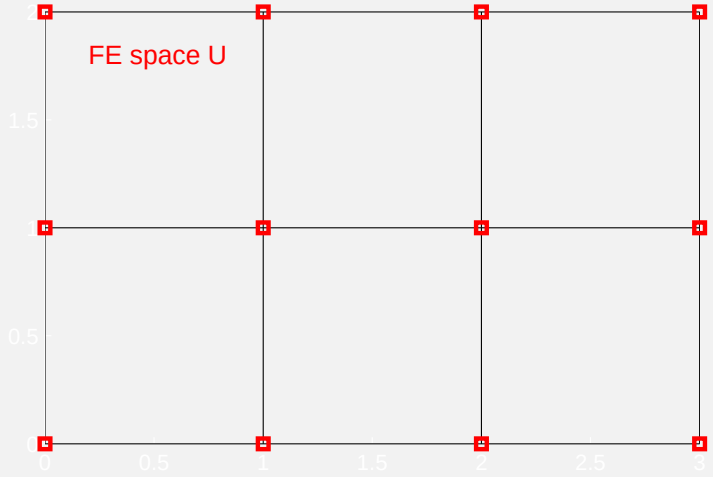
We now consider the space of piecewise bilinear functions with rectangular supports U_h and two types of hierarchical refining:

$$V_{0,h} = U_h \oplus W_{0,h} \qquad \text{a} \qquad V_{1,h} = U_h \oplus W_{1,h}.$$

The space $W_{0,h}$ involves piecewise bilinear functions with smaller supports (h-hierarchy).

The space $W_{1,h}$ involves polynomials of higher order (Q_2) (p-hierarchy).

The numbers of degrees of freedom for $V_{0,h}$ and for $V_{1,h}$ are equal.



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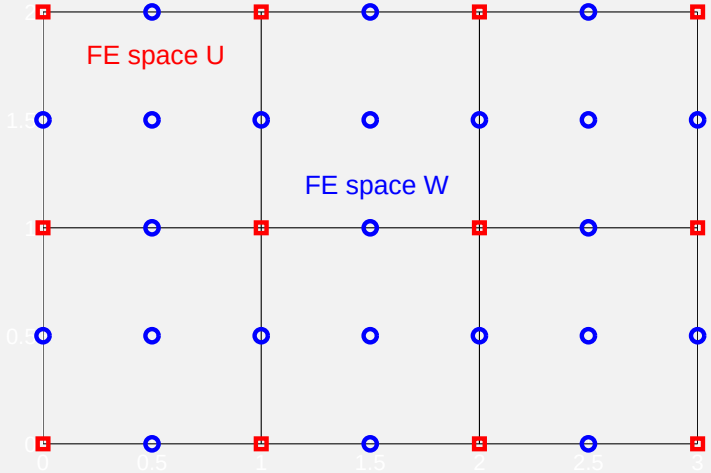


CBS constants

CBS constants γ for h-hierarchy and for p-hierarchy are

$$\gamma_0 \leq \sqrt{\frac{3}{4}} \qquad \text{a} \qquad \gamma_1 \leq \sqrt{\frac{5}{6}},$$

if $a(.,.)$ is a generalized laplacian with coefficients piecewise constant on macroelements.

If the operator $a(.,.)$ and the geometry of the elements are izotropic, the CBS constants for h-hierarchy and for p-hierarchy are

$$\gamma_0 \leq \sqrt{\frac{3}{8}} \qquad \text{a} \qquad \gamma_1 \leq \sqrt{\frac{5}{11}}.$$

P-version of hierarchy leads to $\gamma \to 1$ for FEs on triangles and rectangles for general elliptic operator.

CBS constants for hierarchical FEs

Linear elements (Courant) [Jung, Maitre, 1997, Axelsson, Blaheta, 2004] $\gamma \leq \sqrt{\frac{3}{4}}$ for anizotropic equation or elements, $\gamma \leq \sqrt{\frac{1}{2}}$ for izotropic problem, $\gamma \leq \sqrt{\frac{m^2-1}{m^2}}$ for the hierarchy of higher order.

Bilinear elements [P, 2005] $\gamma \leq \sqrt{\frac{3}{4}}$ for the anizotropy, and $\gamma \leq \sqrt{\frac{3}{8}}$ for izotropic problem. For a higher hierarchy, $H = 4h$, $\gamma \leq \sqrt{\frac{5}{6}}$ when the problem is anizotropic.

Nonconforming elements (Crouzeix-Raviart) [Blaheta, Margenov, Neytcheva, 2004] lead to $\gamma \leq \sqrt{\frac{3}{4}}$, [Kraus, Margenov, Synka, 2007].

Bilinear nonconforming elements (Rannacher-Turek) [Georgiev, Kraus, Margenov, 2006], various schemes of hierarchy lead e.g. to $\gamma \leq \sqrt{\frac{3}{8}}$ or $\gamma \leq \sqrt{\frac{5}{12}}$.

Preconditioning for p-hierarchical linear elements [Beuchler, 2003].

Optimal preconditioning for the block B_W

We solve the system

$$\begin{pmatrix} B_W & B_{WU} \\ B_{UW} & B_U \end{pmatrix} \begin{pmatrix} \tilde{u}_W \\ \tilde{u}_U \end{pmatrix} = \begin{pmatrix} F_W \\ F_U \end{pmatrix}$$

preconditioned by the matrix

$$M = \begin{pmatrix} B_W & 0 \\ 0 & B_U \end{pmatrix}.$$

Big jumps in coefficients may destroy good conditioning of B_W.

In 1999 Axelsson and Padiy discovered that a tri-diagonal matrix can be found (after a proper reordering the elements in W) in such a way that

$$\kappa(C_W^{-1} B_W)$$

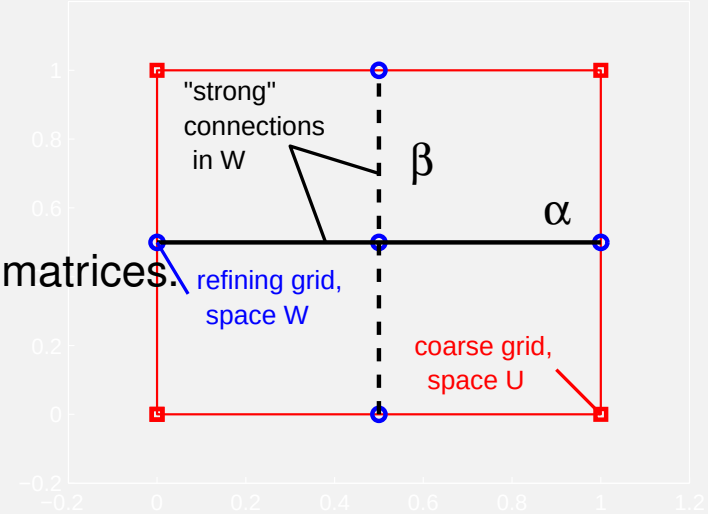
is bounded independently on the coefficients. For linear elements

$$\kappa(C_W^{-1} B_W) \leq \frac{1 + \sqrt{\frac{7}{15}}}{1 - \sqrt{\frac{7}{15}}} \approx 5.31.$$

Optimal preconditioning for the block B_W for h-hierarchy

The same idea can be used for bilinear elements.

Matrix C_W is built elementwise - certain off-diagonal elements are omitted from the macroelement stiffness matrices.



If $\frac{A_{11}d_y^2}{A_{22}d_x^2} > 1$ then the "strong connections" α are kept.

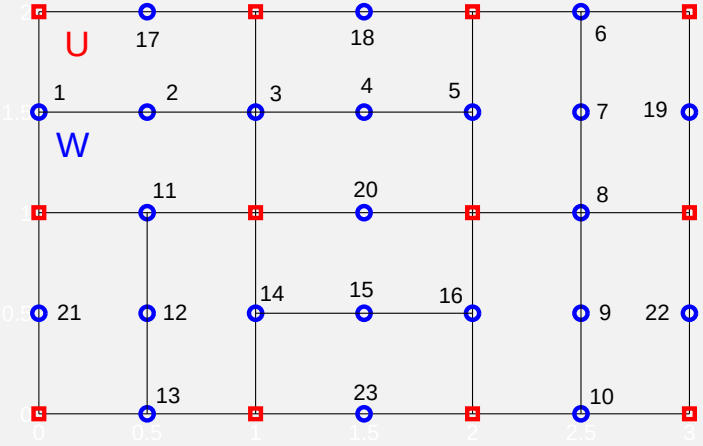
(A_{ij} are the coefficients in the equation;

size of a macroelement is $d_x \times d_y$.)

We also have

$$\kappa(C_W^{-1}B_W) \leq \frac{5+\sqrt{19}\cos(\phi/3)}{5-\sqrt{19}\cos(\phi/3-\pi/3)} \approx 4.81,$$

where $\cos(\phi) = 467/(8\sqrt{19^3})$.



An example of renumbering the elements in W.

Optimal preconditioning for the block B_W for multiple h-hierarchy

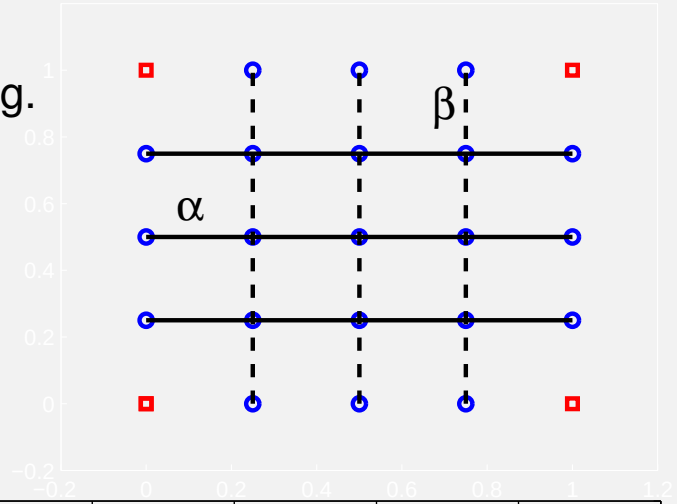
A higher hierarchy allows also to use this preconditioning.

The Figure shows an example of the choice

of the ”strong” connections involved

in the macroelement stiffness matrix

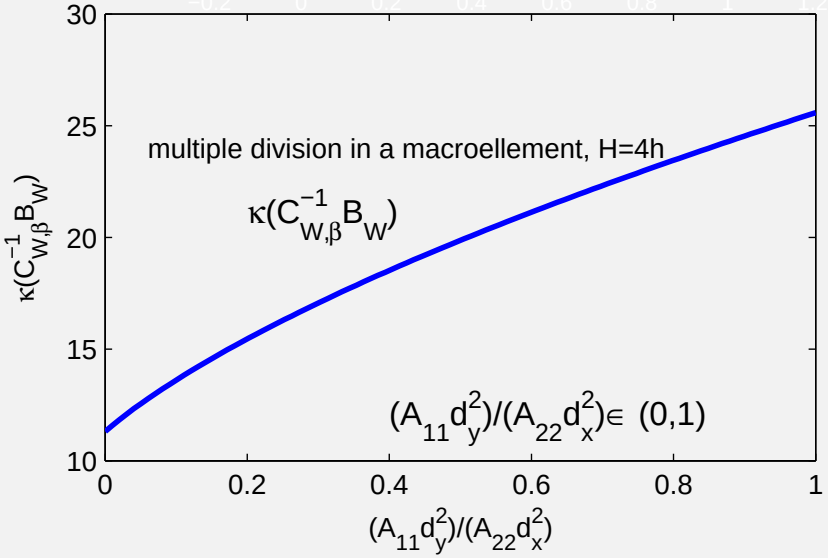
in case $\frac{A_{11}d_y^2}{A_{22}d_x^2} > 1$ (α) or < 1 (β).



In the Graph - the condition number of $C_W^{-1}B_W$

for $\frac{A_{11}d_y^2}{A_{22}d_x^2} \in (0,1)$ when the coarse and the fine

meshes are in the relation $H = 4h$



Optimal preconditioning for the block B_W for p-hierarchy

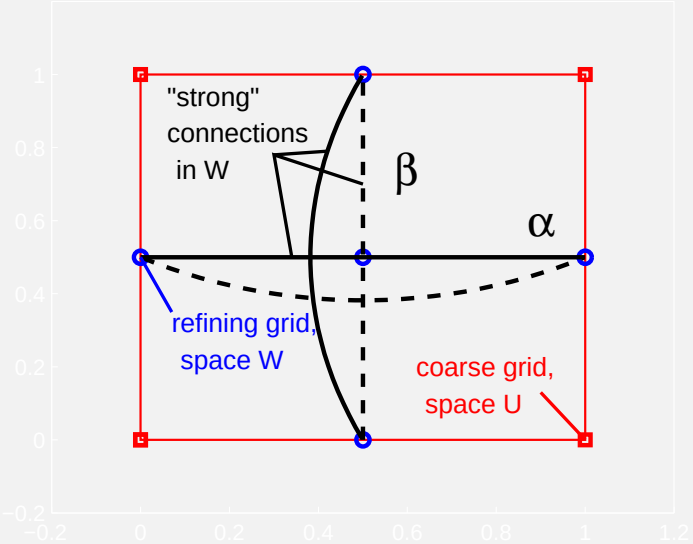
The Figure shows the choice of the ”strong” connections for p-hierrachy.

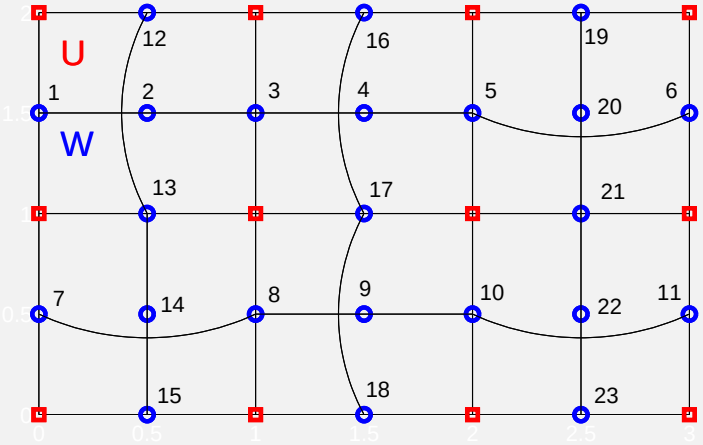
It can be reached $\kappa(C_W^{-1} B_W) \leq 8.2$, when ”connections” α or β are kept.

Once again, the criterion is $\frac{A_{11} d_y^2}{A_{22} d_x^2} > 1$ for α etc.

The analytical estimate exploiting ovals of Cassisni yields $\kappa(C_W^{-1} B_W) \leq 11.2$.

An example of renumbering the elements in W in order to obtain the tri-diagonal matrix C_W.





Conclusion

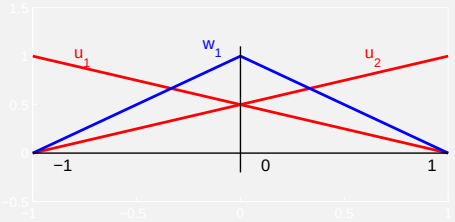
1. New estimates of CBS constant for hierarchical FEs on rectangles for h- and p-hierarchy.
2. Optimal preconditioning for the block B_W.
3. Hierarchical bases enable the a posteriori error estimates.

Quiz

The uniform estimate of the CBS constant γ for hierarchical piecewise cubic polynomials (splines) on a line and for the scalar product $a(u,v) = \int_c^d u'v' \, \mathrm{d}x$ is $\gamma \leq \sqrt{\frac{5}{8}}$.

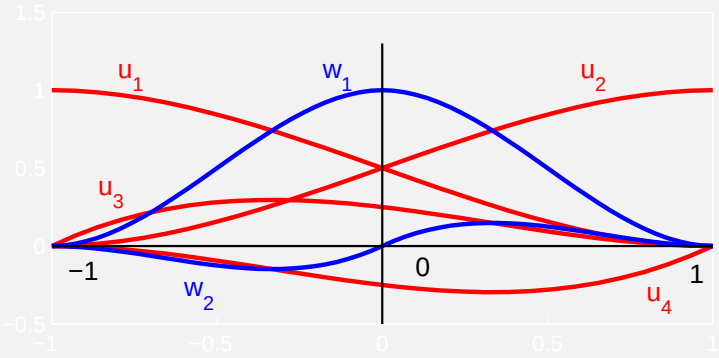
The uniform estimate of the CBS constant γ for these elements and for the the scalar product $a(u,v) = \int_c^d u''v'' \, \mathrm{d}x$ is

a) $\gamma = 0$ ... inherited from linear elements on a line ...



b) $\gamma \leq \sqrt{\frac{3}{4}}$ ... similar to linear elements in 2d

c) $\gamma \to 1$ when $h \to 0$



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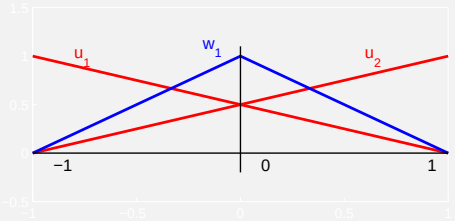

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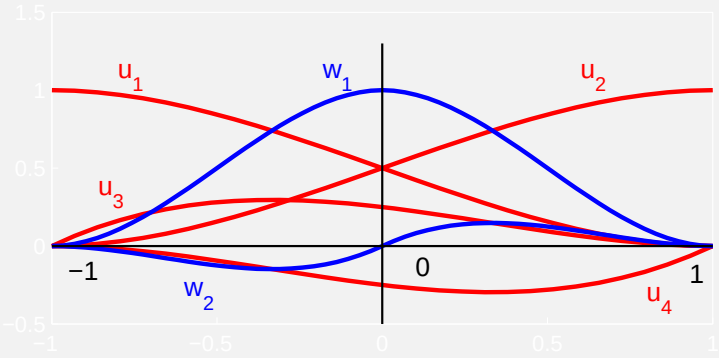

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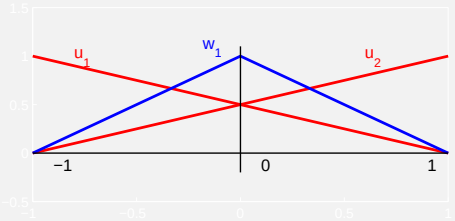

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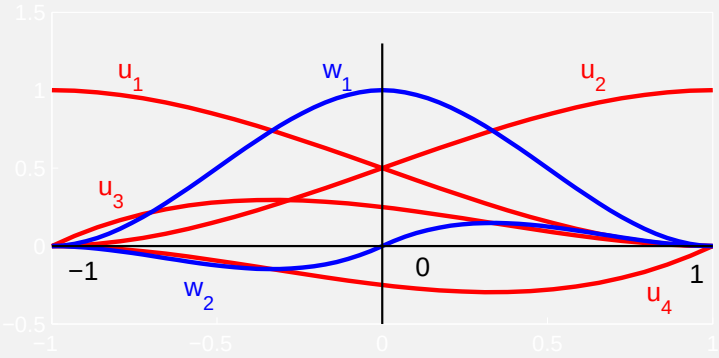

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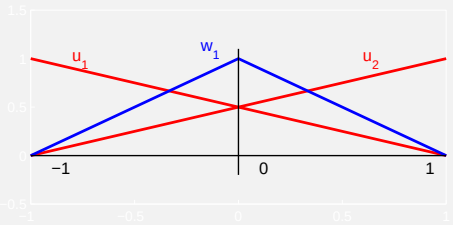

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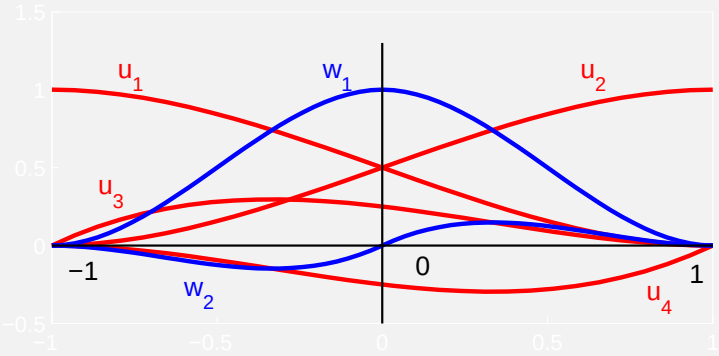

The answer is $\gamma = 0$.

Thank you!