Parallel Algebraic Schwarz Methods with Aggregation

R. Blaheta
Institute of Geonics AS CR, Ostrava,
& VŠB-TU Ostrava, Czech Republic
www.ugn.cas.cz

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Outline

- FEM solution of selfadjoint elliptic problems
- local FE problems and AS(δ) preconditioner
- two-level AS2(δ) preconditioner
- two-level AS2(0) preconditioner
- AS2(δ) preconditioner with aggregation
- AS2(0) preconditioner with aggregation
- hybrid preconditioners
- numerical results

The Problem

Generally, a selfadjoint elliptic boundary value problem :

$$
\begin{aligned}
-\sum_{ij} \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial u}{\partial x_j}\right) &= f \qquad \text{in } \Omega \\
u &= \hat{u} \qquad \text{on } \Gamma_0 \subset \partial\Omega \\
\sum_{ij} k_{ij}\frac{\partial u}{\partial x_j} n_i &= \hat{f} \qquad \text{on } \Gamma_1 \subset \partial\Omega
\end{aligned}
$$

FEM discretization gives an algebraic problem

$$Au = b, \quad u, b \in R^n$$

with a matrix A, which is SPD, ill conditioned, sparse and large scale (dimension $n \sim 10^5 - 10^7$).

Particularly,
a simple model:

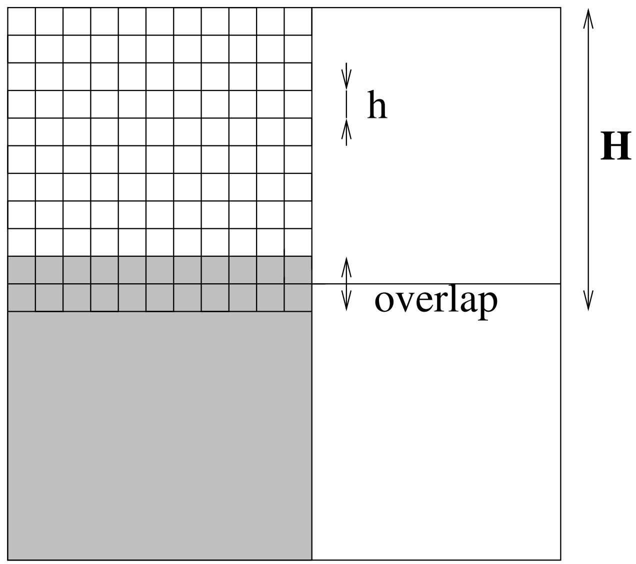
$\Omega = \langle 0, 2\rangle \times \langle 0, 3\rangle$

$-\triangle u = f$ in Ω

$u = 0$ on $\partial\Omega$

h=1/30, n=5100

Domain Decomposition and Local FE Spaces



triangulation	$\mathcal{T}_h, \;\; \Omega = \{E : \; E \in \; \mathcal{T}_h\}$
partition	$\mathcal{T}_h = \bigcup \mathcal{T}^0_{h,k} = \bigcup \mathcal{T}^\delta_{h,k}$
h	size of triangulation
H	size of subdomains
δ	size of overlap

subdomain	$\Omega^\delta_k = \left\{E : \; E \in \; \mathcal{T}^\delta_h\right\}$
DD	$\Omega = \Omega_1 \cup \ldots \cup \Omega_m$
FE space	$V_h = \left\{v \in H^1_D : \; v\|_E \in P_1\right\}$
local FE space	$V_{h,k} = \{v \in V_h : \; v = 0 \text{ on } \Omega \backslash \Omega_k\}$
SD	$V_h = V_{h,1} \cup \ldots \cup V_{h,m}$

Construction of

a preconditioner:

$R_k, \;\; R_k^T$

$A_k = R_k A R_k^T$

parallel FE or

algebraic construct.

Additive Schwarz Preconditioner

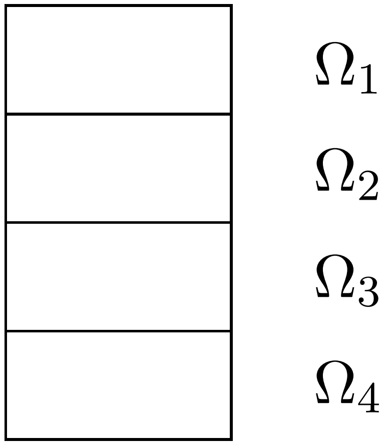
AS Preconditioner:

$$B_A = \sum_k R_k^T\, A_k^{-1}\, R_k$$

Model problem: $-\triangle u = f$ in $\Omega = \langle 0,2\rangle \times \langle 0,3\rangle$

$u = 0$ on $\partial\Omega$

DD type



#subdomains:	2	4	8	12	16	24
$\delta = 1$	16	22	30	36	42	45
$\delta = 2$	12	16	22	23	31	37
$\delta = 3$	10	13	17	22	25	30

Numbers of iterations for $\varepsilon = 10^{-3}$.

AS preconditioner is not numericaly scalable, the number of iterations grow as nuber of subdomains grow.

Analysis of AS(δ) Preconditioner

Matsokin, Nepomnyaschikh 1985, Lions 1988, Dryja, Widlund 1987,1989

$V = V_1 + \ldots + V_m$

D1: $\forall v \in V \ \exists v_k \in V_k : v = v_1 + \ldots + v_m, \quad \sum_k \| v_k \|_A^2 \leq K_0 \| v \|_A^2.$

D2 : $\forall v \in V \ \forall v_k \in V_k : v = v_1 + \ldots + v_m, \quad \| v \|_A^2 \leq K_1 \sum_k \| v_k \|_A^2.$

D1,**D2** $\Rightarrow \ \lambda_{\min}(B_A A) \geq 1/K_0, \ \lambda_{\max}(B_A A) \leq K_1,$ cond $(B_A A) \leq K_0 K_1$

D1:

- ex. partition of unity $\theta_1, \ldots, \theta_m$

  $\sum \theta_k = 1$ on Ω

  $\theta_k \in C^\infty(R^d), \theta_k = 0$ on $R^d \backslash \Omega_k$

  $\| \nabla \theta_k \|_{L^\infty(\Omega)} \leq c/\delta$

  $\Rightarrow$

  - $v \in V_h \Rightarrow v = \sum_k \Pi_h(\theta_k v)$
  - $K_0 = C(1 + \delta^{-2})$

- ex. interpolation $\Pi_h : C(\Omega) \to V_h$

D2: If $\mathcal{E} = (\varepsilon_{kl}), \ \varepsilon_{kl} = \cos(V_k, V_l)_A$, then $K_1 \leq \rho(\mathcal{E}) \leq \max \sum_l \varepsilon_{kl}$.

Potentially m-independent.

AS Preconditioner - inexact subproblem solvers

Model problem: $-\triangle u = f$ in $\Omega = \langle 0,2\rangle \times \langle 0,3\rangle$

$u = 0$ on $\partial\Omega$

AS Preconditioner:

$$B_A = \sum_k R_k^T \widetilde{A}_k^{-1} R_k$$

$$\widetilde{A}_k = LL^T$$

is an incomplete factorization of A, here: cholinc('0') MATLAB procedure

#subdomains:	2	4	8	12	16	24
AS($\delta = h$)	16	22	30	36	42	45
inexact	47	49	52	54	55	59
AS($\delta = 2h$)	12	16	22	23	31	37
inexact	50	51	51	53	53	55
AS($\delta = 3h$)	10	13	17	22	25	30
inexact	55	55	55	54	54	55

Numbers of iterations for $\varepsilon = 10^{-3}$.

Two-level AS2(δ) Preconditioner

AS2 Preconditioner:

$$B_{A2} = \sum_k R_k^T A_k^{-1} R_k$$

$$+ R_0^T A_0^{-1} R_0$$

A_0 arises from discretization on $V_0 = V_H$, $H > h$.

$$V_h = V_0 + \ldots + V_m$$

$- \triangle u = f$ in $\Omega = \langle 0, 2 \rangle \times \langle 0, 3 \rangle$ $u = 0$ on $\partial\Omega$

#subdomains:	2	4	8	12	16	24
AS($\delta = h$)	16	22	30	36	42	45
AS2, H=3h	7	8	8	9	9	9
AS($\delta = 2h$)	12	16	22	23	31	37
AS2, H=3h	7	7	8	8	7	8

Numbers of iterations for $\varepsilon = 10^{-3}$.

Numericaly scalable, but we have to construct coarse grid and assembly A_0. Difficulties: (1) coarse grid construction, (2) to balance A_0 with A_k, (3) components are not derived from A itself (black box features of AS).

Two-level AS2(0) Preconditioner

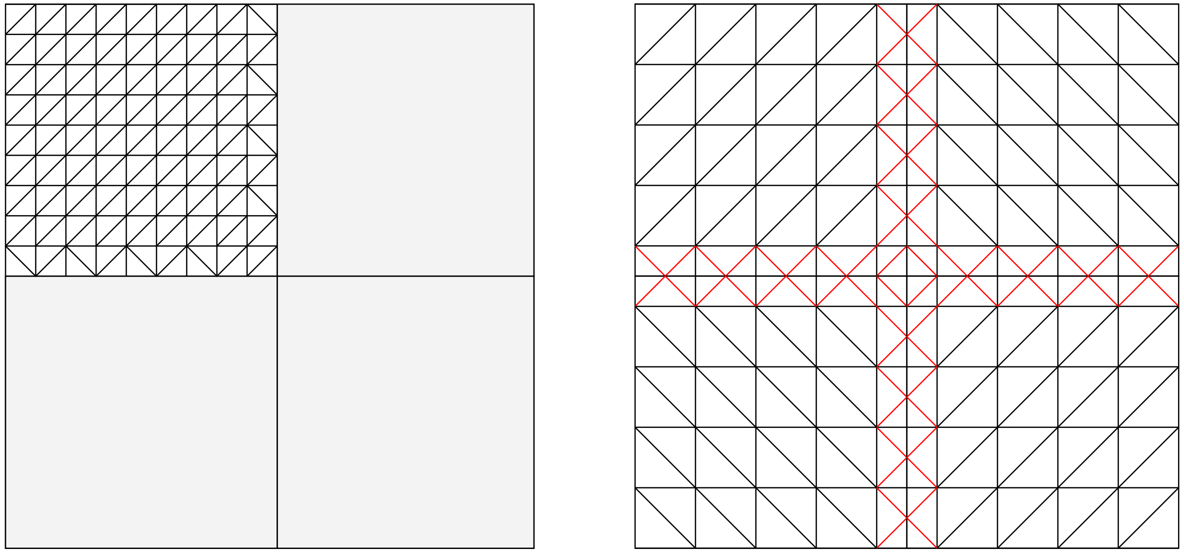
$\mathcal{T}_h \to \mathcal{T}^0_{h,k} \to \Omega^0_k \to V_k \subset V_h$

$V_k = \{v \in V_h : v|_{\Omega \setminus \Omega_k} = 0\}$

$\mathcal{T}_H \to V_0 = V_H$

$V_h = V_0 + \sum V_k$

$\mathcal{T}_H$ should contain DOF on interface $\Gamma = \bigcup_{k \neq l} \partial\Omega_k \cap \partial\Omega_l$



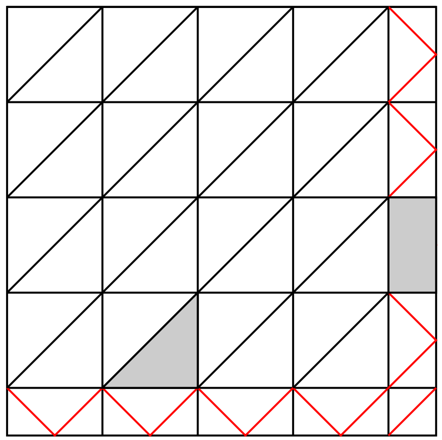
$$W_k = \{v \in V_k : \ v(x) = 0 \ if \ x \in N(\mathcal{T}_H)\} \ \Rightarrow V = V_0 \oplus \sum W_k = V_0 \oplus W_0$$

$$v = v_0 + \sum w_k \Rightarrow$$

$$(1-\gamma)\left[\|v_0\|^2_A + \sum \|w_k\|^2_A\right] \leq \|v\|^2_A \leq (1+\gamma)\left[\|v_0\|^2_A + \sum \|w_k\|^2_A\right]$$

$$\gamma = \cos\left(V_0, W_0\right)_A, \quad K_0 = 1/\left(1-\gamma\right), \ K_1 = 1+\gamma$$

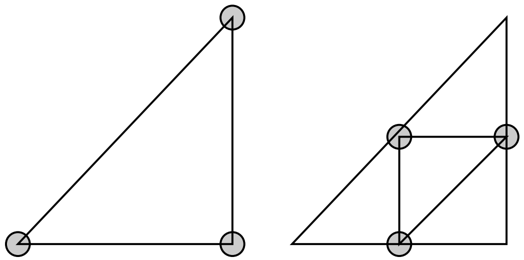
CBS constant - boundary macroelements



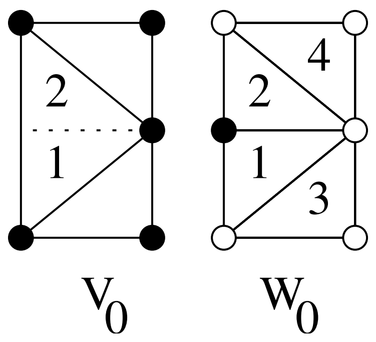
$$\gamma = \sup\left\{ \frac{a(v,w)}{\sqrt{a(v,v)}\sqrt{a(w,w)}} \;:\; v \in V_0, w \in W_0,\; v,w \neq 0 \right\}$$

$$a\,(v,w) = \sum_E a_E\,(v,w) = \sum_E \int_E \langle D\,\nabla v,\, \nabla w\rangle\, dx$$

$$\gamma = \max \gamma_E, \;\; \text{E - inner/interface macroelement}$$



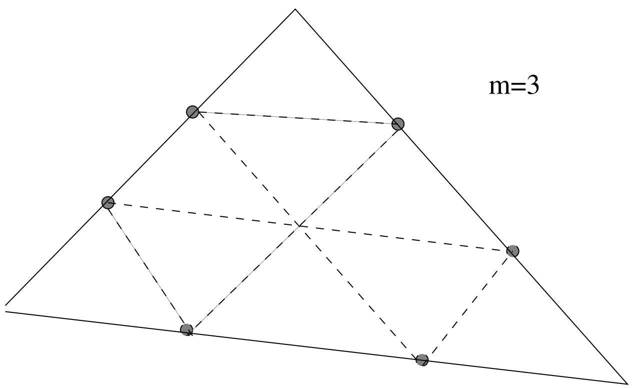
Interface: $v \in V_0(E) \to \nabla v = (\delta_x,\, \delta_y)$ in $T_1,\, T_2$.
$w \in W_0(E) \to \nabla w = (d_x,\, d_y)$ in T_1, $\nabla w = (d_x,\, -d_y)$ in T_2, $\nabla w = 0$ in T_3, T_4. If $D = \mathrm{diag}[k_x,\, k_y]$, then



$$\left.\begin{array}{r} a_E\,(v,w) = 2k_x\delta_x\, d_x\,\Delta \\ a_E\,(v) \leq 2k_x\delta_x^2\,\Delta \\ a_E\,(w) \leq 2(k_x d_x^2 + k_y d_x^2)\Delta \end{array}\right\} \Rightarrow \gamma_E \leq \sqrt{\frac{k_x}{k_x+k_y}}.$$

CBS constant - inner macroelements

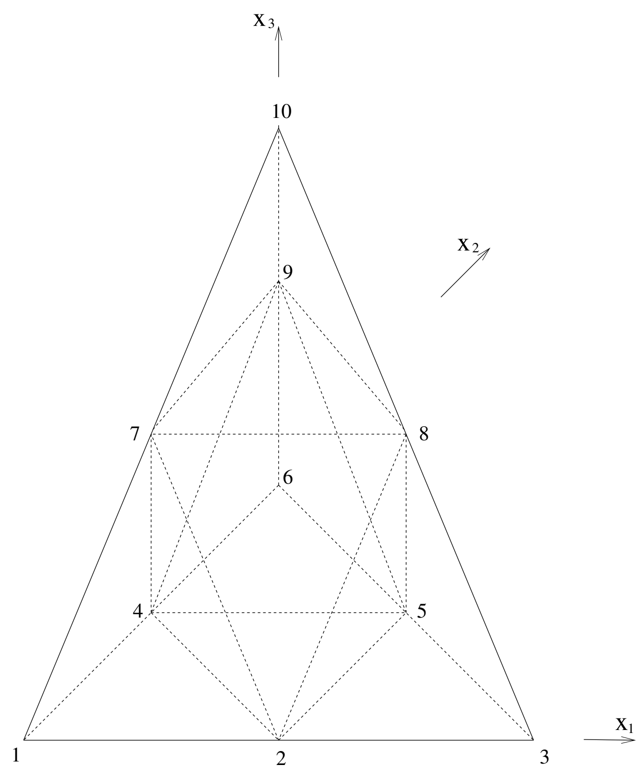
Maitre, Musy 1981:
$\gamma = \sqrt{1/2}$ for isotropic Laplacian
and rectangular elements, $m_p = 2$.



- m_p fold refinement!
- anisotropic Laplacian
- general elasticity $c_{i,j,k,l}$!
- arbitrary element shape

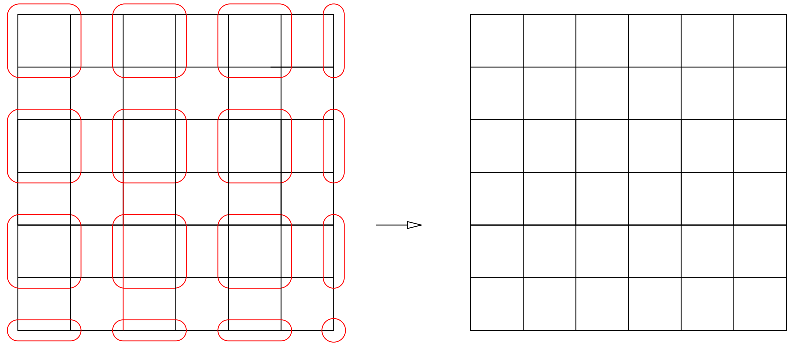
$\gamma \le \sqrt{\frac{m_p^2 - 1}{m_p^2}}$ Axelsson, RB 2001

$m_p = 2$, Achchab, ... NLAA 2001
m_p^3 tetrahedra



$\gamma \le \sqrt{1 - \frac{2}{m_p^4 + m_p^2}}$ RB NLAA 2003

AS2(δ)-AG Preconditioner with Aggregation



$$\Omega = \Omega_1 \cup \ldots \cup \Omega_m$$

$$V_h = \text{span } \left\{\phi_i^h\right\}_{i=1}^n$$

$\{1, \ldots, n\} = J_1 \cup \ldots \cup J_N$, disjoint

$$V_0 = \text{span}\{\psi_i\}, \quad \psi_i = \sum_{j \in J_i} \phi_j^h$$

$$c_1 H^d \le |\text{supp}\psi_i| \le c_2 H^d, \; H \le \{\mathbf{H}, kh\}$$

$$Q : V_h \to V_0, \quad Qv = \sum \alpha_k(v)\,\psi_k$$

$$\alpha_k = \frac{1}{\mu(\text{supp}\psi_k)} \int_{\text{supp}\psi_k} u(x)\,dx$$

Then:

- $| Qv |^2_{H^1(\Omega)} \le c\frac{H}{h} | v |^2_{H^1(\Omega)}$
- $\| v - Qv \|_{L_{2(\Omega)}} \le CH | v |_{H^1(\Omega)}$

  (weak approximation property)

- $V_h = V_0 + V_1 + \ldots + V_m$
- $v \in V \;\Rightarrow\; v = v_0 + v_1 + \ldots + v_m$
- $v_0 = Qv, \quad v_k = \Pi_h\left(\theta_k(v - v_0)\right)$
- $K_0 = C\left(1 + h^{-1}H + \delta^{-2}H^2\right)$
- $K_1 \le 2(1 + \rho(\mathcal{E}))$

$\text{cond}\,(G_A A) \le K_0 K_1$, etc.

Blaheta 1986,1989, Braess 1994, Vaněk, Mandel, Brezina 1996, Brezina 1997, Jenkins et al. 2001

AS2(0)-AG Preconditioner

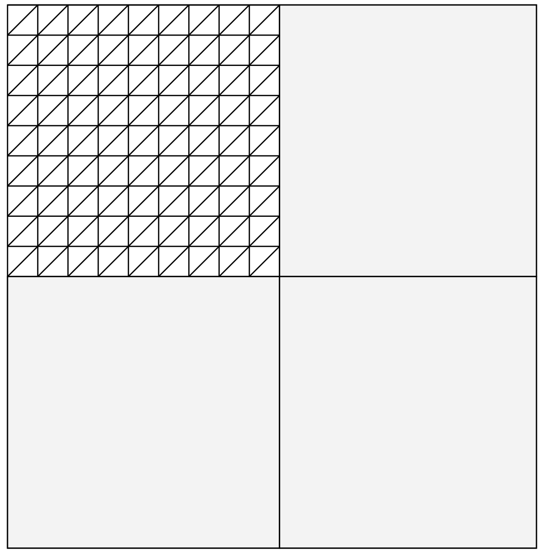
$\mathcal{T}_h \rightarrow \mathcal{T}^0_{h,k} \rightarrow \Omega^0_k \rightarrow V_k \subset V_h$

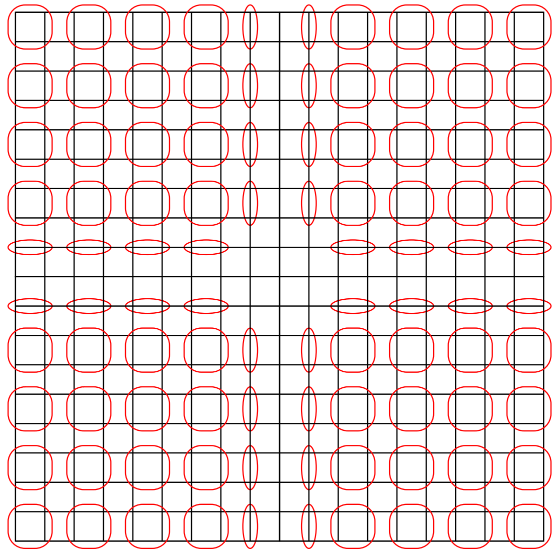
$V_k = \{v \in V_h : v|_{\Omega \setminus \Omega_k} = 0\}$

V_0 created by aggregation except the inner interface

$\Gamma = \bigcup_{k \neq l} \partial\Omega_k \cap \partial\Omega_l$

$V_h = V_0 + \sum V_k$





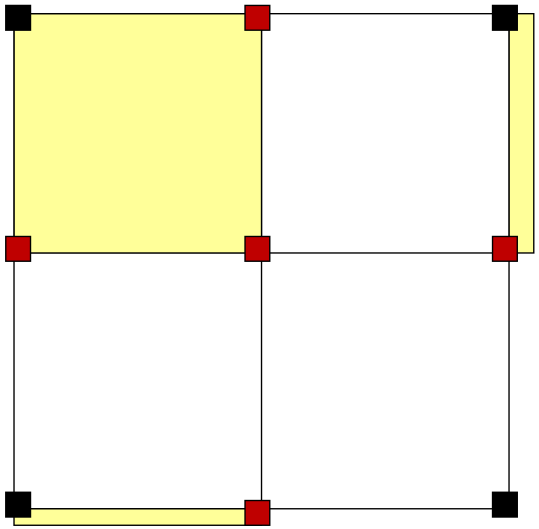
not a special grid

$C \subset N_h$ - for each aggregate of nodes G_i there is one point in $G_i \cap C$

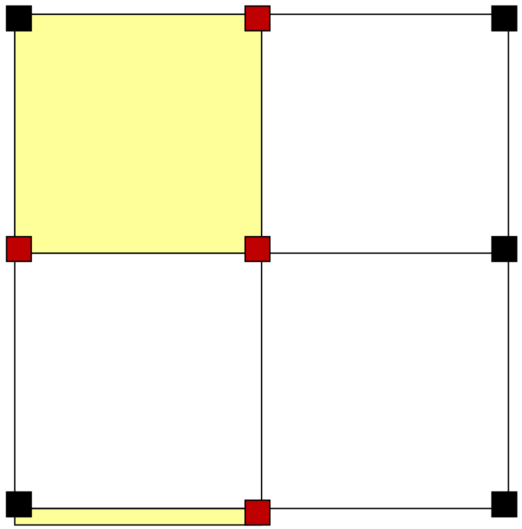
$W_k = \{v \in V_k : \ v(x) = 0 \ if \ x \in C\} \ \Rightarrow V = V_0 \oplus \sum W_k = V_0 \oplus W_0$

$\gamma = \cos(V_0, W_0)_A, \quad K_0 = 1/(1-\gamma), \ K_1 = 1 + \gamma$

Analysis of AS2(0)-AG Preconditioner



inner macroelement



interface macroelem.

CBS constants for isotropic Laplacian:

inner macroelement

γ =0.8966

interface macroelement

γ =0.8916

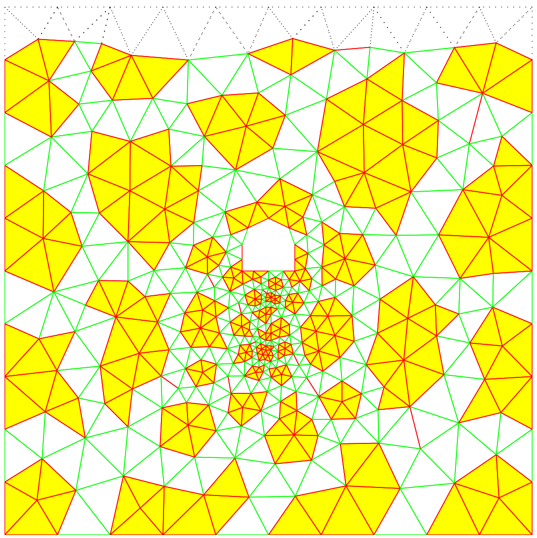
Influence of anisotropy: k=0,1,2,3,6

$$R_\alpha^T \begin{bmatrix} 1 & \\ & 10^k \end{bmatrix} R_\alpha :$$

0°	0.8966	0.9080	0.9123	0.9128	0.9129
45°	0.8966	0.9237	0.9903	0.9990	1.000
90°	0.8966	0.9080	0.9123	0.9128	0.9129

AS2(δ)-SA Preconditioner and other possibilities

1. Aggregation / Agglomeration on general grids accounting strength of the couplings



2. SA -Smoothed Aggregations
3. EA -Enhanced Aggregations

$\Omega = \langle 0,2\rangle \times \langle 0,3\rangle$, 2D plane strain, $\nu = 0.3$. Zero displ. on $\partial\Omega$, h=1/30

#subdomains:	4	16	24
c-grid H=3h, AP	8	8	9
AS2($\delta = 2$)-AG(2h)	17	20	20
+ rotations	16	18	19
AS2($\delta = 0$)-AG(2h)	16	17	18
smooth. AS2(2)-AG(2h)	12	14	14

Numbers of iterations for $\varepsilon = 10^{-3}$.

Hybrid Preconditioners

$B_{A2}: \; r \rightarrow g,$

$$g_k = R_0^T A_0^{-1} R_0 \, r + \sum_1^m R_k^T A_k^{-1} R_k \, r$$

$B_{SH}: \; r \rightarrow g,$

$$\bar{g} = R_0^T A_0^{-1} R_0 \, r$$

$$\bar{\bar{g}} = \bar{g} + \sum_1^m R_k^T A_k^{-1} R_k (r - A\bar{g})$$

$$g = \bar{\bar{g}} + R_0^T A_0^{-1} R_0 (r - A\bar{\bar{g}})$$

$B_H: \; r \rightarrow g, \quad g = \bar{\bar{g}}$

$B_{SH} = B_0 + (I - B_0 A)(\sum_1^m B_k)(I - AB_0)$, where $B_k = R_k^T A_k^{-1} R_k$

$\sigma(B_{SH} A) \subset \langle \min\{1, \; \lambda_{\min}(B_{A2} A)\}, \; \lambda_{\max}(B_{A2} A) \rangle$

CG with Hybrid Preconditioner

$u^0 = R_0^T A_0^{-1} R_0\, b, \quad r^0 \;=\; b - Au^0$

for $i = 0, 1, \ldots$ **until** $\| \, r^i \, \| \leq \varepsilon \; \| \, b \, \|$ **do**

$w \;=\; (I - AB_0) r^{\,i}$ (if A_0^{-1} is exact, then $w \;=\; r^{\,i}$)

$w \;=\; \sum_1^m B_k w$

$g^i \;=\; (I - B_0 A) w$

$\sigma_i = \langle r^i ,\, g^i \rangle$

if $i = 0$ then $\beta_i = \sigma_i / \sigma_{i-1}$

if $i = 0$ then $v^i \;=\; g^i$ else $v^i \;=\; g^i + \beta_i \, v^{i-1}$

$w^i \;=\; Av^i,\ \alpha_i \;=\; \sigma_i / \langle w^i, v^i \rangle$

$u^{i+1} \;=\; u^i + \alpha_i v^i$

$r^{i+1} \;=\; r^i - \alpha_i w^i$

end

GPCG -nonsymmetric/nonlinear preconditioners

given $u^0 \rightarrow r^0 = b - Au^0,\ g^0 = G(r^0),\ v^0 = g^0$

for $i = 0, 1, \ldots$ **until** $\| r^i \| \leq \varepsilon \| b \|$ **do**

$$w^i = Av^i$$

$$\alpha_i = \sigma_i / \langle w^i, v^i \rangle$$

$$u^{i+1} = u^i + \alpha_i v^i$$

$$r^{i+1} = r^i - \alpha_i w^i$$

$$v^{i+1} = g^{i+1} = G(r^{i+1})$$

for $k = 1, \ldots, \min\{i+1, s\}$ **do**

$$\beta_{i+1}^{(k)} = (\langle g^{i+1}, r^{i+2-k} \rangle - \langle g^{i+1}, r^{i+1-k} \rangle)/\sigma_{i+1-k}$$

$$v^{i+1} = v^{i+1} + \beta_{i+1}^{(k)} v^{i+1-k}$$

end $\Rightarrow \sigma_{i+1}$

end

Extra:

$s - 1\times$ vector storage

$r^i,\ r^{i-1}, \ldots, r^{i+2-s} (r^0)$

$s\times$ inner products

$\langle g^{i+1}, r^{i+1-k} \rangle$,

$k = 1, \ldots, \min\{i+1, s\}$

For s=1: one extra inner product

Schwarz preconditioner with aggregation

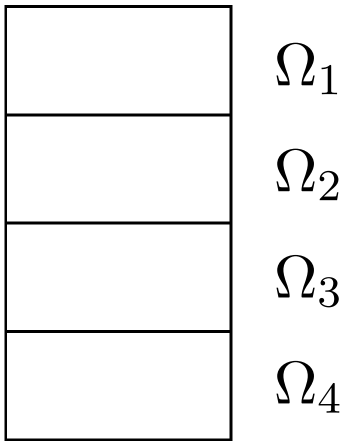
Model problem:

$\Omega = \langle 0, 2\rangle \times \langle 0, 3\rangle$

$- \triangle u = f$ in Ω

$u = 0$ on $\partial\Omega$

h=1/30, n=5100



Overlap 2h, #subdomains:	4	16	24
c-grid H=3h, AP	7	7	8
c-grid H=3h, HP	6	6	6
aggreg. 2h, AP	13	17	17
aggreg. 2h, HP	10	11	11
interface & aggreg. 2h, AP	14	14	14
interface & aggreg. 2h, HP	8	8	8
smooth. aggreg. 2h, AP	10	11	11
smooth. aggreg. 2h, HP	7	7	8

Numbers of iterations for $\varepsilon = 10^{-3}$. AP=additive preconditioner, HP=hybrid preconditioner + GPCG[1]

Schwarz precond. with aggregation: Elasticity

Model problem:

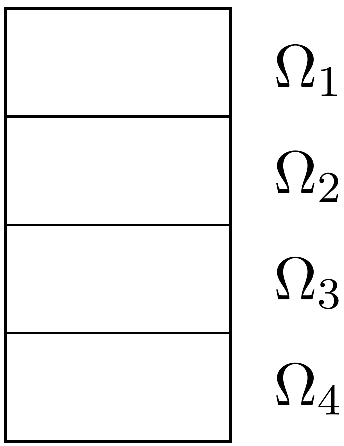
$\Omega = \langle 0, 2\rangle \times \langle 0, 3\rangle$

2D plane strain,

$\nu = 0.3.$

zero displ. on $\partial\Omega$,

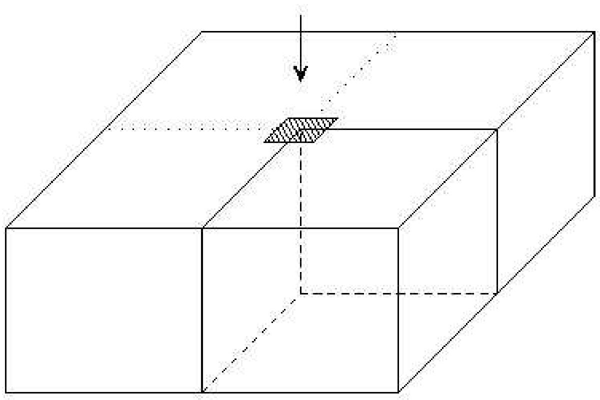
h=1/30, n=5100



Overlap 2h, #subdomains:	4	16	24
c-grid H=3h, AP	8	8	9
c-grid H=3h, HP	6	7	8
aggreg. 2h, AP	17	20	20
aggreg. 2h, HP	12	13	14
interface & aggreg. 2h, AP	16	17	18
interface & aggreg. 2h, HP	8	9	9
smooth. aggreg. 2h, AP	12	14	14
smooth. aggreg. 2h, HP	9	10	10

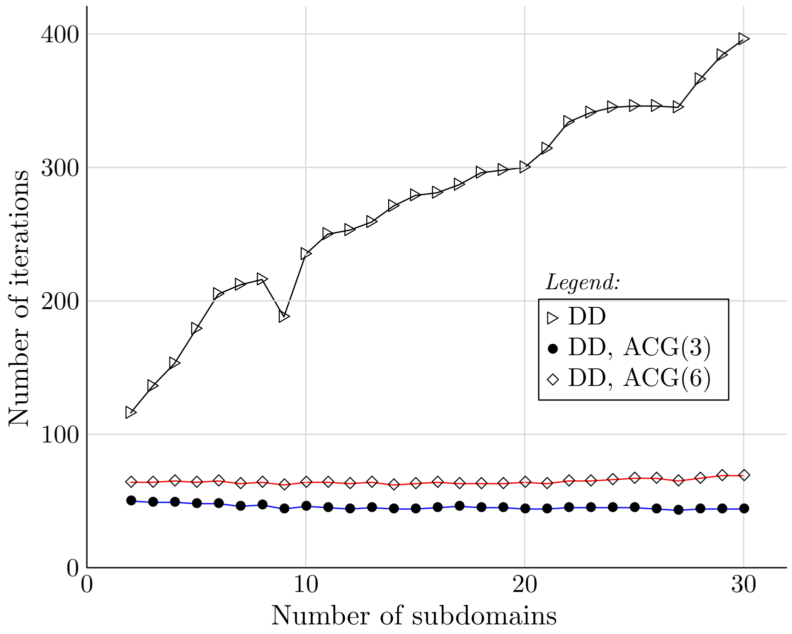
Numbers of iterations for $\varepsilon = 10^{-3}$. AP=additive preconditioner, HP=hybrid preconditioner + GPCG[1]

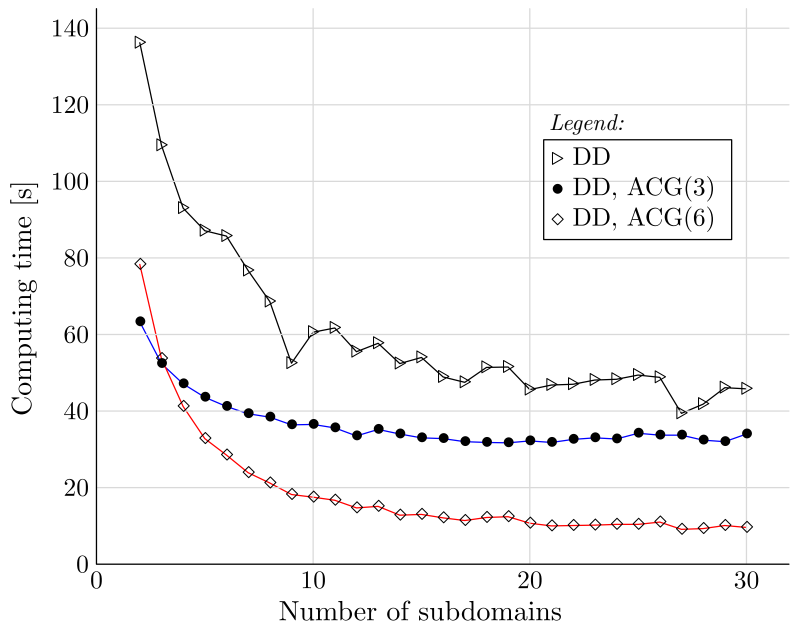
3D Example - elasticity - parallel Schwarz method



3D model problem: **square footing benchmark, 1 594 323 DOFs.**

The additive Schwarz method, AGG(3), AGG(6), Linux cluster with 16 computing nodes (2 AMD Athlon/2600, 3 GB). Myrinet L9 2Gb/s and FastEthernet.





Conclusions

- Schwarz preconditioners
  - importance of a coarse approximation
  - algebraic construction
  - lot of variants AS2(δ), AS2(0), ...
- hybrid preconditioners and their implementation
- numerical results

Other issues

- subproblem solvers (inner iterations, balance)
- robustness
- implementation - parallel, black box

Thank you for your attention.