Numerical illustration of theoretical results for non-linear semi-coercive beam problem

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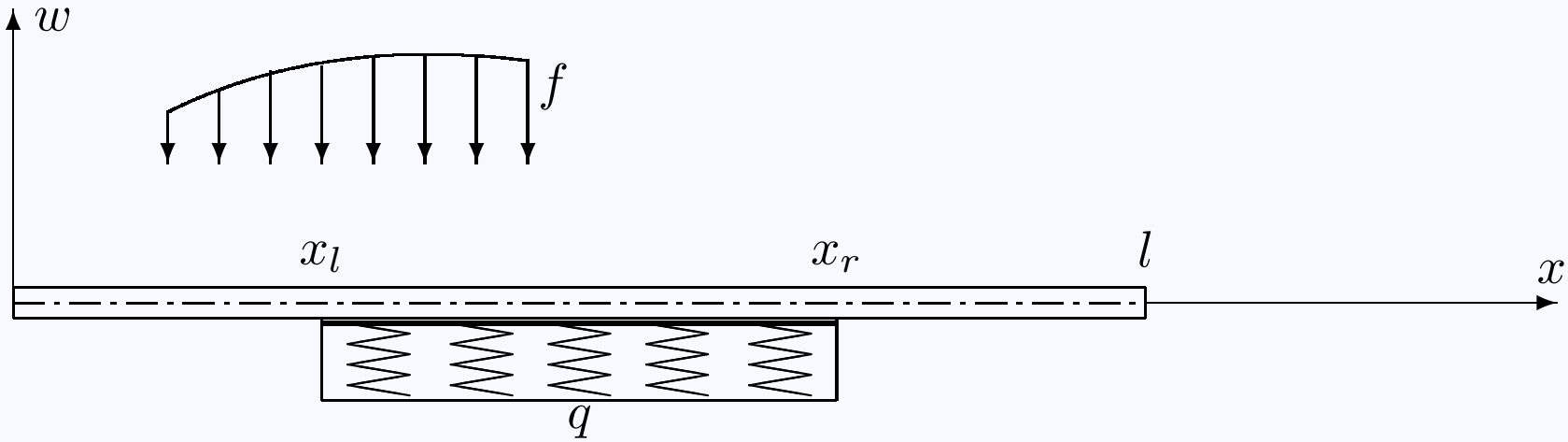
Outline

1. Mathematical model.
2. Problem approximation.
3. Numerical methods.
4. Connections with dual methods.
5. Numerical Examples.

1 Mathematical model

1.1 Scheme



1.2 Variational formulation

Total energy functional:

$$J(w) := \frac{1}{2}\left(a(w,w) + b(w^-,w^-)\right) - L(w), \quad w \in H^2(\Omega),$$

$$w^-(x) := \min\{0,\ w(x)\}, \qquad L \in (H^2(\Omega))^*,$$
$$a(w,v) := \int_\Omega EIw''v''\ dx, \qquad b(w,v) := \int_{\Omega_s} qwv\ dx.$$

Minimization problem:

$$(P) \qquad ?w^* \in H^2(\Omega): \quad J(w^*) \le J(w) \quad \forall w \in H^2(\Omega)$$

Non-linear variational equation:

$$?w^* \in H^2(\Omega): \quad a(w^*,v) + b((w^*)^-,v) = L(v) \quad \forall v \in H^2(\Omega)$$

1.3 Solvability and dependence on load

Condition for existence and uniqueness of the solution:

$$F < 0 \quad \text{and} \quad x_l < T < x_r, \quad F := L(1) \quad \text{and} \quad T := \frac{L(x)}{L(1)}.$$

Dependence on the load: Let $\delta, \xi, \eta > 0$ and

$$\mathcal{S}_{\delta,\xi,\eta} := \{L \in V^* \mid T \in (x_l + \delta, x_r - \delta)\},\ F < -\xi < 0,\ \|L\|_* \leq \eta\}.$$

Then

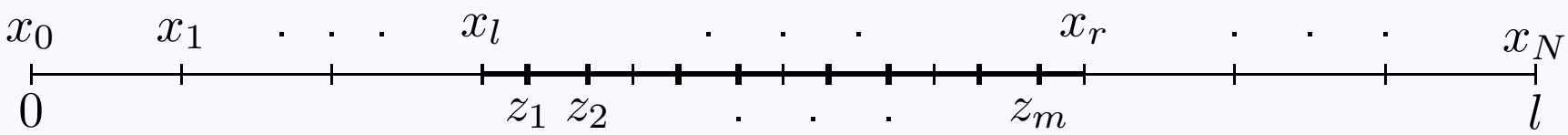
$$\exists c = c(\delta, \xi, \eta) > 0 : \quad \|w_1^* - w_2^*\|_{2,2} \leq c\|L_1 - L_2\|_*, \quad \forall L_1, L_2 \in \mathcal{S}_{\delta,\xi,\eta},$$

where $w_i^* = w_i^*(L_i)$ solve the problem (P) with respect to the loads L_i, $i = 1, 2$.

$$T \to \partial\Omega_s \quad \text{or} \quad \frac{F}{\|L\|_*} \to 0 \qquad \Longrightarrow \qquad c \to +\infty$$

2 Problem approximation

Partition τ_h of $\overline{\Omega} = [0, l]$ and points of numerical quadrature:



Approximation of the bilinear form b:

$$b_h(v_1, v_2) := \sum_{i=1}^{m} r_i v_1(z_i) v_2(z_i), \quad v_1, v_2 \in H^2(\Omega),$$

Approximation of $H^2(\Omega)$:

$$V_h \subset H^2(\Omega), \quad V_h := \{v_h \in C^1(\overline{\Omega}) \mid \ v_h|_{(x_{j-1}, x_j)} \in P_3, \ j = 1, 2, \ldots, N\}.$$

Problem approximation:

$$(P_h) \qquad \begin{cases} \text{find } w_h^* \in V_h : \; J_h(w_h^*) \le J_h(v_h) \quad \forall v_h \in V_h, \\ J_h(v_h) := \frac{1}{2} a(v_h, v_h) + \frac{1}{2} b_h(v_h^-, v_h^-) - L(v_h), \end{cases}$$

Solvability condition:

$$F < 0 \quad \text{and} \quad z_1 < T < z_m.$$

Convergence results:

$$\begin{aligned} \|w^* - w_h^*\|_{2,2} &\le c_1 h^2 \|w^*\|_{4,2} && w^* \in H^4(\Omega) \cap \mathcal{V}_M, \\ \|w^* - w_h^*\|_{2,2} &\le c_2 h \|w^*\|_{3,2} && w^* \in H^3(\Omega), \\ \lim_{h \to 0} \|w^* - w_h^*\|_{2,2} &= 0 && w^* \in H^2(\Omega), \end{aligned}$$

where

$$\mathcal{V}_M = \{v \in H^2(\Omega) |\; \exists p \le M, \exists z_1, \ldots, z_{2p} \in \overline{\Omega}_s : \; \{x \in \overline{\Omega}_s |\; v^-(x) = 0\} = \bigcup_{i=1}^{p} [z_{2i-1}, z_{2i}]\}$$

Algebraic formulation:

- n, m - number of unknowns and number of springs,
- $v_h \in V_h \;\mapsto\; v \in \mathbb{R}^n$,
- $a(u_h, v_h) \;\mapsto\; (Ku, v)_n, \;\; K \in \mathbb{R}^{n\times n}$,
- $b_h(u_h, v_h) \;\mapsto\; (DBu, Bv)_m, \;\; B \in \mathbb{R}^{m\times n}, \;\; D \in \mathbb{R}^{m\times m}$,
- $L(v) \;\mapsto\; (f, v)_n, \;\; f \in \mathbb{R}^n$,

$$(\mathbb{P}) \qquad \begin{cases} ?w^* \in \mathbb{R}^n : \; J(w^*) \le J(w) \quad \forall w \in \mathbb{R}^n, \\ J(w) := \frac{1}{2}(Kw, w)_n + \frac{1}{2}\left(D(Bw)^-, (Bw)^-\right)_m - (f, w)_n, \end{cases}$$

or

$$(\mathbb{P}) \qquad ?w^* \in \mathbb{R}^n : \;\; Kw^* + B^T D(Bw^*)^- = f$$

3 Numerical methods

$\mathcal{A} \subset \{1, \dots, m\}$ ... represents the choice of springs, $\mathcal{A} \mapsto A \in \mathbb{R}^{m \times m}$

$v \in \mathbb{R}^n \mapsto \mathcal{A}(v) := \{i \mid (Bv)_i < 0\} \mapsto A(v) \in \mathbb{R}^{m \times m}$

Auxilliary linear problem with bilateral elastic springs:

$$(\mathbb{P}^A) \qquad ?w = w(A) \in \mathbb{R}^n : \quad Kw + B^T DABw = f$$

Auxilliary non-linear "projected" problem for $v \in \mathbb{R}^n$:

$$(\mathbb{P}^v) \quad ?c = c(v) \in \mathbb{R}^2 : \quad J(v + Rc) \leq J(v + Ra) \quad \forall a \in \mathbb{R}^2, \;\; R \in \mathbb{R}^{n \times 2}, \; KR = 0$$

$$\text{or} \quad G^T D(Bv + Gc)^- = e,$$

$$G := BR = \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_m \end{pmatrix}^T, \quad e := R^T f = F \begin{pmatrix} 1 \\ T \end{pmatrix},$$

Algorithm 1

Initialization

$$w^{(0)} = 0,$$
$$\mathcal{A}^{(0)} = \{1, \dots, m\} \mapsto A^{(0)}.$$

Iteration $k = 0, 1, \dots$

$$s^{(k)},\ \ w^{(k)} + s^{(k)} \text{ solves } (\mathbb{P}^{A^{(k)}}),$$
$$\alpha^{(k)} = arg \min_{0 \le \alpha \le 1} J(w^{(k)} + \alpha s^{(k)}),$$
$$w^{(k+1)} = w^{(k)} + \alpha^{(k)} s^{(k)},$$
$$A^{(k+1)} = A(w^{(k+1)}).$$

Algorithm 2

Initialization

$$w^{(0)} = Rc^{(0)},\ c^{(0)} \text{ solves } (\mathbb{P}^0),$$
$$A^{(0)} = A(w^{(0)}),$$

Iteration $k = 0, 1, \dots$

$$s^{(k)},\ \ w^{(k)} + s^{(k)} \text{ solves } (\mathbb{P}^{A^{(k)}}),$$
$$\alpha^{(k)} = arg \min_{0 \le \alpha \le 1} J(w^{(k)} + \alpha s^{(k)}),$$
$$\tilde{w}^{(k)} = w^{(k)} + \alpha^{(k)} s^{(k)},$$
$$c^{(k)},\ c^{(k)} \text{ solves } (\mathbb{P}^{\tilde{w}^{(k)}}),$$
$$w^{(k+1)} = \tilde{w}^{(k)} + Rc^{(k)},$$
$$A^{(k+1)} = A(w^{(k+1)}).$$

$$w^{(k)} \mapsto w_h^{(k)} \in V_h, \quad w_h^{(k)} \to w_h^* \text{ in } H^2(\Omega) \text{ uniformly w.r.t. } h$$

Connections with dual methods

Dual formulation of the problem $(\mathbb{P})$:

$$(\mathbb{D}) \qquad \begin{cases} ?\lambda^* \in \Lambda : \quad \mathcal{F}(\lambda^*) \leq \mathcal{F}(\lambda) \quad \forall \lambda \in \Lambda, \\ \Lambda := \{\lambda \in \mathbb{R}^m \mid \lambda \leq 0,\ G^T D\lambda = e\}, \\ \mathcal{F}(\lambda) = \frac{1}{2}(Kw(\lambda), w(\lambda))_n + \frac{1}{2}(D\lambda, \lambda)_m, \\ Kw(\lambda) = f - B^T D\lambda \end{cases}$$

Dual formulation of the problem $(\mathbb{P}^A)$:

$$(\mathbb{D}^A) \qquad \begin{cases} ?\lambda(A) \in \Lambda^A : \quad \mathcal{F}(\lambda(A)) \leq \mathcal{F}(\lambda) \quad \forall \lambda \in \Lambda^A, \\ \Lambda^A := \{\lambda \in \mathbb{R}^m \mid \lambda_i = 0,\ i \notin \mathcal{A},\ G^T D\lambda = e\} \end{cases}$$

$\lambda^* = (Bw^*)^-$, w^* solves $(\mathbb{P})$, $\lambda(A) = ABw(A)$, $w(A)$ solves $(\mathbb{P}^A)$

Properties of Λ:

- closed, convex, non-empty and bounded set on $\mathbb{R}^m$, if $F < 0$ and $z_1 < T < z_m$
- $T \to z_1$ or $T \to z_m$ or $F \to 0 \quad \Longrightarrow \quad diam(\Lambda) \to 0$
- Projection P of $\mathbb{R}^m$ onto Λ: $\quad (D(\eta - P(\eta)), \lambda - P(\eta))_m \leq 0 \quad \forall \lambda \in \Lambda$,

$$P(Bv) = (Bv + Gc)^- \quad \forall v \in \mathbb{R}^n,\ c \in \mathbb{R}^2 \text{ solves } (\mathbb{P}^v)$$

$\Rightarrow$ $(Bw^*)^-, (Bw^{(k)})^- \in \Lambda$, where $w^{(k)}$ are generated by Algorithm 2

$\Rightarrow$ Algorithm 2 has better convergence properties for unstable loads than Algorithm 1

4 Numerical examples

Length of the beam: $l = 1\ m$

Equidistant partitions

Stopping criterion: $\frac{\|r^{(k)}\|_n}{\|f\|_n} \leq 10^{-6}, \quad r^{(k)} := f - Kw^{(k)} - B^T DB(w^{(k)})^-$

Numerical quadratures:

$$NQ_1 : \qquad \int_{-1}^{1} \phi(\xi)\ d\xi \approx 2\phi(0)$$

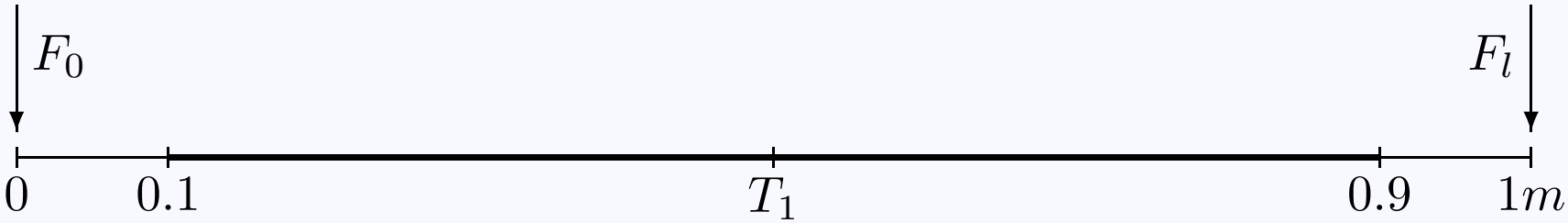
$$NQ_2 : \qquad \int_{-1}^{1} \phi(\xi)\ d\xi \approx \phi(-1) + \phi(1)$$

$$NQ_3 : \qquad \int_{-1}^{1} \phi(\xi)\ d\xi \approx \phi(-\sqrt{3}/3) + \phi(\sqrt{3}/3)$$

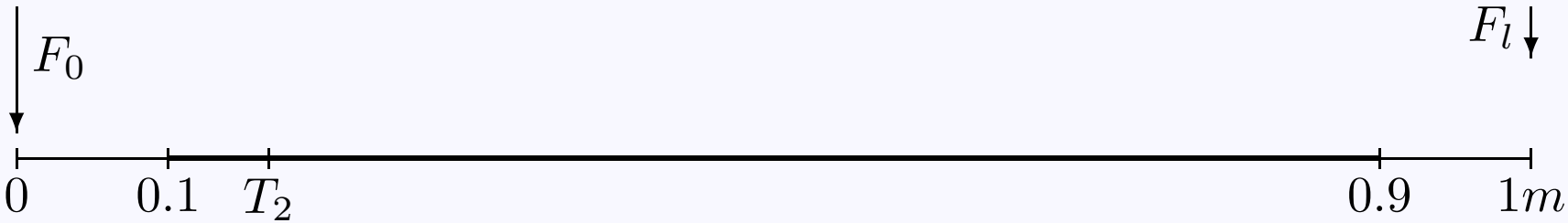
4.1 Comparision of the algorithms

Parameters: $x_l = 0.1\ m,\ x_r = 0.9\ m,\ EI = 5 * 10^5\ Nm^2,\ q = 5 * 10^8\ Nm^{-2},\ NQ_3$

Example 1 (stable). $F_0 = -5000\ N$, $F_l = -5000\ N$, $T_1 = 0.5\ m$



Example 2 (unstable). $F_0 = -5000\ N$, $F_l = -1000\ N$, $T_2 = 0.1667\ m$

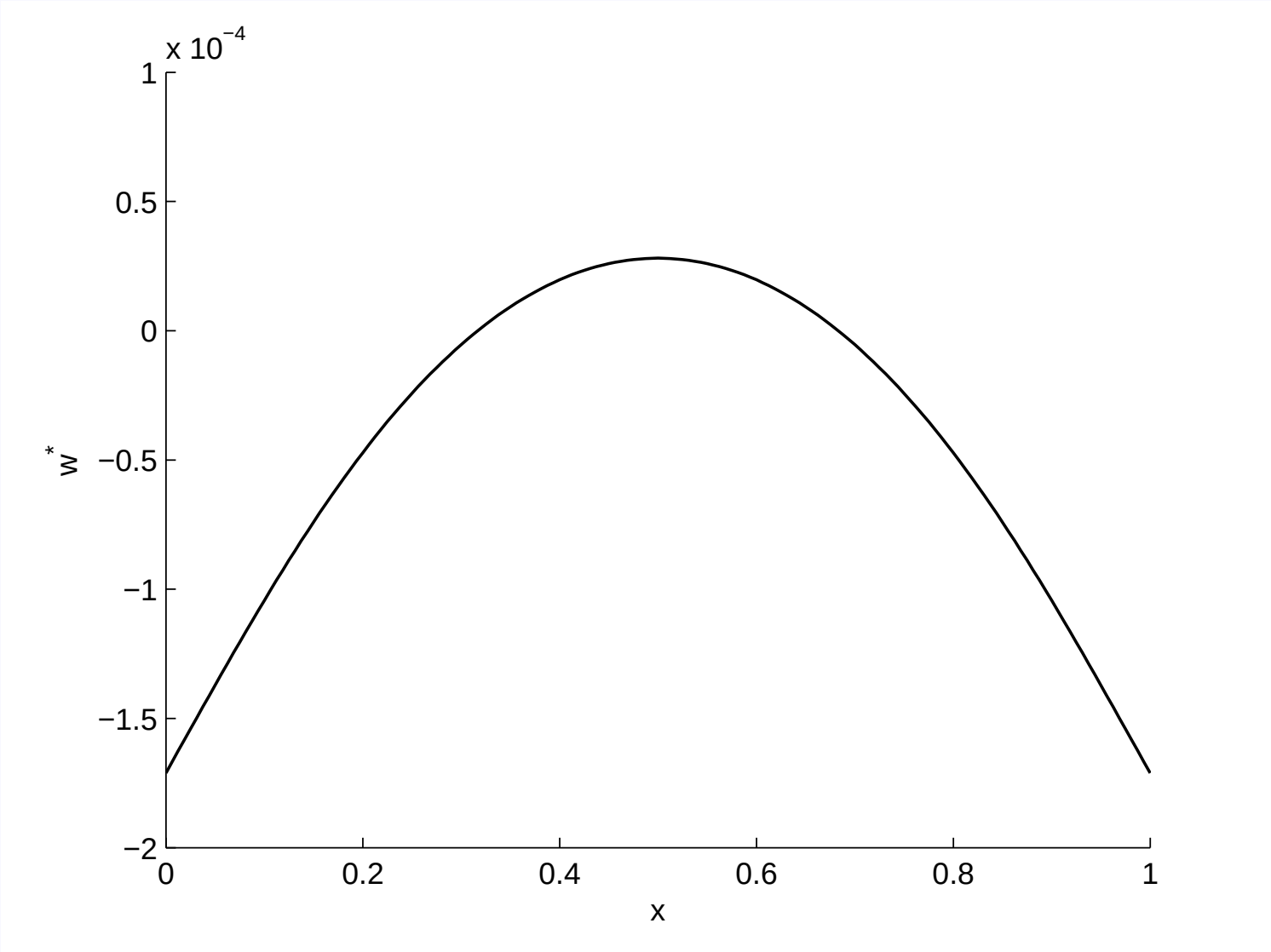


Ex. 1	40	80	160	320	640	1280
ALG1	4	3	4	4	4	4
ALG2	3	3	3	3	3	3

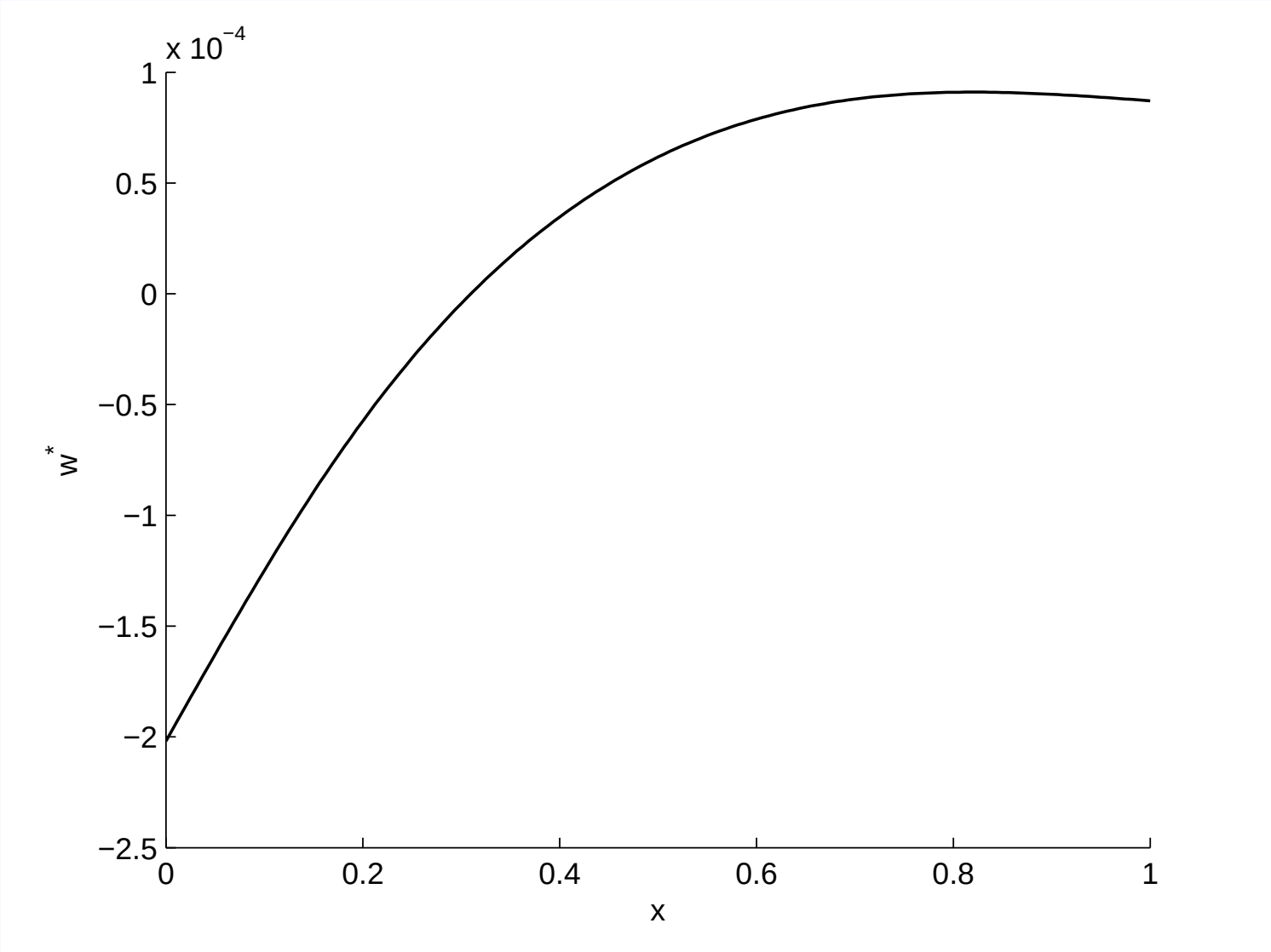
Ex. 2	40	80	160	320	640	1280
ALG1	6	6	7	8	7	8
ALG2	2	2	2	2	2	2

Table 1: Numbers of iterations.

Example 1



Example 2



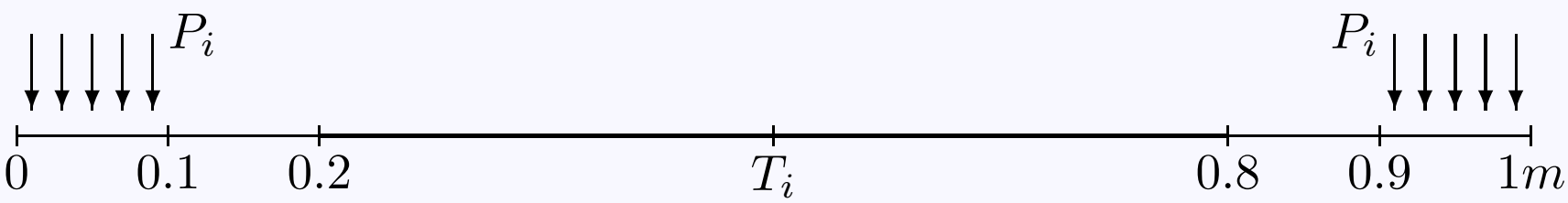
4.2 Load stability

Parameters: $x_l = 0.2\ m$, $x_r = 0.8\ m$, $EI = 20\ Nm^2$, $q = 1.2 * 10^6\ Nm^{-2}$, NQ_3

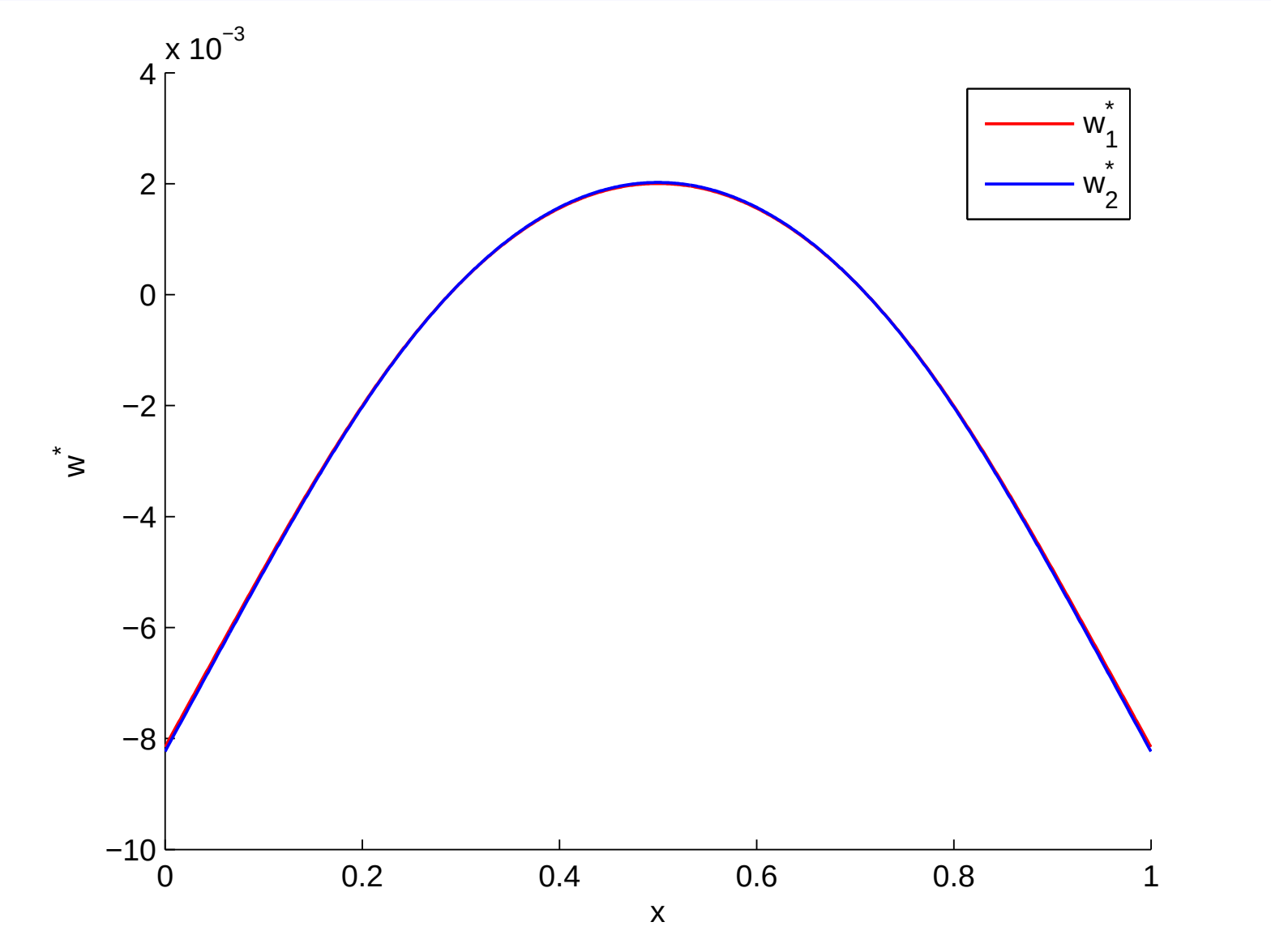
Investigated term:

$$c_p = \frac{\frac{\|w_1^* - w_2^*\|_A}{\|w_1^*\|_A}}{\frac{\|f_1 - f_2\|_n}{\|f_1\|_n}}, \quad \|v\|_A^2 := (Av, v)_n, \quad A = K + B^T DB$$

Example 1 (stable). $P_1 = -100Nm^{-1}$, $P_2 = -101Nm^{-1}$

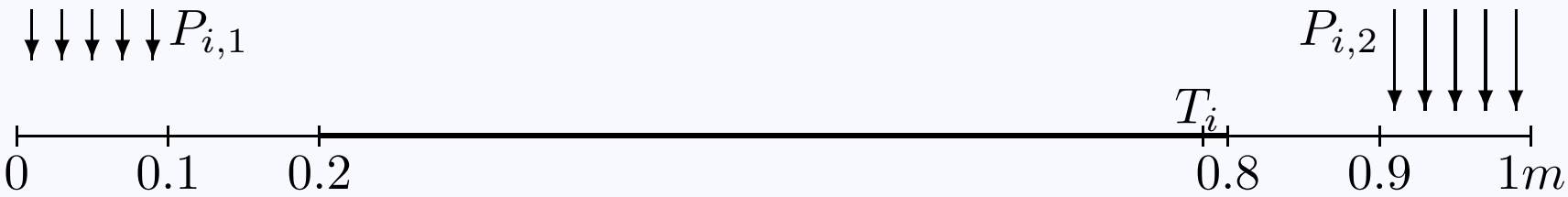


$$c_p = 1.00$$

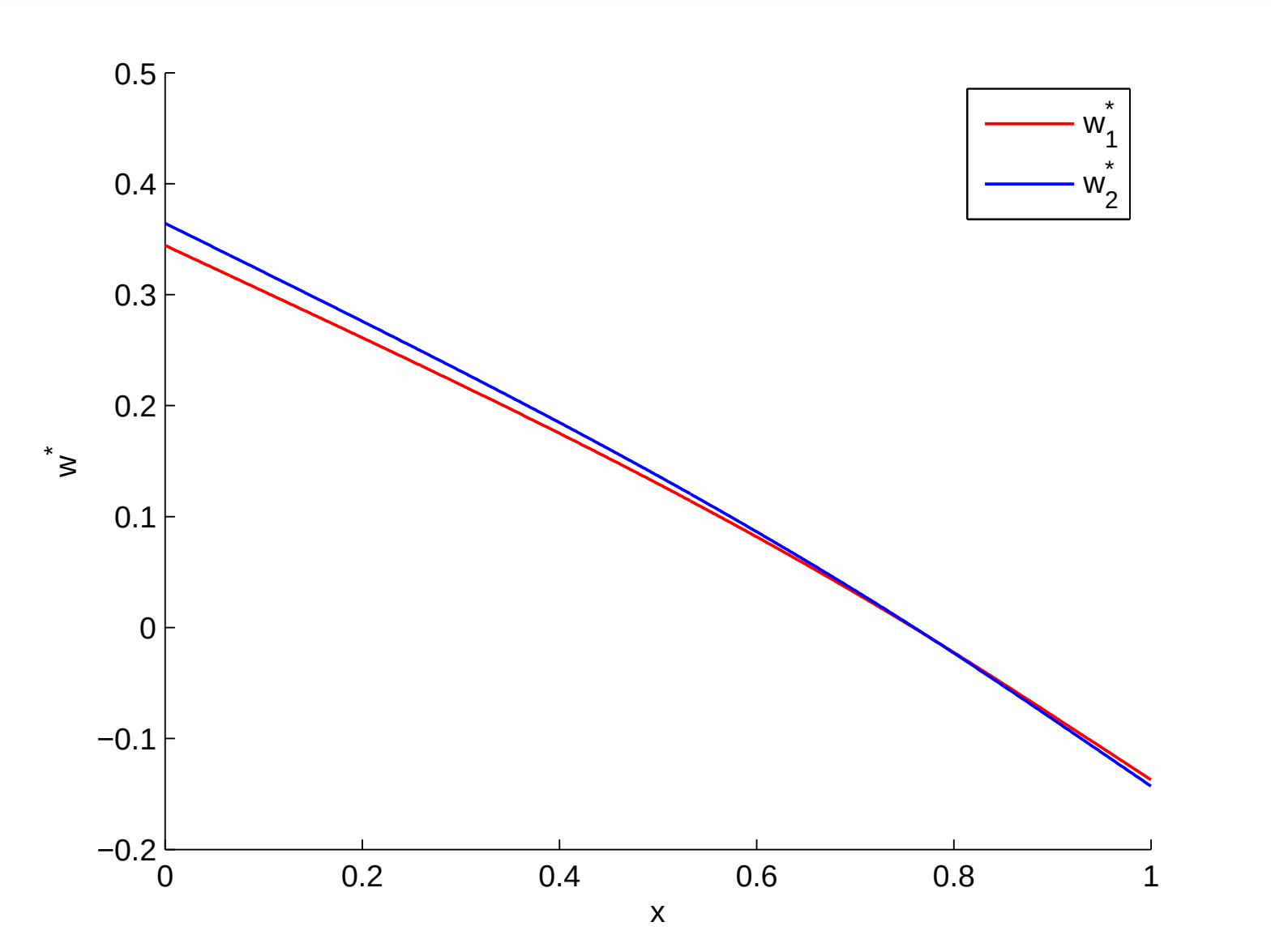

x 10^−3
4
2
0
−2
−4
−6
−8
−10
w*
0
0.2
0.4
0.6
0.8
1
x
w*1
w*2

Example 2 (unstable).

$$P_{1,1} = -100Nm^{-1}, \;\; P_{1,2} = -450Nm^{-1}, \;\; T_1 = 0.7864m, \;\; F_1 = -55.0N,$$
$$P_{2,1} = -100Nm^{-1}, \;\; P_{2,2} = -451Nm^{-1}, \;\; T_2 = 0.7867m, \;\; F_2 = -55.1N$$



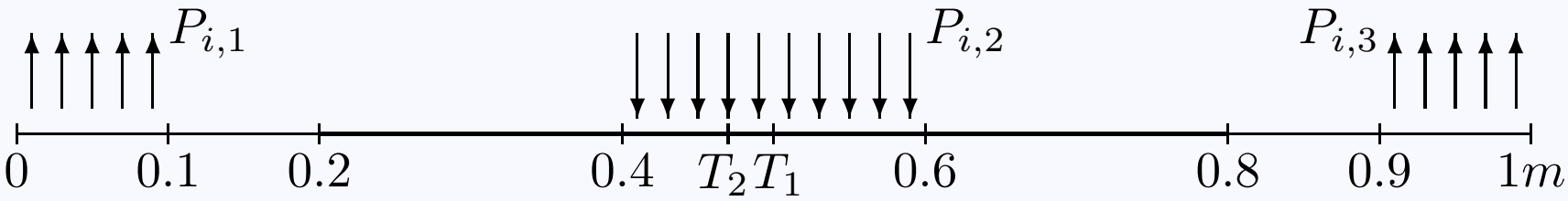
$$c_p = 25.83$$


0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
w*
0
0.2
0.4
0.6
0.8
1
x
w*1
w*2

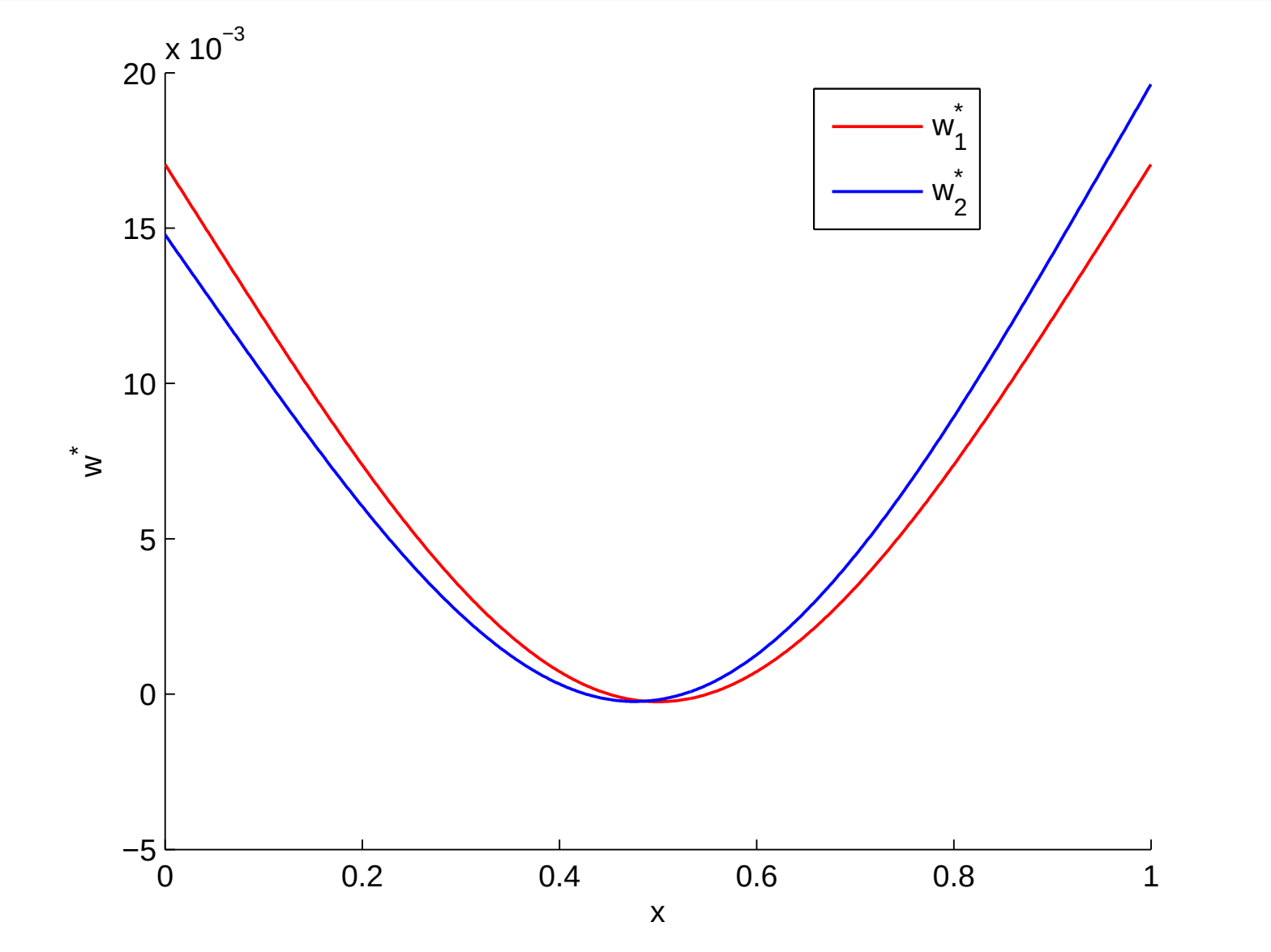
Example 3 (unstable).

$$P_{1,1} = 100Nm^{-1}, \;\; P_{1,2} = -110Nm^{-1}, \;\; P_{1,3} = 100Nm^{-1},$$
$$T_1 = 0.5000m, \;\; F_1 = -2.0N,$$

$$P_{2,1} = 100Nm^{-1}, \;\; P_{2,2} = -110Nm^{-1}, \;\; P_{2,3} = 101Nm^{-1},$$
$$T_2 = 0.4763m, \;\; F_2 = -1.9N$$



$$c_p = 44.53$$

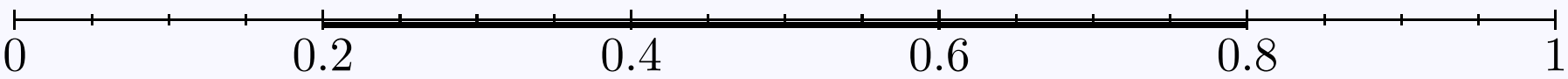

x 10^−3
20
15
10
5
0
−5
w*
0
0.2
0.4
0.6
0.8
1
x
w*1
w*2

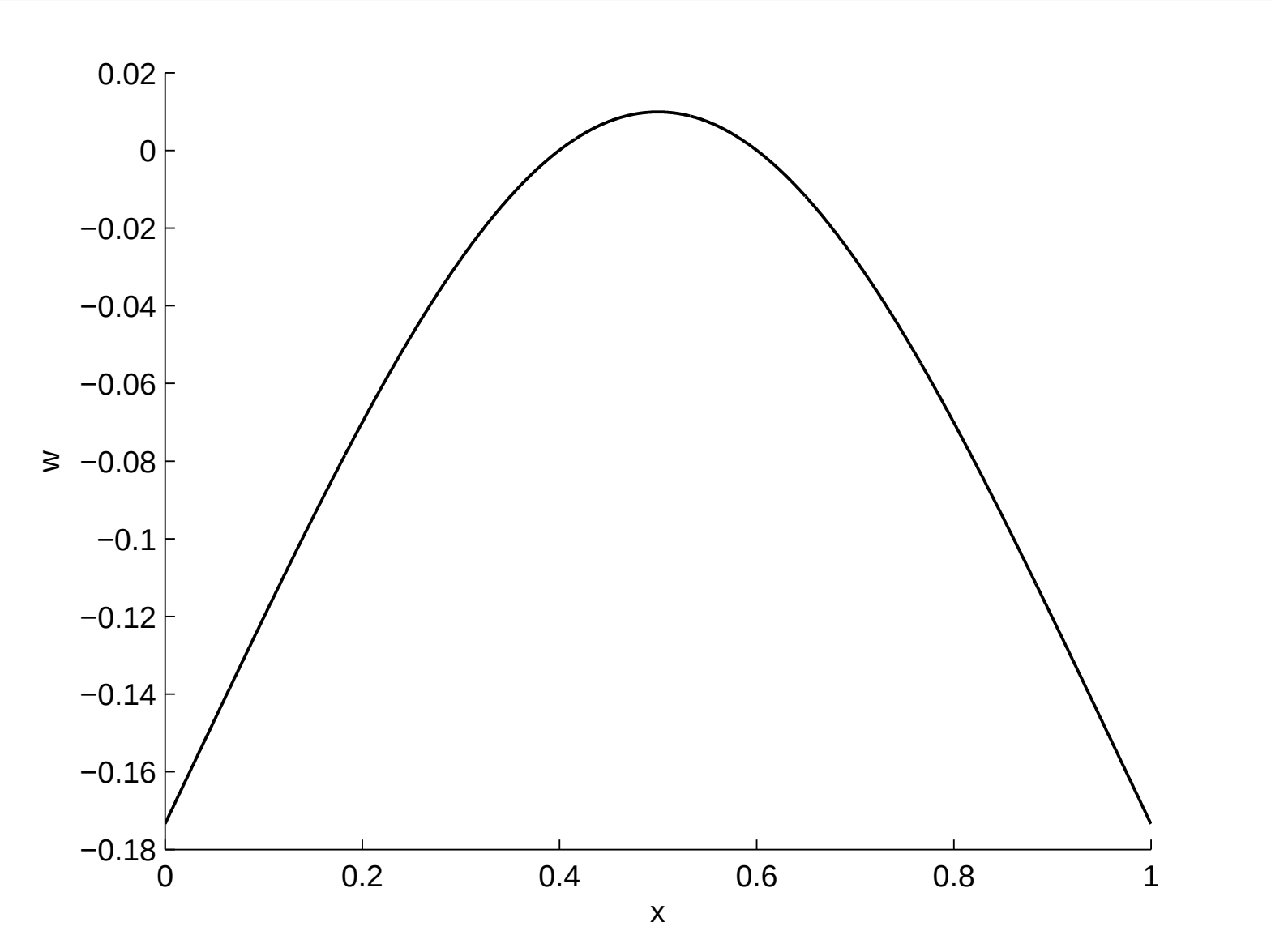
4.3 Verification of convergence results

$$(P) \quad \begin{cases} w^{(4)}(x) + \chi((1/5, 4/5))w^{-}(x) = f(x), & x \in (0,1), \\ w''(0) = w'''(0) = w''(1) = w'''(1) = 0, \end{cases}$$

$$f(x) = \begin{cases} -384\left(x - \frac{1}{2}\right)^2 + 32, & x \in (0, \frac{1}{5}) \cup (\frac{2}{5}, \frac{3}{5}) \cup (\frac{4}{5}, 1), \\ -\frac{16}{15}\left(x - \frac{1}{2}\right)^6 + \frac{4}{3}\left(x - \frac{1}{2}\right)^4 - & \\ -385\left(x - \frac{1}{2}\right)^2 + \frac{148816}{15000000} + 32, & x \in (\frac{1}{5}, \frac{2}{5}) \cup (\frac{3}{5}, \frac{4}{5}) \end{cases}$$

$$w(x) = -\frac{16}{15}\left(x - \frac{1}{2}\right)^6 + \frac{4}{3}\left(x - \frac{1}{2}\right)^4 - \left(x - \frac{1}{2}\right)^2 + \frac{148816}{15000000}$$



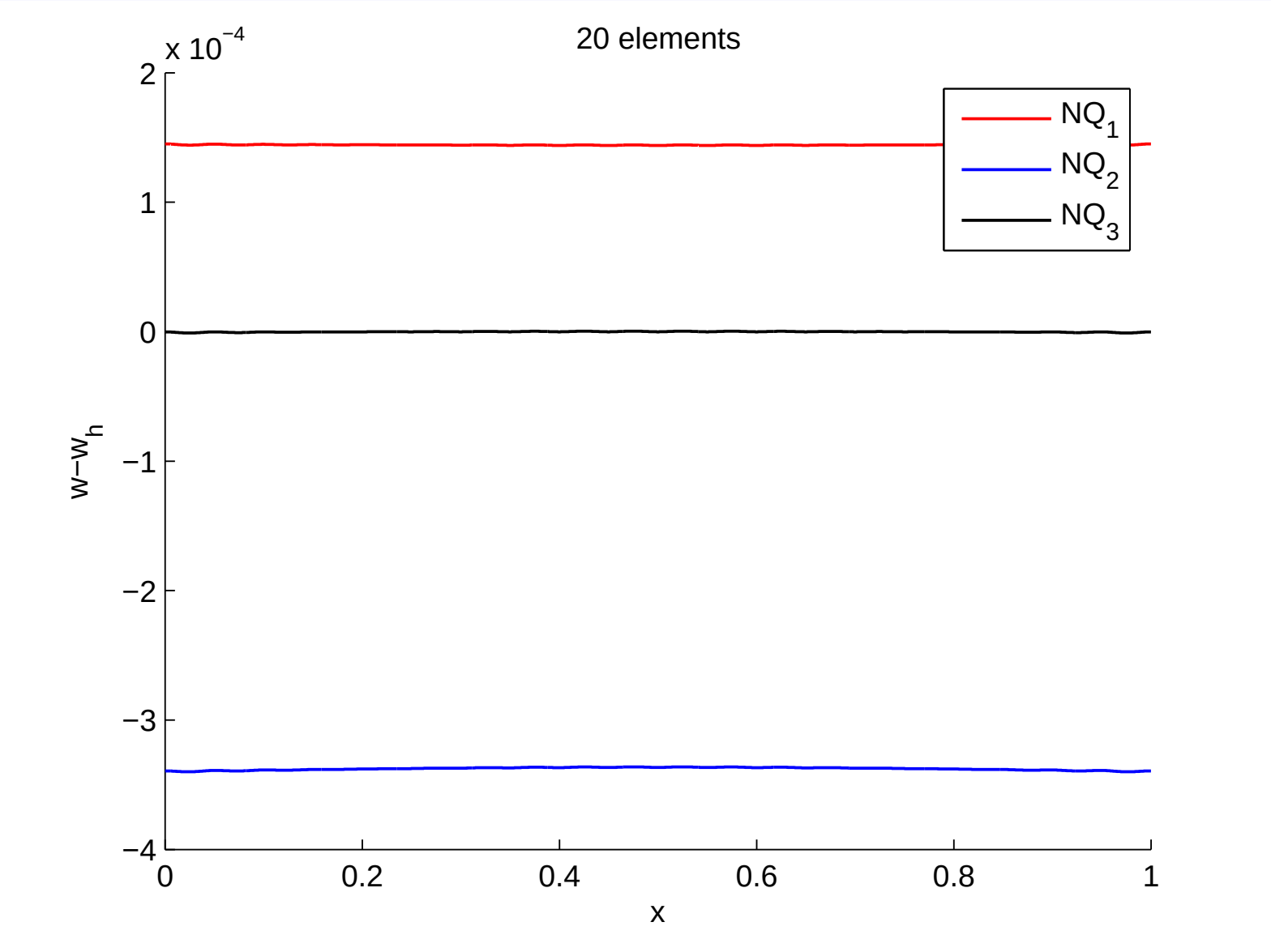

0.02
0
−0.02
−0.04
−0.06
−0.08
−0.1
−0.12
−0.14
−0.16
−0.18
w
0
0.2
0.4
0.6
0.8
1
x

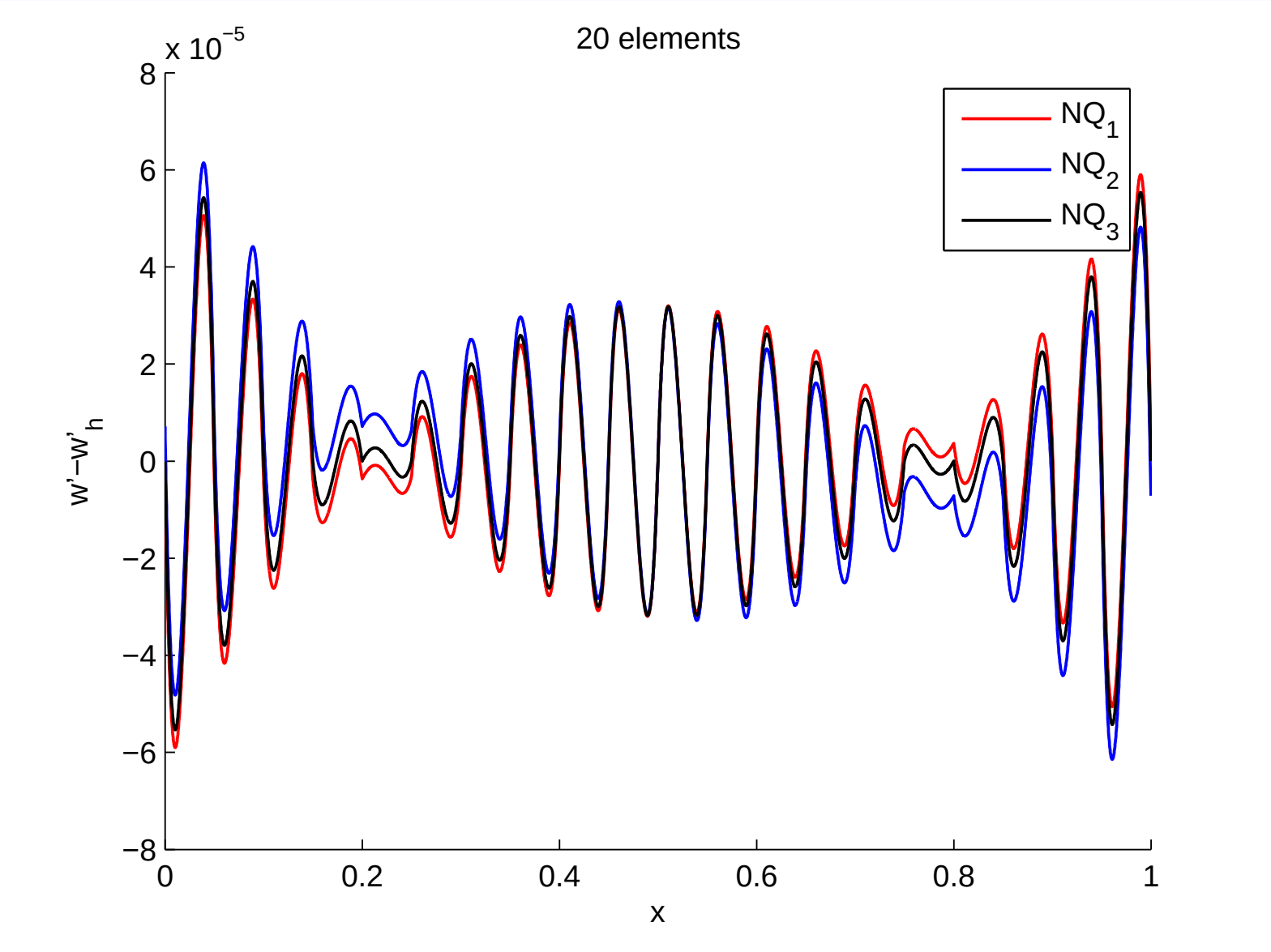
Investigated terms:

$$c_1 = \|w - w_h\|_{2,2}/h^2, \quad \|v\|_{2,2}^2 := \int_0^1 (v'')^2 \, dx + \int_{1/5}^{4/5} v^2 \, dx,$$
$$c_2 = |w - w_h|_{0,2}/h^2, \quad |v|_{0,2}^2 := \int_0^1 v^2 \, dx$$

c_1	10	20	40	80	160
NQ1	1.0422	1.0606	1.0651	1.0663	1.0666
NQ2	1.0423	1.0606	1.0652	1.0663	1.0664
NQ3	1.0422	1.0606	1.0651	1.0663	1.0666

c_2	10	20	40	80	160
NQ1	0.0575	0.0577	0.0577	0.0577	0.0585
NQ2	0.1559	0.1351	0.1289	0.1302	0.1388
NQ3	0.0005	0.0001	0.0000	0.0000	0.0017


20 elements
x 10^−4
2
1
0
−1
−2
−3
−4
w−w_h
0
0.2
0.4
0.6
0.8
1
x
NQ_1
NQ_2
NQ_3


20 elements
x 10^−5
8
6
4
2
0
−2
−4
−6
−8
w'−w'_h
0
0.2
0.4
0.6
0.8
1
x
NQ_1
NQ_2
NQ_3

Thank you for your attention.