On a posteriori error estimates in the finite volume method including the algebraic error

A diffusion model problem

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Real-world problem
$\downarrow$
Mathematical model
$\downarrow$ $\longrightarrow$ *discretization error*
Discrete problem
$\downarrow$ $\longrightarrow$ *algebraic error*
Computed solution

Question: How the error in the iterative process affects the error in the PDE?

We construct a posteriori error estimate including the error in the iterative method and thus get:

- the bound for the overall (including both discretization and algebraic) error of the computed solution,
- the stopping criterion for an iterative solver.

Arioli, Loghin, and Wathen [2005], Arioli [2004], Arioli, Noulard, and Russo [2001], Deuflhard [1994], Becker, Johnson, and Rannacher [1995].

Continuous problem

Find $p : \Omega \to \mathbb{R}^d$ such that

$$-\nabla \cdot (\mathbf{S}\nabla p) = f \quad \text{in } \Omega, \qquad p = g \quad \text{on } \Gamma := \partial\Omega,$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a polygonal/polyhedral domain, $\mathbf{S}$ is a diffusion tensor, f is a source term and g prescribes Dirichlet boundary conditions.

Weak formulation: find $p \in H^1_\Gamma(\Omega)$ such that

$$\mathcal{B}(p, \varphi) = (f, \varphi) \qquad \forall \varphi \in H^1_0(\Omega),$$

where

$$\mathcal{B}(p, \varphi) := (\mathbf{S}\nabla p, \nabla \varphi), \qquad |||\varphi|||^2 := \mathcal{B}(\varphi, \varphi).$$

Assumptions:

- $\mathbf{S}$ is a bounded symmetric positive definite tensor,
- $f \in L^2(\Omega)$, $g \in H^{1/2}(\Omega)$

$\Rightarrow$ weak formulation has a unique solution.

Notation and assumptions

Partition of the domain:

- Let $\mathcal{T}_h$ be a conforming partition of Ω into closed simplices.
- Let $\mathcal{E}_K$ be the set of sides of K.
- Let $\mathcal{E}_h = \cup_{K \in \mathcal{T}_h} \mathcal{E}_K$ be the set of all sides of $\mathcal{T}_h$ and let $\mathcal{E}_h$ be partitioned as $\mathcal{E}_h = \mathcal{E}_h^{\mathrm{int}} \cup \mathcal{E}_h^{\mathrm{ext}}$ into two disjoint sets of interior and exterior sides.

Additional assumptions:

- $\mathbf{S}$ is element-wise constant, and for each $K \in \mathcal{T}_h$ there are positive constants $\underline{c}_{\mathbf{S},K}$ and $\overline{c}_{\mathbf{S},K}$ such that

$$\underline{c}_{\mathbf{S},K} \mathbf{v}^T \mathbf{v} \leq \mathbf{v}^T \mathbf{S}_K \mathbf{v} \leq \overline{c}_{\mathbf{S},K} \mathbf{v}^T \mathbf{v} \qquad \forall \mathbf{v} \in \mathbb{R}^d.$$

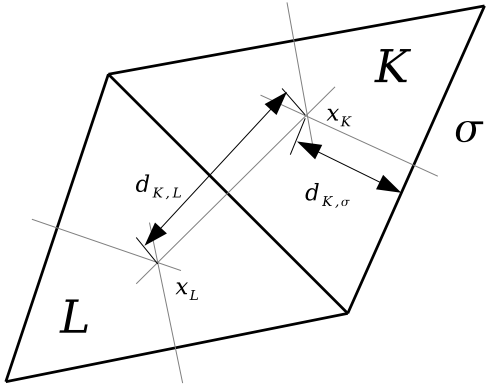
Finite volume method

Cell-centered finite volume scheme (see, e.g., Eymard, Gallouët, and Herbin [2000]): find $p_h \in P_0(\mathcal{T}_h)$ such that

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = f_K |K| \qquad \forall K \in \mathcal{T}_h.$$

Particular choice of $F_{K,\sigma}$ (for $\mathbf{S} = s\mathbf{Id}$, strictly Delaunay $\mathcal{T}_h$):

$$F_{K,\sigma} = \begin{cases} -\, s_\sigma \dfrac{|\sigma|}{|d_{K,L}|}(p_L - p_K) & \text{for } \sigma = \sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}, \\ -\, s_K \dfrac{|\sigma|}{|d_{K,\sigma}|}(g_\sigma - p_K) & \text{for } \sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{ext}}. \end{cases}$$



Postprocessing and Oswald interpolation

- p_h is only piecewise constant; to get a more regular solution, look for $\tilde{p}_h \in P_2(\mathcal{T}_h)$ such that (see Vohralík [2007b])

$$
\begin{aligned}
-\nabla \cdot (\mathbf{S}\nabla \tilde{p}_h) &= \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} && \forall K \in \mathcal{T}_h, \\
\tilde{p}_h(\mathbf{x}_K) &= p_K && \forall K \in \mathcal{T}_h, \\
-\mathbf{S}\nabla \tilde{p}_h|_K \cdot \mathbf{n} &= \frac{1}{|\sigma|} F_{K,\sigma} && \forall \sigma \in \mathcal{E}_K,\ K \in \mathcal{T}_h.
\end{aligned}
$$

- Modified Oswald interpolation operator $\mathcal{I}_{Os}^{\Gamma} : P_k(\mathcal{T}_h) \to P_k(\mathcal{T}_h) \cap H^1(\Omega)$ is defined via (see, e.g., Achdou, Bernardi, and Coquel [2003])

$$
\mathcal{I}_{Os}^{\Gamma}(\varphi)(\mathbf{x}) = \frac{1}{\mathrm{card}(\mathcal{T}_{\mathbf{x}})} \sum_{K \in \mathcal{T}_{\mathbf{x}}} \varphi|_K(\mathbf{x}), \quad \varphi \in P_k(\mathcal{T}_h),
$$

(for suitable nodes $\mathbf{x}$ of each $K \in \mathcal{T}_h$) and

$$
\mathcal{I}_{Os}^{\Gamma}(\varphi)|_{\Gamma} = g.
$$

System of algebraic equations in FVM

The set of conservation equations

$$\sum_{K \in \mathcal{T}_h} F_{K,\sigma} = f_K |K| \qquad \forall K \in \mathcal{T}_h$$

leads to the system of linear algebraic equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \qquad \mathbf{A} \in \mathbb{R}^{N \times N},\ \mathbf{b} \in \mathbb{R}^N,\ N := \mathrm{card}(\mathcal{T}_h).$$

The iterative solver gives an approximation $\mathbf{x}^*$:

$$\mathbf{A}\mathbf{x}^* = \mathbf{b} - \mathbf{r}, \qquad \mathbf{r} := \mathbf{b} - \mathbf{A}\mathbf{x}^*,$$

where $\mathbf{x}^*$ corresponds to some approximation p_h^* of p_h and $\mathbf{r}$ corresponds to the piecewise constant residual function r. We get:

$$\sum_{K \in \mathcal{T}_h} F^*_{K,\sigma} = f_K |K| + r_K \qquad \forall K \in \mathcal{T}_h.$$

A posteriori error estimate

We use the abstract framework derived in Vohralík [2007a].

Let $\tilde{p}_h^*$ be the postprocessed approximation to the FVM solution p_h, and let $\mathbf{t}_h \in RT_0(\mathcal{T}_h) \cap H(\mathrm{div}, \Omega)$ be such that

$$\nabla \cdot \mathbf{t}_h|_K = r_K/|K| \qquad \forall K \in \mathcal{T}_h.$$

Then

$$|||p - \tilde{p}_h^*||| \leq \eta_{\mathrm{NC}} + \eta_{\mathrm{R}} + \eta_{\mathrm{IT}},$$

where

- $\eta_{\mathrm{NC}} := |||\tilde{p}_h^* - \mathcal{I}_{Os}^{\Gamma}(\tilde{p}_h^*)|||$ is the *nonconformity estimator*,
- $\eta_{\mathrm{R}} := \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_{P,K}}{\underline{c}_{\mathbf{S},K}} h_K^2 \|f - f_K\|_K^2 \right\}^{1/2}$ is the *residual estimator*,
- $\eta_{\mathrm{IT}} := \sup\{(\mathbf{t}_h, \nabla\varphi);\ \varphi \in H_0^1(\Omega),\ |||\varphi||| = 1\}$ is the *iteration error estimator*.

Evaluation of η_{IT}

Using algebraic residual vector:

$$\eta_{\mathrm{IT}} \leq \eta_{\mathrm{IT}}^{(1)} := h_\Omega \left\{ \frac{c_{F,\Omega}}{\underline{c}_{\mathbf{S},\Omega}} \sum_{K \in \mathcal{T}_h} \frac{r_K^2}{|K|} \right\}^{1/2},$$

- not very effective, usually a large overestimate.

Relation with the $\mathbf{A}$-norm of the error $\mathbf{x} - \mathbf{x}^*$:

$$\eta_{\mathrm{IT}} \approx \eta_{\mathrm{IT}}^{(2)} := \|\mathbf{x} - \mathbf{x}^*\|_{\mathbf{A}}$$

- very effective estimate, but does not hold exactly in FV,
- $\eta_{\mathrm{IT}}^{(2)}$ can be estimated in CG, see, e.g., Strakoš and Tichý [2002], Meurant and Strakoš [2006].

Using "arbitrary" $\mathbf{t}_h$:

$$\eta_{\mathrm{IT}} \leq \eta_{\mathrm{IT}}^{(3)} := \|\mathbf{S}^{-1/2}\mathbf{t}_h\|.$$

Particular construction of $\mathbf{t}_h$

Local construction: let $\mathcal{T}_h = \{K_1, \ldots, K_N\}$ and for each K_i there is a side $\sigma \in \mathcal{E}_{K_i}$ such that $\sigma \notin \mathcal{E}_h^{\mathrm{ext}}(\cup_{j=1}^{i-1} K_i)$.

The restrictions of $\mathbf{t}_h$ are found by solving the following problems for $i = 1, \ldots, n$:

1. find $\mathbf{z}_i \in RT_0(K_i)$ satisfying

$$\|\mathbf{z}_i\|_{K_i} = \min_{\tilde{\mathbf{z}} \in RT_0(K_i)} \|\tilde{\mathbf{z}}\|_{K_i}$$

$$\text{w.r.t.} \quad \nabla \cdot \mathbf{z}_i = r_{K_i}/|K_i|, \ \mathbf{z}_i \cdot \mathbf{n} = \mathbf{z}_j \cdot \mathbf{n} \text{ on } \sigma \in \mathcal{E}_{K_i} \cap \mathcal{E}_{K_j}, \ j < i,$$

2. set $\mathbf{t}_h|_{K_i} = \mathbf{z}_i$.

Model example

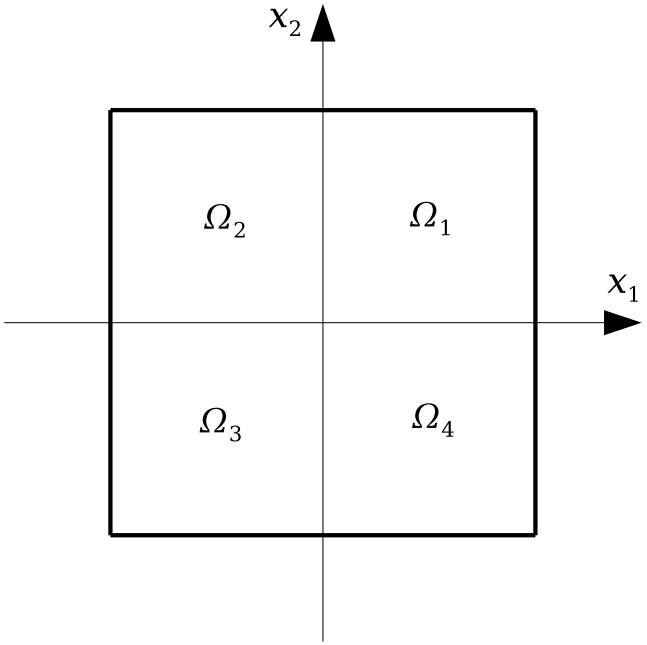
Consider the problem

$$-\nabla \cdot (s \nabla p) = 0 \quad \text{on } \Omega = (-1,1)^2,$$

with

- a nonhomogeneous diffusion coefficient s constant in each quadrant,
- boundary conditions given by the analytical solution of the form

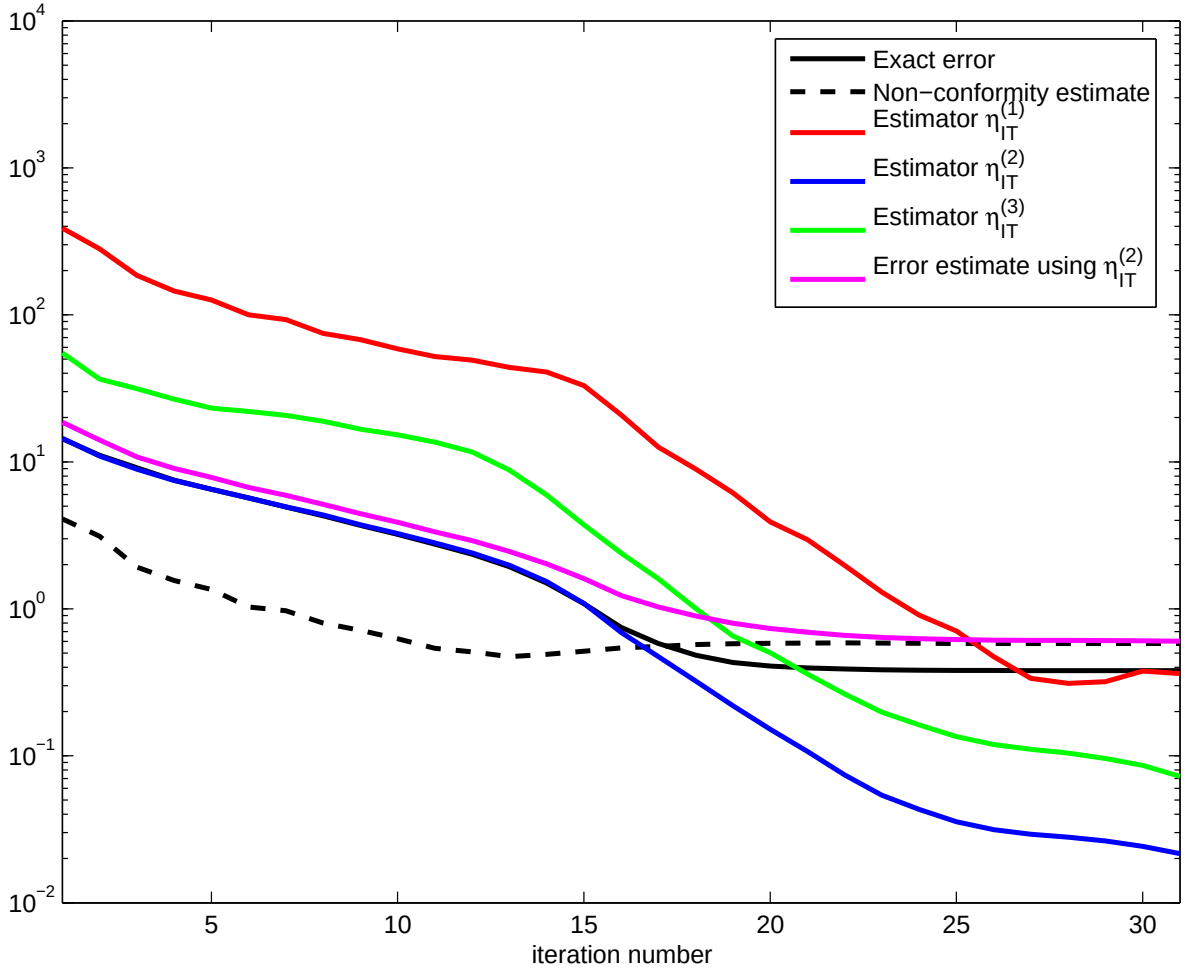
$$p(\varrho, \vartheta)|_{\Omega_i} = \varrho^{\alpha}[a_i \sin(\alpha\vartheta) + b_i \cos(\alpha\vartheta)].$$



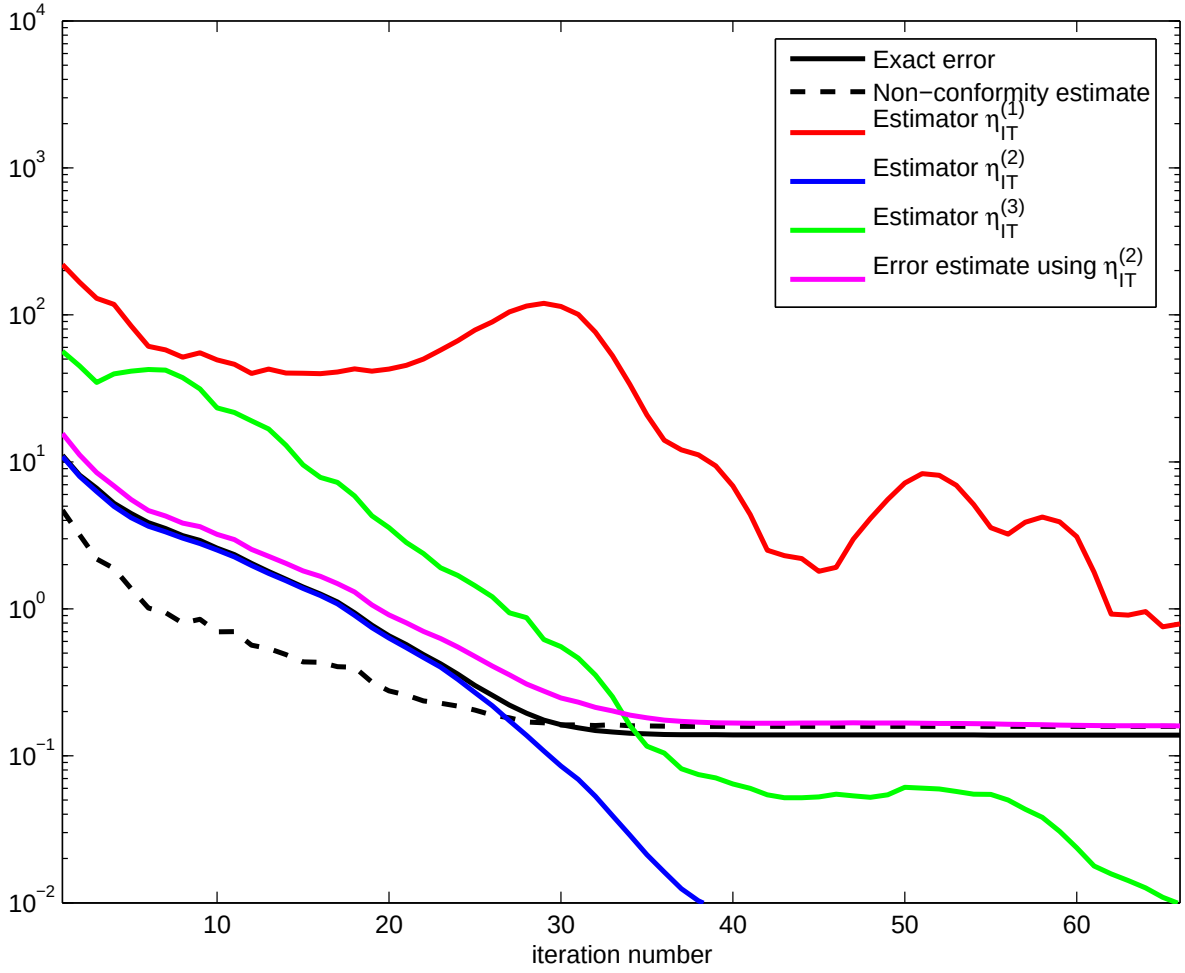
1. $s_1 = s_3 = 5,\ s_2 = s_4 = 1,\ \alpha \approx 0.535,$
2. $s_1 = s_3 = 100,\ s_2 = s_4 = 1,\ \alpha \approx 0.127.$

- We present results where the initial mesh with 112 elements was refined uniformly or adaptively using the a posteriori error estimate;
- initial estimate for the CG method is set to zero;
- preconditioned by the diagonal scaling.

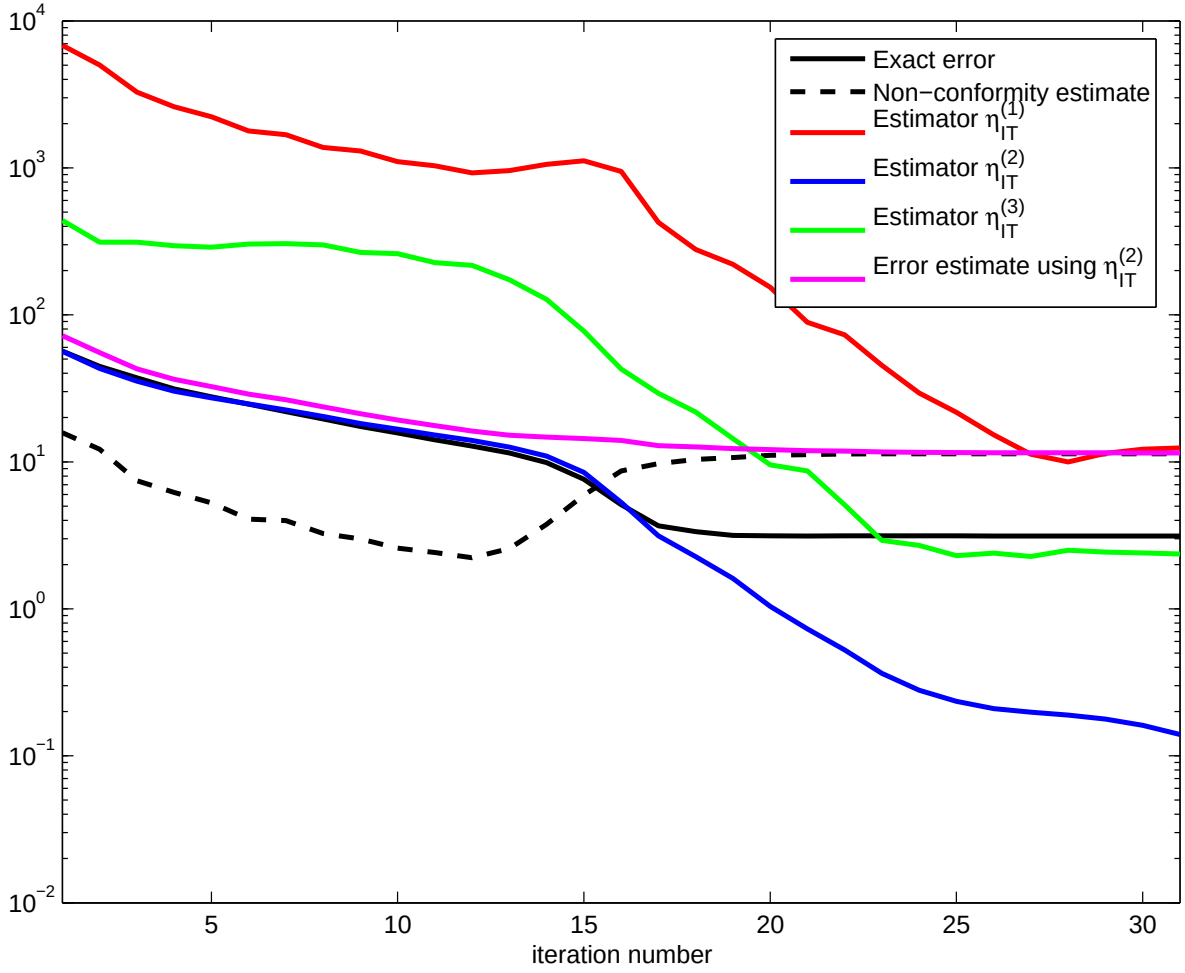
Model example 1 ($\alpha \approx 0.535$) – uniform refinement, $N = 1510$



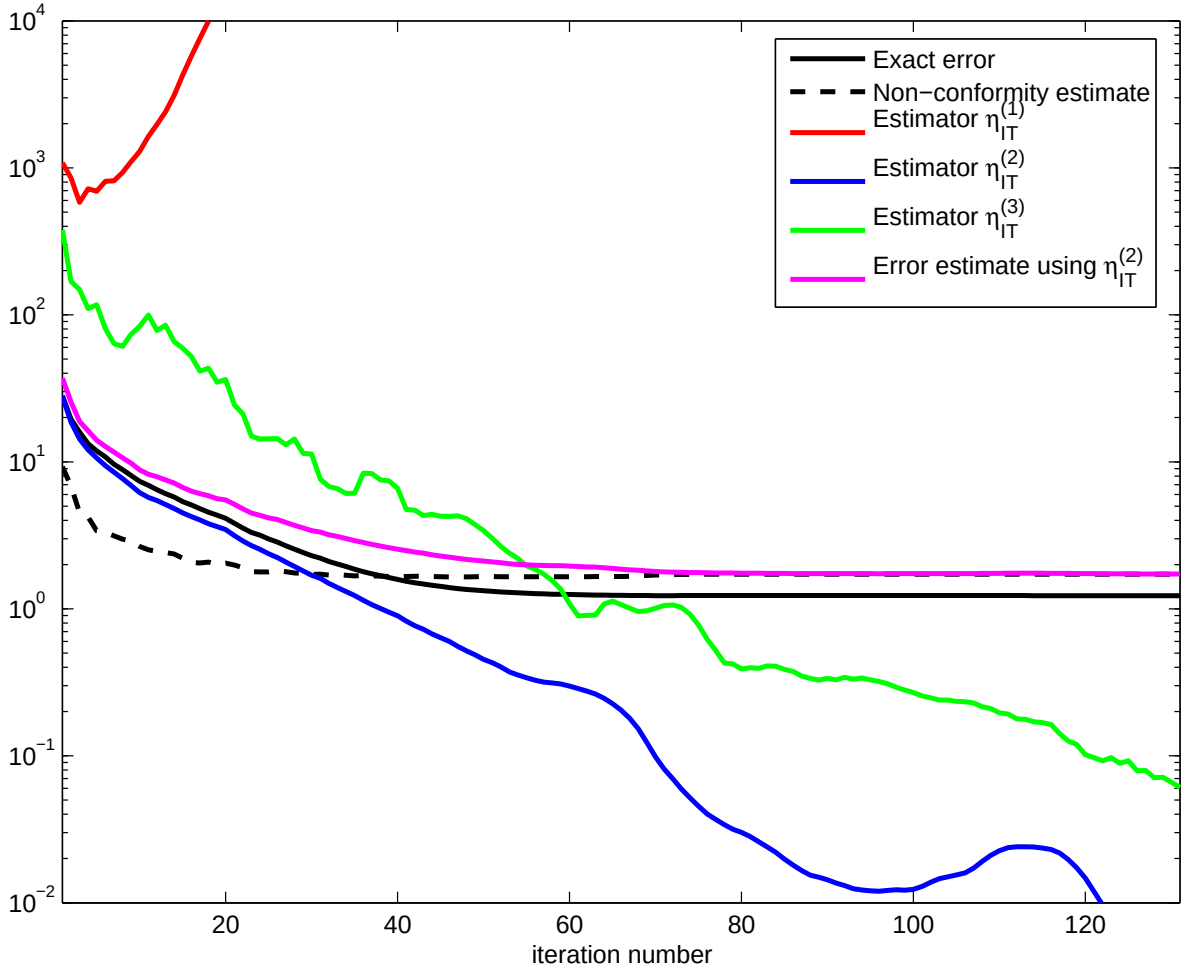
Model example 1 ($\alpha \approx 0.535$) – adaptive refinement, $N = 1812$

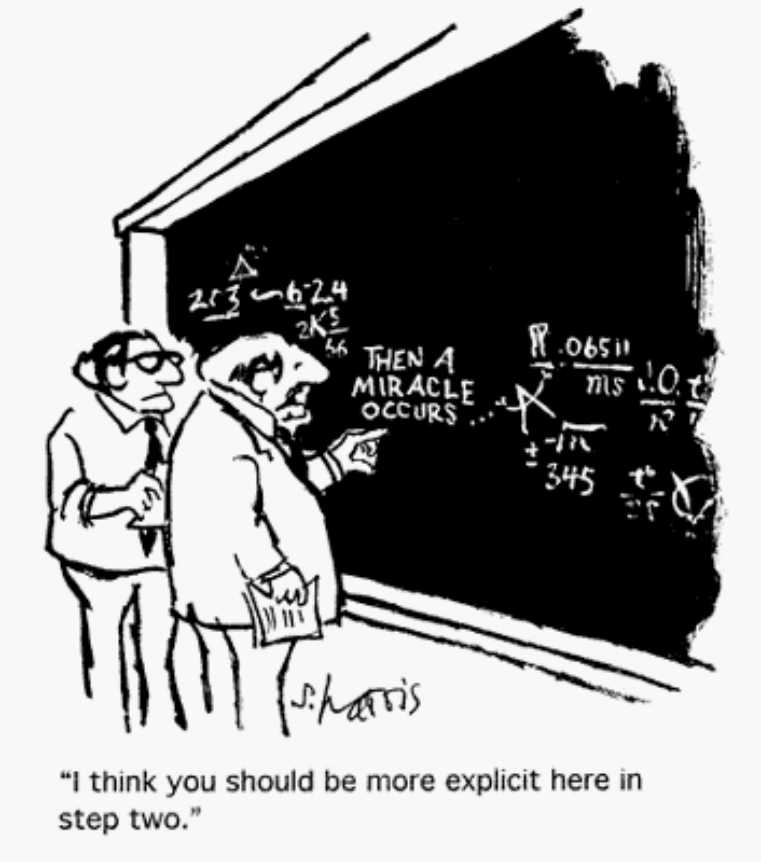


Model example 2 ($\alpha \approx 0.127$) – uniform refinement, $N = 1510$



Model example 2 ($\alpha \approx 0.127$) – adaptive refinement, $N = 1510$





"I think you should be more explicit here in step two."

For more information, see the paper in preparation...

P. Jiránek, Z. Strakoš, M. Vohralík,
A posteriori error estimates including iterative solvers error: guaranteed upper bound for the discretization error and stopping criterion.

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