Multi-level hierarchical preconditioning (An introduction)

Radim Blaheta, Petr Byczanski

Institute of Geonics AS CR, Ostrava, Czech Republic
www.ugn.cas.cz

SNA '06
January 16-20, 2006

Outline of the talk

- **Hierarchical decompositions and AMLI preconditioners**
  HB, additive and multiplicative preconditioners, sub-solvers
- **Analysis of HD methods through CBS constant**
  2D and 3D, robustness w.r.t. anisotropy and element shape
- **Fully algebraic AMLI with aggregation or agglomeration**
  decomposition, analysis, CBS, further development
- **AMLI for nonconforming FE's**
  Two ways of decomposition, CBS, optimal preconditioners

Solved problems

BVP in $\Omega \Rightarrow$ find : $\quad u \in V(\Omega) : \;\; a(u,v) = b(v) \quad \forall v \in V(\Omega)$,

$$a(u,v) = \int_\Omega \langle D\nabla u, \nabla v\rangle \, dx,$$

D ... SPD, piecewise constant, *physical anisotropy*.

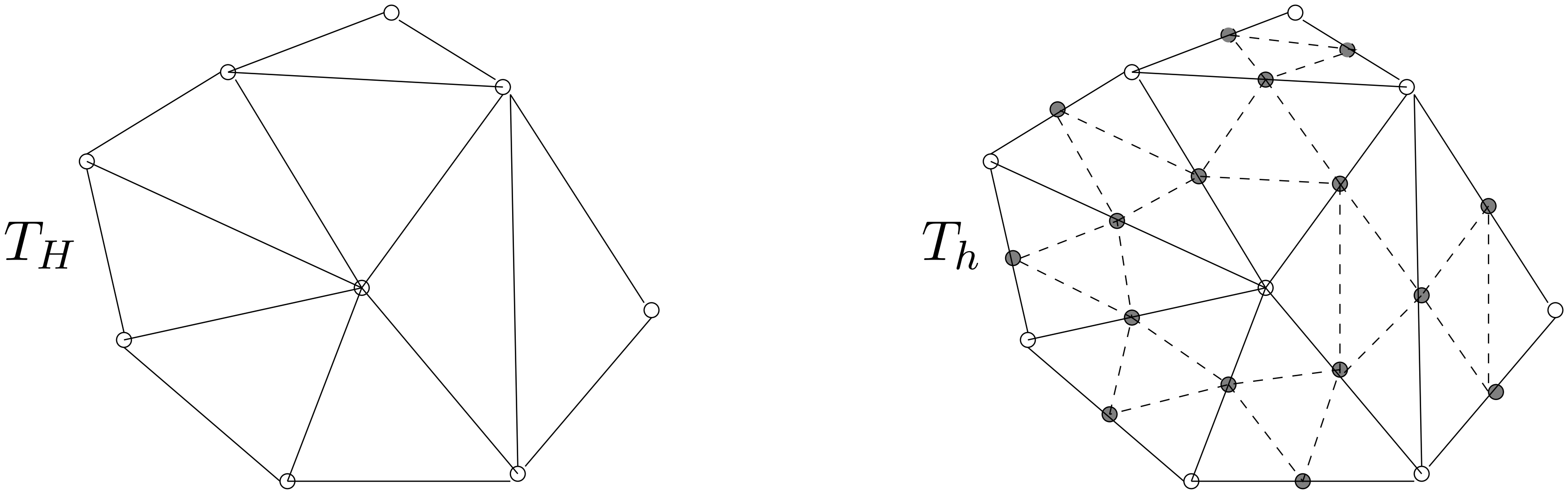
$$V = \left\{ v \in H^1(\Omega), v(x) = 0 \text{ on } \Gamma_0 \right\}$$

FEM: mostly linear triangular/tetrahedral FE. *Numerical anisotropy*.

$V_h = \mathrm{span}\{\phi_i^k\} \subset V(\Omega)$, $\quad u_h = \sum_1^n u_i \phi_i^k$, $\quad u = [u_i]$

NLA: $\quad Au = b$, $\;A$ is SPD. We are interested in *solvers with good complexity (optimal order), robustness and scalability (parallelizable)*.

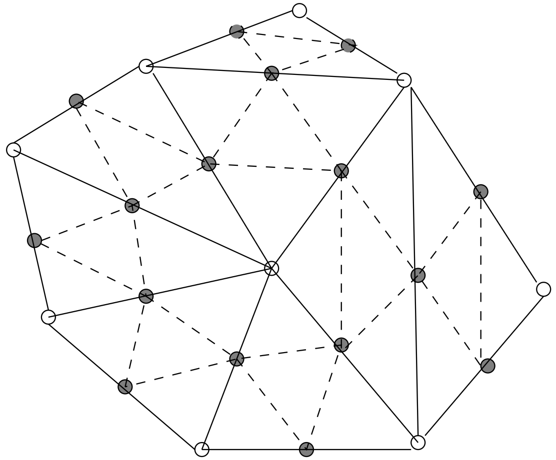
Hierarchical FEM (geometrical hierarchy)

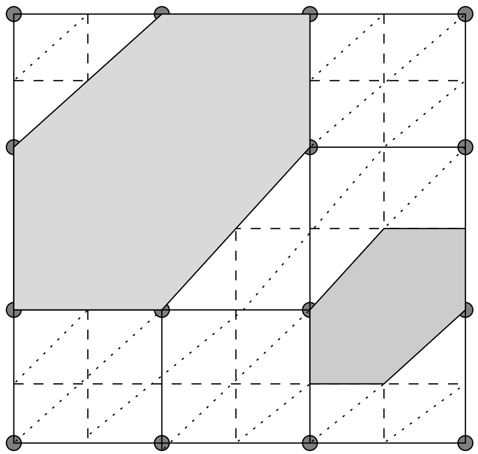


Grid hierarchy: FE discretization & system solution

- AMLI : coarse grid space + fine grid complement
- Multigrid: coarse grid space + full fine grid space with cheap solver
- Two-level Schwarz methods: coarse grid space + local fine grid spaces
- Local grid refinement techniques

Nodal and hierarchical bases (NB, HB)





- **Grid $\mathcal{T}_h$, nodes $\mathcal{N}_h = \{x_1, \ldots, x_n\}$, nodal basis $\{\phi_i^h\}_{i=1}^n \subset V_h$**
- **Grid $\mathcal{T}_H$, nodes $\mathcal{N}_H \subset \mathcal{N}_h$, nodal basis $\{\phi_i^H\} \subset V_H$**
- $\mathcal{N}_h = \mathcal{N}_H \cup \mathcal{N}_h^+ \Rightarrow$ **hierarchical basis:**

$$\{\bar{\phi}_i^h\} \subset V_h\colon\ \bar{\phi}_i^h = \begin{cases} \phi_i^h & \text{if } x_i \in \mathcal{N}_h^+ \\ \phi_i^H & \text{if } x_i \in \mathcal{N}_H \end{cases}$$

FE matrices in NB and HB

Order nodes as: nodes in $\mathcal{N}_h^+$ **first** and nodes in $\mathcal{N}_H$ **next**.
Similarly, order both nodal and hierarchical basis functions.

Transformation of bases:

$$\bar{\phi}_i^h = \sum_k J_{ik}\, \phi_i^h, \qquad J = (J_{ij}) = \begin{bmatrix} I_1 & 0 \\ I_{21} & I_2 \end{bmatrix} \begin{matrix} N\left(V_h^+\right) \\ N\left(V_H\right) \end{matrix},$$

I_{21} from interpolation, $(I_{21})_{kl} = \bar{\phi}_k^h(x_l)\ ,\ \ x_l \in \mathcal{N}_h^+$.

$$\overline{A} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{12}^T & \overline{A}_{22} \end{bmatrix} = \left[a\left(\sum_k J_{ik}\, \phi_i^h,\ \sum_l J_{jl}\, \phi_l^h \right)\right] = JAJ^T$$

- $\overline{A}$ is denser than A,
- $\overline{A}_{11} = A_{11},\ \ \overline{A}_{22} = A_H$!!! there is a block with global information
- $\overline{S} = \overline{A}_{22} - \overline{A}_{21}\, \overline{A}_{11}^{-1}\, \overline{A}_{12} = S = A_{22} - A_{21} A_{11}^{-1} A_{12}$

Hierarchical decomposition of NB and HB matrices

NB:

$V_h = \text{span}\{\phi_i^{\mathbf{h}}\}$

HB: $V_h = V_H + V_h^+ =$

$$\text{span}\{\phi_i^{\mathbf{h}} : \ x_i \in N\left(V_h^+\right)\} + \text{span}\{\phi_i^{\mathbf{H}} : \ x_i \in N\left(V_H\right)\}$$

$$A_{NB} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{matrix} N\left(V_h^+\right) \\ N\left(V_H\right) \end{matrix}$$

both diagonal blocks with only local information

$$A_{HB} = \begin{bmatrix} A_{11} & \tilde{A}_{12} \\ \tilde{A}_{12}^T & A_H \end{bmatrix} \begin{matrix} N\left(V_h^+\right) \\ N\left(V_H\right) \end{matrix}$$

first block with local information, second with global information.

Two-level multiplicative preconditioner

$$A = A^{(k+1)} = \begin{bmatrix} I & \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$B_M = B_M^{(k+1)} = \begin{bmatrix} I & \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ & I \end{bmatrix}$$

$g = B_M^{-1} r$ can be **implemented** as follows:

$$\begin{array}{lll} \textit{step I} & g_1 \leftarrow A_{11}^{-1} r_1 \\ \textit{step II} & g_2 \leftarrow \overline{A}_{22}^{-1}(r_2 - A_{21}g_1) \\ \textit{step III} & g_1 \leftarrow g_1 - A_{11}^{-1}A_{12}g_2 = A_{11}^{-1}(r_1 - A_{12}g_2) \end{array}$$

Computational cost:

2x solution of systems with A_{11} ; 1x solution of system with $\bar{A}_{22}$; multiplication by A_{21}, A_{12}; 2x vector subtration.

Two-level additive preconditioner

$$\overline{A} = J\,A\,J^T = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \rightarrow \overline{B}_A = \begin{bmatrix} \overline{A}_{11} & \\ & \overline{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & \\ & A_H \end{bmatrix}$$

$$B_A \sim A = J^{-1}\overline{A}\,J^{-T} \Rightarrow B_A = J^{-1}\,\overline{B}_A\,J^{-T}, \qquad \sigma(B_A^{-1}A) = \sigma(\overline{B}_A^{-1}\overline{A})$$

$g = B_A^{-1}\,r = J^{\,T}\left(\overline{B}_A\right)^{-1} J\,r$ can be **implemented** as follows:

$$\begin{array}{ll} step\,I & g_2 \leftarrow \overline{A}_{22}^{-1}(r_2 + I_{21}r_1) \\ step\,II & g_1 \leftarrow \overline{A}_{11}^{-1}r_1 + I_{21}^T\,g_2 \end{array}$$

Computational cost:

1x solution of systems with $\overline{A}_{11} = A_{11}$; 1x solution of system with $\overline{A}_{22}$;
2x vector addition, 1x multiplication with I_{21}, 1x multiplication with I_{21}^T.

Approximate two-level preconditioners

$\widetilde{B}_A$ and $\widetilde{B}_M$ arise when $\overline{A}_{kk} v_k = w_k$ are **solved approximately**

$$v_k = \text{appsolve}\left(\overline{A}_{kk},\, w_k\right) = \text{solve}\left(B_{kk},\, w_k\right)$$

by exploiting approximation $B_{kk} \sim \overline{A}_{kk}$ or inner iterations.

Choices for A_{11}:

- $B_{11} = \text{diag}(A_{11})$, $B_{11} = \text{ilu}(A_{11})$, $B_{11} = \text{milu}(A_{11})$,
- inner iterations controled by $\text{maxit} = \eta,\ \text{acc} = \varepsilon$ with simple preconditioner.

Choices for $\overline{A}_{22}$:

- $B_{22} = B_A^c$ or $B_{22} = B_M^c$ where $B_A^c,\ B_M^c$ are again hierarchical preconditioners on the coarser level,
- inner iterations with preconditioners $B_A^c,\ B_M^c$.

Multi-level preconditioners

$$\begin{array}{cccc} \mathcal{T}_1 \longrightarrow & \mathcal{T}_2 \longrightarrow & \ldots\ldots\ldots & \longrightarrow \mathcal{T}_m \\ A^{(1)} \longrightarrow & A^{(2)} \longrightarrow & \ldots\ldots\ldots & \longrightarrow A^{(m)} \\ & I_{12}^{(2)} \longrightarrow & \ldots\ldots\ldots & \longrightarrow I_{12}^{(m)} \end{array}$$

for $k = 2, \ldots, m$
  define $\mathcal{N}_k = \mathcal{N}_{k-1} \cup \mathcal{N}_k^+$
  define $B_A^{(k)}$ or $B_M^{(k)}$

for $k = 2, \ldots, m$

define **multilevel preconditioner** $\mathbf{B}^{(k)} = \mathbf{B}_A^{(k)}$ or $\mathbf{B}^{(k)} = \mathbf{B}_M^{(k)}$

if $k = 2$ then $\mathbf{B}^{(k)} = B^{(k)}$

if $k > 2$ then $\mathbf{B}^{(k)}$ is approximate two-level preconditioner $\widetilde{B}^{(k)}$ with

- different appsolve(A_{11})
- appsolve(A_{22}) given by ν_k iterations of inner Chebyschev or CG iterations with preconditioner $\mathbf{B}^{(k-1)}$.

Analysis of two level preconditioners

Let $\gamma < 1$ *be the smallest constant such that*:

$$\left\langle \overline{A}_{12}\bar{v}^{(2)},\, \bar{v}^{(1)} \right\rangle \;\le\; \gamma \sqrt{\left\langle \overline{A}_{11}\bar{v}^{(1)},\, \bar{v}^{(1)} \right\rangle} \sqrt{\left\langle \overline{A}_{22}\bar{v}^{(2)},\, \bar{v}^{(2)} \right\rangle}.$$

$$\begin{aligned}
&\left\langle \overline{B}_A\bar{v},\, \bar{v} \right\rangle = \left\langle A_{11}\bar{v}^{(1)},\, \bar{v}^{(1)} \right\rangle + \left\langle A_{22}\bar{v}^{(2)},\, \bar{v}^{(2)} \right\rangle \\
&\left\langle \overline{A}\bar{v},\, \bar{v} \right\rangle = \left\langle \overline{B}_A\bar{v},\, \bar{v} \right\rangle + 2\left\langle A_{12}\bar{v}^{(2)},\, \bar{v}^{(1)} \right\rangle \qquad \Rightarrow \quad \varkappa(\overline{B}_A^{-1}\overline{A}) = \varkappa(B_A^{-1}A) \le \frac{1+\gamma}{1-\gamma} \\
&(1-\gamma)\overline{B}_A \le \overline{A} \le (1+\gamma)\overline{B}_A
\end{aligned}$$

$$B_M^{-1}A = \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & S^{-1}\overline{A}_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$\sigma(B_M^{-1}A) \subset \sigma(I) \cup \sigma(\overline{A}_{22}^{-1}S) \;=\; \sigma(I) \cup \sigma(\overline{A}_{22}^{-1}\bar{S}) \subset \langle 1-\gamma^2, 1\rangle$$

$$\Rightarrow \quad \varkappa\left(B_M^{-1}A\right) \le 1/(1-\gamma^2) = \tfrac{1}{(1+\gamma)^2}\,\text{estimate}(\varkappa(B_A^{-1}A)).$$

Analysis of approximate two level preconditioners

Let A be SPD, γ is as above and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants:

$$\begin{array}{l} \alpha_1 A_{11} \leq B_{11} \leq \beta_1 A_{11} \\ \alpha_2 \overline{A}_{22} \leq B_{22} \leq \beta_2 \overline{A}_{22} \end{array} . \quad \text{Then} \quad \mathrm{cond}(\widetilde{B}_A^{-1} A) \leq \frac{4}{1-\gamma^2} \frac{\frac{1}{2}(\beta_1+\beta_2)}{2/\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)}.$$

Let A be SPD, γ is as above and α, β are positive constants:

$$\begin{array}{l} A_{11} \leq B_{11} \leq (1+\alpha) A_{11} \\ \overline{A}_{22} \leq B_{22} \leq (1+\beta) \overline{A}_{22} \end{array} . \quad \text{Then} \quad \mathrm{cond}(\widetilde{B}_M^{-1} A) \leq \frac{1}{1-\gamma^2}(1+\alpha+\beta).$$

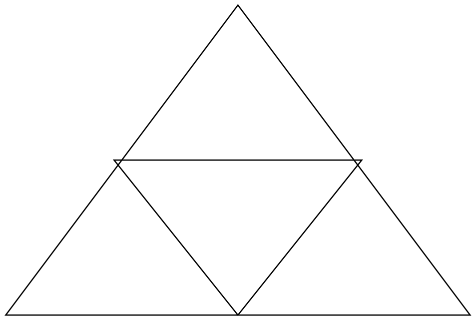
The above estimates can be found e.g. in Axelsson 2002. Another estimate showing desirable properties of B_{11} (small difference $B_{11}v - A_{11}v$ for smooth vectors v) can be found e.g. in Notay 1997.

Constant γ and strengthened CBS inequality

$\langle \overline{A}_{12}\bar{v}^{(2)}, \bar{v}^{(1)} \rangle = a(v_2, v_1)$, where $v_2 \in V_H$ and $v_1 \in V_h^+ = W_h$.

$V_h = V_H \oplus V_h^+, \ \gamma = \cos(V_H, V_h^+) = \sup \frac{a(u,v)}{\sqrt{a(u,u)}\sqrt{a(v,v)}} \ u \in V_H, \ v \in V_h^+$

Local investigation γ:



Let $E = \cup e_k$ be a macroelement,

$u \in V_H \Rightarrow u\,|_E \in P_1, \ i.e. \ u\,|_E \in V_H(E),$

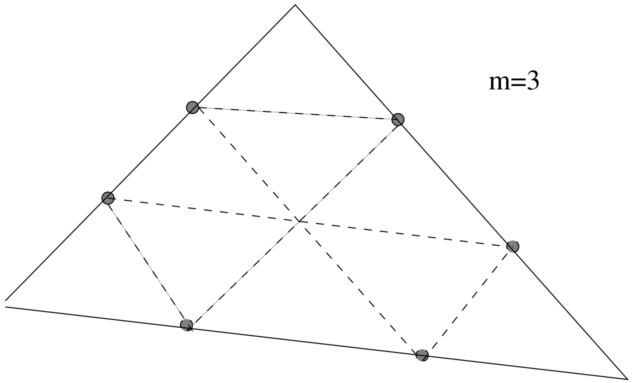
$v \in V_h^+ \Rightarrow v\,|_{e_k} \in P_1, \ v = 0 \ in \ vertices \ of \ E ... V_h^+(E)$

$$a(u,v) = \sum_E \int_E \langle D \,\mathrm{grad}\, u, \mathrm{grad}\, v \rangle \, dx = \sum_E a_E(u_E, v_E)$$

Theorem:

$$\gamma \le \max_E \gamma_E, \quad \gamma_E = \sup \frac{a_E(u,v)}{\sqrt{a_E(u,u)}\sqrt{a_E(v,v)}} : \quad \begin{array}{c} u \in V_H(E), \ u \in V_h^+(E) \\ u \ne const., \ v \ne 0 \end{array}$$

CBS in 2D



How γ depends on ?

- meshsize h
- refinement multiplicity m
- coefficient anisotropy
- element shape
- coefficient jumps

Investigation of γ

- *Maitre, Musy 1981:* $\gamma = \sqrt{3/4}$ 2D Laplacian, arbitrary element shape ($\gamma = \sqrt{3/8}$ for isosceles, $\gamma = \sqrt{1/2}$ for rectangular triangles)
- *Margenov 1994:* $\gamma = \sqrt{3/4}$ 2D elasticity E, ν, rectangular triangles
- *Achchab, Maitre 1996:* $\gamma = \sqrt{3/4}$ 2D isotrop. elasticity, general triangles

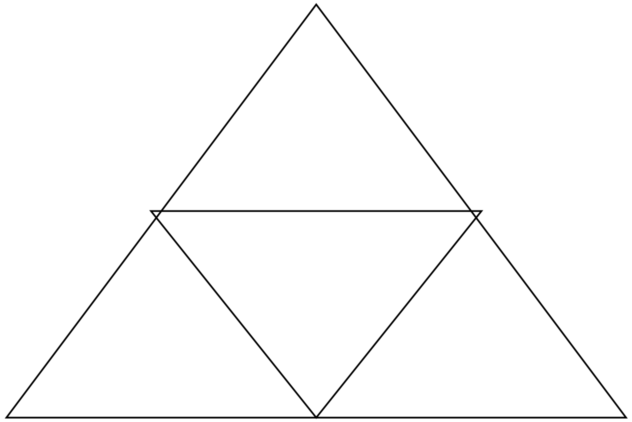
CBS in 2D - recent results

- *Axelsson 1999*: 2D, anisotropic Laplacian, reference triangle
- *Axelsson, Blaheta* (KUN Rep. 2001, Appl. Math. 2004):
  2D, reference element !, anisotropic Laplacian and general elasticity

$$\left.\begin{array}{l} - \ m \text{ fold refinement !} \\ - \ \text{anisotropic Laplacian} \\ - \ \text{general elasticity } c_{i,j,k,l} \text{ !} \\ - \ \text{arbitrary element shape} \end{array}\right\} \sqrt{\frac{m^2-1}{m^2}}$$

An idea of simple estimate of the CBS in 2D

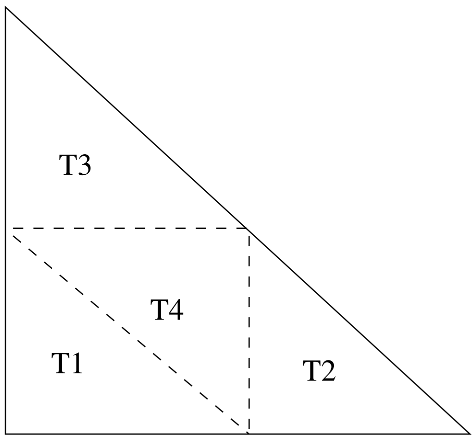
$$a_E(u,v) = \int_E \langle D \operatorname{grad} u, \operatorname{grad} v \rangle \, dx$$



$$\widetilde{a}_R(\widetilde{u},\widetilde{v}) = \int_R \left\langle \widetilde{D} \operatorname{grad} \widetilde{u}, \operatorname{grad} \widetilde{v} \right\rangle d\widetilde{x}$$

affine mapping to R (reference)

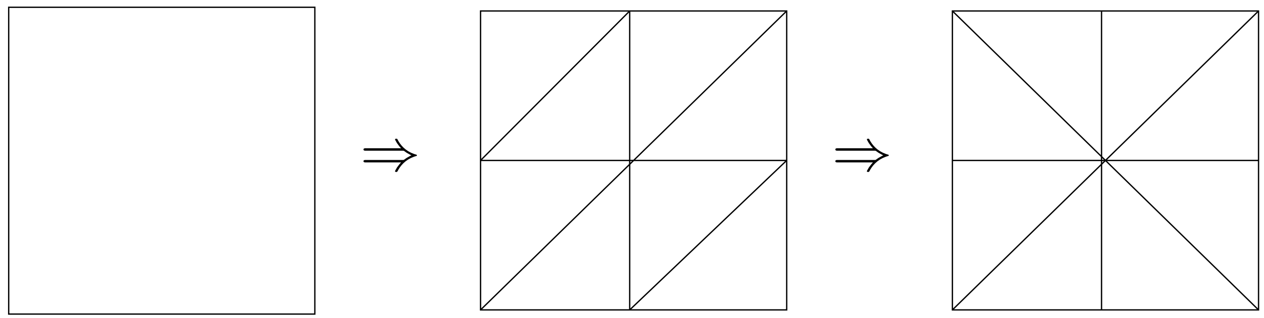
$F : E \to R,\ Fx = Gx + g$



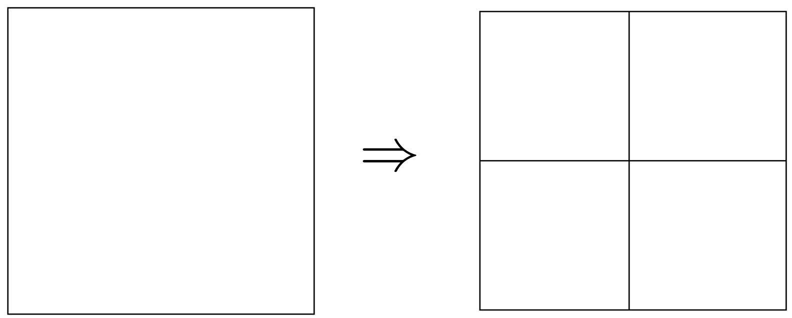
$\widetilde{D} = G^{-T} D G^{-1} |G|\,,\ \widetilde{a}_R(\widetilde{u},\,\widetilde{v}) = \sum \int_{T_k} \left\langle \widetilde{D}\delta,\, d^{(k)} \right\rangle d\widetilde{x}$. $\widetilde{D}\delta$ constant on E.

Relations for $d^{(k)} = \nabla \widetilde{v}$ like $d_1^{(1)} = -d_1^{(2)}$ and $d_1^{(3)} = d_1^{(4)}$, also $d^{(1)} + d^{(2)} + d^{(3)} = d^{(4)}$ then enable to proof the estimate

Other hierarchical decompositions

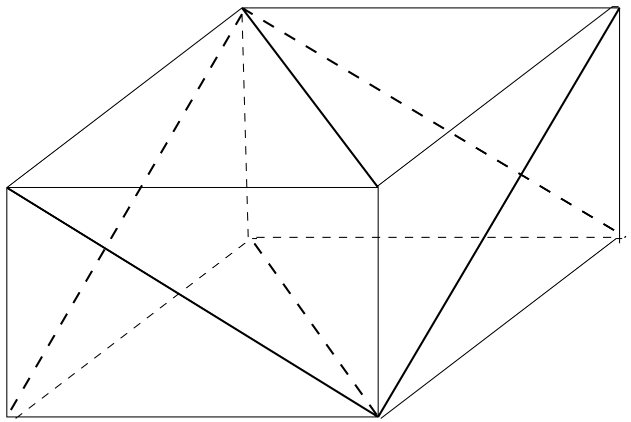


- linear-linear FE: for symmetric triangulation $\gamma \to 1$
- quadratic-linear FE:
  $S^{(2)} = \frac{4}{3} S^{(1)} - \frac{1}{3} A_H^{(1)}$ gives $\gamma_2^2 = \frac{4}{3} \gamma_1^2$. Thus $\gamma = \gamma_2 \to 1$.

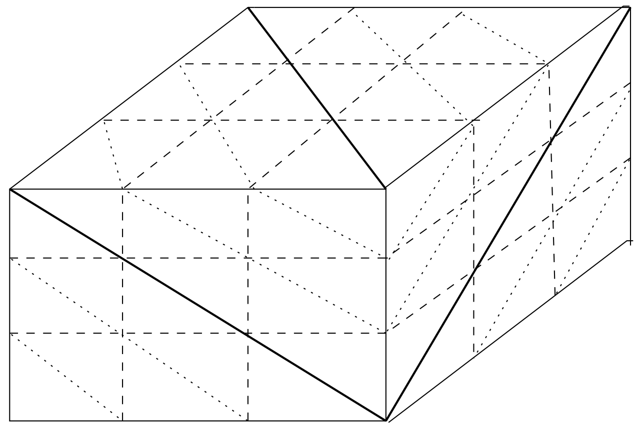


- bilinear-bilinear FE (Pultarová 2004) $\gamma^2 \leq \frac{3}{4}$ for orthotropic Laplacian. Moreover, $\gamma^2 = \frac{3}{8}$ in the isotropic case.

Hierarchical decomposition in 3D



Kuhn's division of cube into 6 tetrahedra



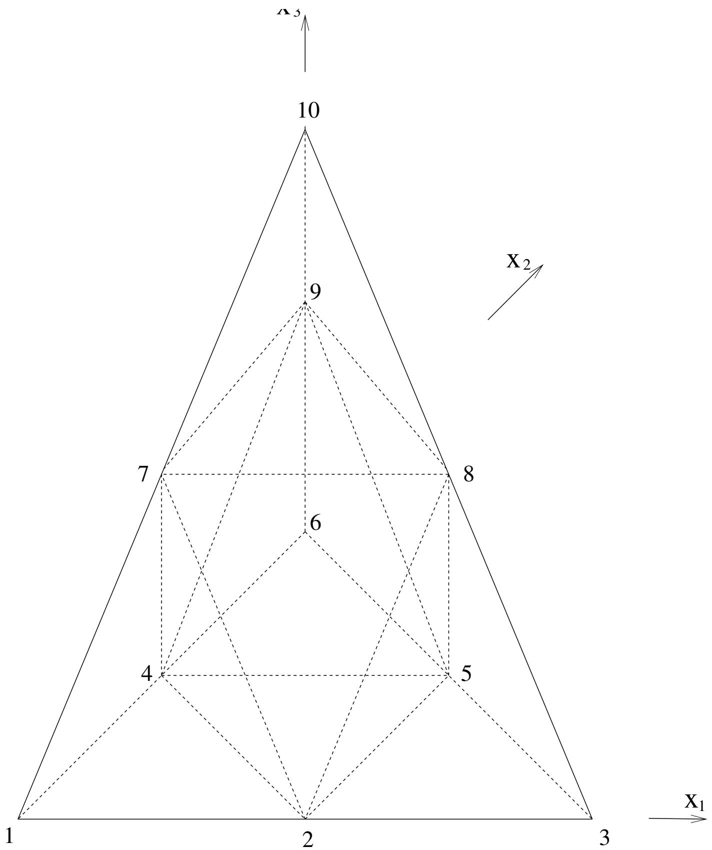
$m = 3 \longrightarrow m^3$ cubes

each cube $\longrightarrow$ 6 tetrahedra

$\Longrightarrow$ **decomposition of a big tetrahedron into m^2 smaler tetrahedra**

CBS in 3D

- *Junk, Maitre 1999:* 3D isotropic elasticity
- *Achchab, Axelsson, et al. 2000:* $\gamma = \sqrt{9/10}$ for 3D, isotrop. elasticity



R. Blaheta (NLAA 2003)

- m fold refinement, $m \leq 5$ (25 T)
- anisotropic Laplacian
- general elasticity !
- arbitrary element shape

$$\gamma \leq \sqrt{1 - \frac{2}{m^4 + m^2}}$$

Diagonal approximation to A_{11} block

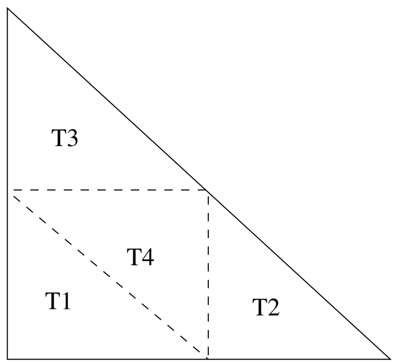
Consider orthotropic form $a\,(u,v) = \int_\Omega \sum k_i \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx$.

$A_{11} \sim \langle A_{11}x, x\rangle = a\left(\sum x_i\phi_i^k, \sum x_i\phi_i^k\right) \quad \sum \ldots$ over $\phi_i^k \in V_h^+$

$D_{11} = \mathrm{diag}\, A_{11},\ \langle D_{11}x, x\rangle = \sum x_i^2 a\left(\phi_i^k, \phi_i^k\right) \quad \sum \ldots$ over $\phi_i^k \in V_h^+$

and rectangular triangular elements. **Local investigation:**

$$V_{1E} = \sum \mathrm{span}\,\{\phi_i\}$$



$$a_E\,(v,v) = k_1\left[2v_3^2 + 2\,(v_1 - v_2)^2\right]\frac{\triangle}{h^2} + k_2\left[2v_2^2 + 2\,(v_1 - v_3)^2\right]\frac{\triangle}{h^2}$$

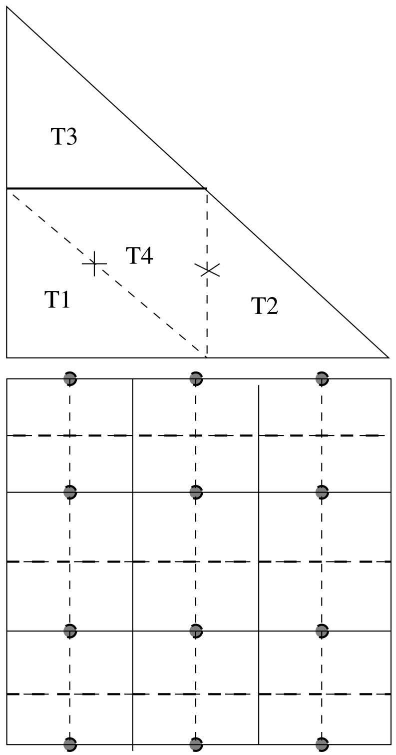
$$d_E(v,v) = \sum v_i^2 a_E\left(\phi_i^k, \phi_i^k\right) = 2\,(k_1 + k_2)\left(v_1^2 + v_2^2 + v_3^2\right)\frac{\triangle}{h^2}$$

The case of anisotropy

$$a_E(v,v) \geq \min(k_1,k_2)\left(v_2^2+v_3^2+v_2^2+(v_1-v_2)^2+v_3^2+(v_1-v_3)^2\right)\frac{\triangle}{h^2}$$

$$\geq \min(k_1,k_2)\left(v_1^2+v_2^2+v_3^2\right)\frac{\triangle}{h^2} \geq \frac{\min(k_1,k_2)}{2(k_1+k_2)}d_E(v,v)$$

Remedy: *if* $k_1 > k_2$ *then* $V_{1E} = \text{span}\{\phi_1,\phi_2\} + \text{span}\{\phi_3\}$



$$d(v,v) = k_1\left[2(v_1-v_2)^2+2v_3^2\right]\frac{\triangle}{h^2}$$

$$+k_2\left[2v_2^2+2v_1^2+2v_3^2\right]\frac{\triangle}{h^2} \quad \text{isotropy}$$

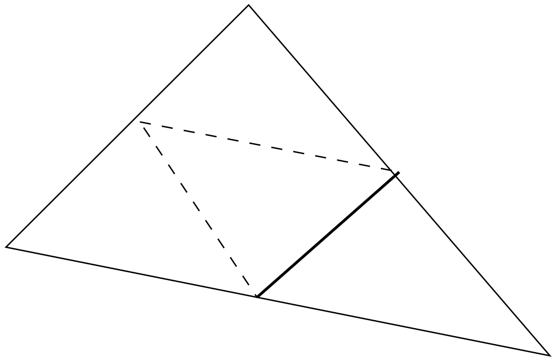
$$a_E(v,v) \geq \frac{1}{2}k_1\left[2v_3^2+2(v_1-v_2)^2\right]\frac{\triangle}{h^2}$$

$$+\frac{1}{2}k_2\left[2v_2^2+2v_3^2+2(v_1-v_3)^2\right]\frac{\triangle}{h^2}$$

$$\geq \frac{1}{8}d(v,v) \quad \text{(robust estimate)}$$

Robust block tridiagonal preconditioner

Axelsson, Padiy, 1999 macroelement construction:



$$A_{11,E} = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \quad B_{11,E} = \begin{bmatrix} \bullet & \circ & \circ \\ & \bullet & \circ \\ \text{sym} & & \bullet \end{bmatrix}$$

only one of $\circ$ can be nonzero !

$$\text{cond}\left(B_{11,E}^{-1} A_{11,E}\right) \le \frac{1+\sqrt{7/15}}{1-\sqrt{7/15}} \doteq 5.31$$

Construction of preconditioner:

- analyzing of strong couplings
- reordering unknowns

} block tridiagonal preconditioner

3D - block tridiagonal preconditioner works for rectangular meshes and orthotropy $k_1 \gg k_2 \sim k_3$ not for $k_1 \sim k_2 \gg k_3$.

Optimal multi-level preconditioners

- Solve A_{11}:
  - $A_{11} \sim \text{diag}(A_{11})$ for isotropic Laplacian
  - $A_{11} \sim \text{block3diag}(A_{11})$ for anisotropic Laplacian in 2D
- Solve $\overline{A}_{22}$:
  - few Chebyshev or GCG iterations with the same ML preconditioner

If everything is well balanced, we can get optimal method !

Axelsson 1981, Axelsson, Gustafsson 1983, Axelsson, Vassilevski 1989, 1990, 1994, ...

Further development - robustness: in 2D we can get optimal method, which is robust w.r.t. anisotropy

Axelsson¸Padiy 1999¸Axelsson 2002

General multi-level solvers

Solve the problem: Au=b with **CG or GPCG[m] method**

Multilevel preconditioner

- Solve A_{11}:
  - $A_{11} \sim$ ilu(A_{11})
  - $A_{11} \sim$ milu(A_{11})
  - inner CG with preconditioner diag(A_{11}), ilu(A_{11}), milu(A_{11}) etc
- Solve $\overline{A}_{22}$:
  - inner CG or GPCG iterations with the same ML preconditioner

Notes: MILU or MIC(0) is a good A_{11} solver¸ for corresponding theory see Notay 1998. The use of inner CG for both $A_{11}$and $\overline{A}_{22}$ requires stabilization by GPCG in the outer iterations.

Implementation issues

Preparatory phase: $\mathcal{T}_1 \longrightarrow \mathcal{T}_2 \longrightarrow \ldots\ldots\ldots \longrightarrow \mathcal{T}_m$

refine triangulation $A^{(1)} \longrightarrow A^{(2)} \longrightarrow \ldots\ldots\ldots \longrightarrow A^{(m)}$

& assembly matrices. $I_{12}^{(2)} \longrightarrow \ldots\ldots\ldots \longrightarrow I_{12}^{(m)}$

$\|$ If the initial triangulation is **distributed** among p **processors** $\Rightarrow$ parallel work on **refinement & assembling** $\Rightarrow$ matrices arise in a distributed form.

Solution phase: for level $k = 2, \ldots, m$

★ implement CG iteration with $\|$ matrix - vector multiplication

★ implement ML preconditioner with main operations

- different appsolve(A_{11}) some of them are highly parallelizable, e. g. diagonal, inner iterations,
- appsolve(A_{22}) by inner iterations is again parallelizable, if k=2 directsolve(A_{22}) is less parallelizable.

More discussion e. g. Neytcheva 1995, Douglas, Haase, Langer: Tutorial 2003

Numerical experiments 1 (B_A)

Model problem: $\Omega = (0,1) \times (0,1)$, $f = f(x, y) = 1 + 2x + 3y$

$$\begin{aligned} -\mathrm{div}(\mathrm{grad} u) &= f \quad in\ \Omega, \\ u &= \quad on\ \Gamma_D = \partial\Omega \end{aligned}$$

Test 1: **Additive** preconditioner, $\varepsilon = 10^{-6}$. **Left:** level, ref.,#dof, inner acc., maxit, A=addit.

1	2	4				25	31	25	25
2	2	9	6	2	A	57	57	55	53
3	2	25	6	2	A	104	109	108	107
4	2	81	6	2	A				
5	2	289	6	2	A				
6		1089	6	2	A				

Above: numbers of iterations

A11 - exact, diag, cholinc, mic(0)

A22(row) - exact, inner CG, V cycle

Numerical experiments 2 ($\mathbf{B}_M$)

Model problem: $\Omega = (0,1) \times (0,1)$, $f = f(x,\, y) = 1 + 2x + 3y$

$$\begin{aligned} -\mathrm{div}(\mathrm{grad}u) &= f \quad in\ \Omega, \\ u &= \qquad on\ \Gamma_D = \partial\Omega \end{aligned}$$

Test 2: **Multiplicative** preconditioner, $\varepsilon = 10^{-6}$. **Left:** level, ref.,#dof, inner acc., maxit, M=multiplicative.

1	2	4			
2	2	9	6	2	M
3	2	25	6	2	M
4	2	81	6	2	M
5	2	289	6	2	M
6		1089	6	2	M

7	74	22	12
8	439	38	13
12	57	21	20

Above: numbers of iterations

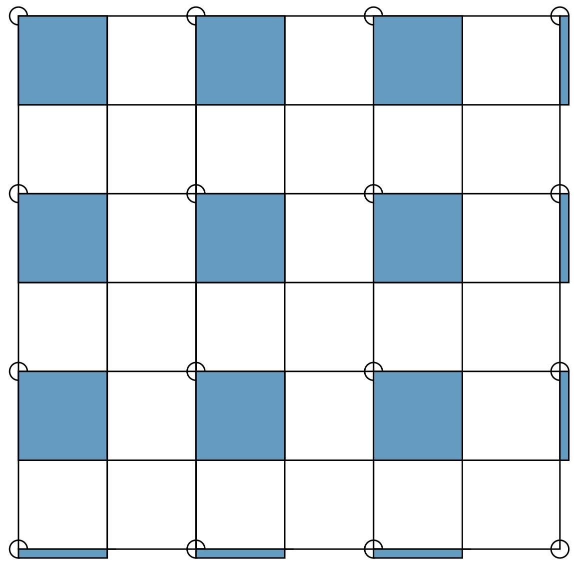
A11 - exact, diag, cholinc, mic(0)

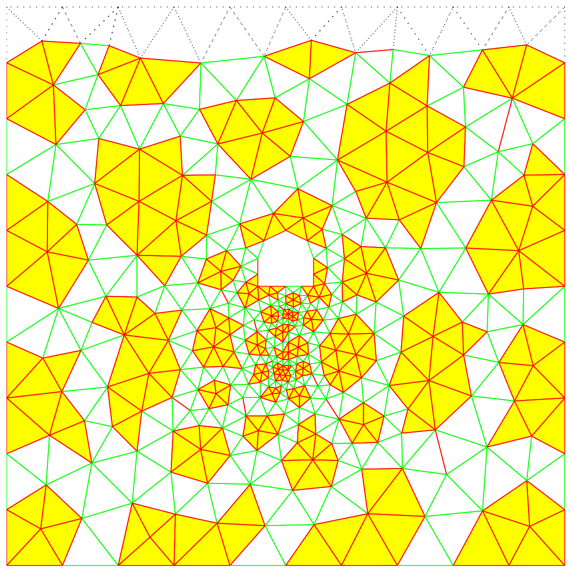
A22(row) - exact, inner CG, V cycle

Algebraic hierarchy: aggregation/agglomeration

- **standard AMLI** uses **geometric hierarchy** of nested FE grids, which has some disadvantages
- using hierarchy of problems created **algebraically** (as in AMG) leads to another AMLI variants
- AMLI with hierarchy of problems created by **aggregation** was introduced in Notay NLAA 2002, 2005 with adaptive aggregation procedure, rescaling aggregated matrices and MIC(0) solver for the pivot block
- AMLI with hierarchy of problems created by **agglomeration** of finite elements and using local Schur complements was introduced in Kraus IMET 2004, NLAA 2005
- next we show AMLI - aggregation method with regular aggregation on regular grids

AMLI with aggregation (algebraic hierarchy)





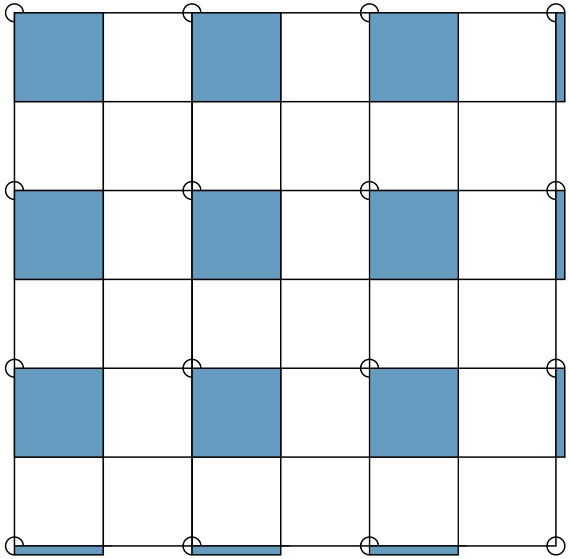
AMLI uses a space decomposition and HB. This can be done also via aggregation of nodes/DOF's. Let us select one node (DOF block) in each aggregation as a C-node (o) and let the remaining nodes be F-nodes.

$$\phi_i^{HB} = \sum_k J_{ik}\,\phi_i^h, \qquad J = (J_{ij}) = \begin{bmatrix} I_1 & 0 \\ I_{21} & I_2 \end{bmatrix} \begin{matrix} F \\ C \end{matrix},$$

I_{21} Boolean, one unity per column

If $i \in G_j$ is a C-node, then $\phi_i^{HB} = \sum_{k \in G_j} \phi_i^h$..

AMLI-aggregation on regular grids



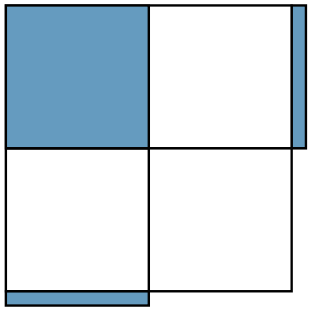
- A11 the same as for geometric hierarchy
- local CBS analysis

$$a(u,v) = \int_\Omega \langle D\mathrm{grad}(u), \mathrm{grad}(v)\rangle \, dx = \sum_E \int_E \langle D\mathrm{grad}(u), \mathrm{grad}(v)\rangle \, dx$$

$$a(u,\, v) = \sum_E a_E(u,\, v),\ \ a(u,\, u) = \sum_E a_E(u,\, u),\ \ a(v,\, v) = \sum_E a_E(v,\, v)$$

CBS constant for decomposition by aggregation

$V_h = V_a \oplus V_a^+$, $\; a\,(u,v) = \sum_E \; a_E\,(u_E,\, v_E)$, E macroelements:



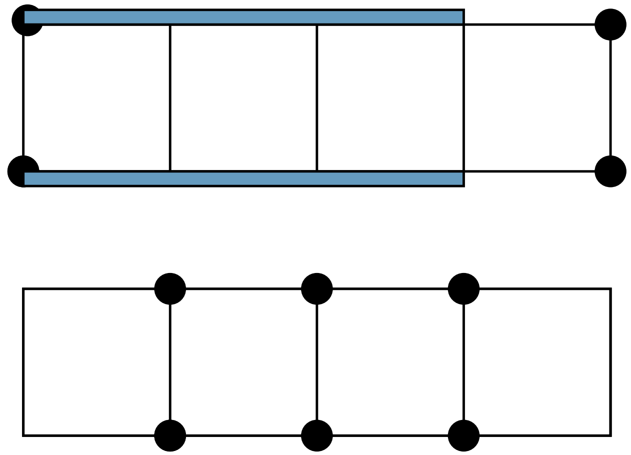
$$D = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 & \\ & 10^k \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

k=	0	1	2	3	4	5	6
0°	.8966	.9080	.9123	.9128	.9129	.9129	.9129
15°	.8966	.9210	.9710	.9961	.9996	1.000	1.000
30°	.8966	.9391	.9879	.9997	.9999	1.000	1.000
45°	.8966	.9453	.9907	.9990	.9999	1.000	1.000

.

CBS constant for aggregation - anisotropy

$V_h = V_a \oplus V_a^+,$
$a(u,v) = \sum_E \; a_E(u_E, v_E),$
E macroelements:



k=	0	1	2	3	4	5	6
0°	.8966	.9080	.9123	.9128	.9129	.9129	.9129
aniso	.9013		.8660		.8660		.8660
90°	.8966	.9080	.9123	.9128	.9129	.9129	.9129
aniso	.9013		.9255		.9258		.9258

Numerical experiments 3 (B_A, B_M)

$\Omega = (0,1) \times (0,1)$, $f = f(x, y) = 1 + 2x + 3y$

$$\begin{aligned} -\mathrm{div}(\mathrm{grad} u) &= f \quad \mathit{in}\ \Omega, \\ u &= \quad \mathit{on}\ \Gamma_D = \partial\Omega \end{aligned}$$

1	2	2500	6	2	A
2	2	676	6	2	A
3	2	196	6	2	A
4	2	64	6	2	A
5		25			

Right: level, coarsening, #dof, inner acc., max. number of inner it., A/M

Test 3: Left: **Additive** preconditioner, $\varepsilon = 10^{-6}$.
Right: **Multiplicative** preconditioner, $\varepsilon = 10^{-6}$.

21	30	23	23	11	126	16	13
63	61	68	66	15	472	25	18
146	173	154	155	38	112	52	51

Above: numbers of iterations. A11(col) - exact, diag, cholinc, mic(0). A22(row) - exact, inner CG, V cycle

Numerical experiments 4

Outer iterations by CG by default. Otherwise GPCG[m].
$\overline{A}_{22}$ system solved by 2 CG iterations with multiplicative hierarchical preconditioner, A_{11} system is solved by different methods.
Overal numbers of outer iterations:

15 for A_{11} system solved exactly,

472 for $A_{11} \sim \mathrm{diag}(A_{11})$,

25 for $A_{11} \sim \mathrm{cholinc}(A_{11})$. Reduces to 23 iterations for GPCG[1].

18 for $A_{11} \sim \mathrm{mic0}(A_{11})$. Reduces to 17 iterations for GPCG[1].

22 for A_{11} system solved by inner CG with diagonal preconditioner and GPCG[1]. For outer CG does not converges!

15 for A_{11} system solved by inner CG with cholinc and GPCG[1],

15 for A_{11} system solved by inner CG with mic(0) and GPCG[1].

HD with aggregation

:-) fully algebraical approach,

:-) regular aggregation on regular grid gives the same A_{11} as HB approach, $\bar{A}_{22}$ with the same stencil, on uniform grid differing from A_H by a scaling factor

:-(CBS constant is bigger and depending on anisotropy

Possible improvements

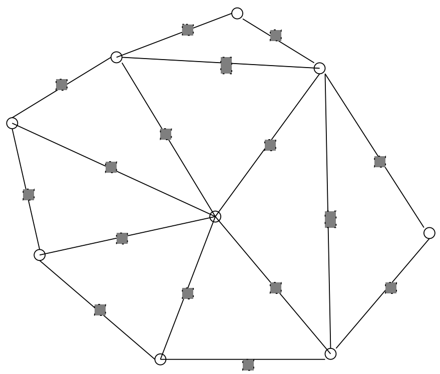
1. problem dependent aggregation,
2. algebraical improvement

Nonconforming FE

BVP: find $u \in V(\Omega): \quad a(u,v) = b(v) \quad \forall v \in V(\Omega)$

$$a(u,v) = \int_\Omega \langle D\nabla u, \nabla v\rangle \, dx, \quad D \ldots 2 \times 2 \quad \text{SPD}$$

$$V = \left\{ v \in H^1(\Omega), v(x) = 0 \text{ on } \Gamma_0 \right\}$$



Crouzeix - Raviart FE:

$\Omega \to$ triangulation $\mathcal{T}_h$

$\mathcal{T}_h \to M_h$ - set of mid points

$[v]$ - jump of v on edges of T

$$\widetilde{U}_h = \{ v \in L_2(\Omega) : v\,|_T \in P_1 \ \forall T \in T_h, \ [v](x) = 0 \ \forall x \in M_h \}$$

$$\widetilde{V}_h = \left\{ v \in \widetilde{U}_h : \ v(x) = 0 \ \forall x \in M_h \cap \Gamma_0 \right\},$$

$$a_h(u,v) = \sum_{T \in T_h} \int_T \langle D\nabla u, \nabla v\rangle \, dx$$

Nonconforming FE

Nonconforming FEM:

find $u_h \in \widetilde{V}_h(\Omega): \quad a_h(u_h, v_h) = b(v_h) \quad \forall v_h \in \widetilde{V}_h(\Omega)$

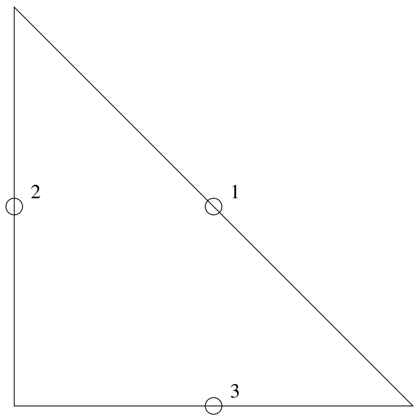
$$u \in H^2(\Omega) \Rightarrow \|u - u_h\| \leq ch\,|u|_2$$

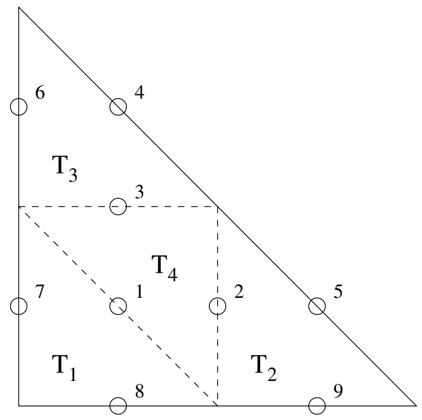
Advantages: locking free elements, sparsity and regularity of the pattern of FE matrices

Hierarchic construction:

$\widetilde{V}_H \not\subseteq \widetilde{V}_h!$

$V_H \subset \widetilde{V}_h$





basis functions $\phi_1 \ldots \phi_9$

Hierarchic preconditioners for CR matrices

FR construction: let denote I - inner nodes $\{1,2,3\}$, S - side nodes

NB $\phi_1 \dots \phi_9 \quad \xrightarrow{J} \quad$ HB $\widetilde{\phi}_1 \dots \widetilde{\phi}_9 \qquad \widetilde{A}_h = J A_h J^T$

$$\widetilde{A}_h = \left|\begin{array}{c|cc|} \widetilde{A}_{II} & \widetilde{A}^1_{IS} & \widetilde{A}^2_{IS} \\ \hline \widetilde{A}^1_{SI} & \widetilde{A}^{11}_{SS} & \widetilde{A}^{12}_{SS} \\ \widetilde{A}^2_{SI} & \widetilde{A}^{21}_{SS} & \widetilde{A}^{22}_{SS} \end{array}\right. \begin{array}{c} \phi_1,.. \\ \phi_4 - \phi_5,.. \\ \phi_4 + \phi_5,.. \end{array}$$

$$B = A_{SS} - A_{SI} A_{II}^{-1} A_{IS}$$

$$B = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}$$

DA construction

NB $\phi_1 \dots \phi_9 \quad \xrightarrow{J} \quad$ HB $\widetilde{\phi}_1 \dots \widetilde{\phi}_9 \qquad \widetilde{A}_h = J A J^T$

$$\widetilde{A}_h = \left|\begin{array}{cc|} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{array}\right. \begin{array}{l} \phi_1,.. \\ \phi_4 - \phi_5,.. \\ \phi_4 + \phi_4 + \phi_5,.. \end{array} \qquad \to \widetilde{A}_{22} = 4A_H$$

AMLI for nonconforming CR FEM

$$\gamma_{FR}: \quad \langle B_{12}x_2, x_1\rangle \leq \gamma_{FR}\sqrt{\langle B_{11}x_1, x_1\rangle}\sqrt{\langle B_{22}x_2, x_2\rangle}$$

$$\gamma_{DA}: \quad \left\langle \widetilde{A}_{12}y_2, y_1\right\rangle \leq \gamma_{DA}\sqrt{\left\langle \widetilde{A}_{11}y_1, y_1\right\rangle}\sqrt{\left\langle \widetilde{A}_{22}y_2, y_2\right\rangle}$$

Theorem (Margenov, Neytcheva, RB, NLAA 2004): *Let us consider nested triangular grids, with coarse triangles consisting from 4 smaller ones. Assume that the coefficient matrix D has no jumps within the coarse triangles. Then*

$$\gamma_{FR} < \sqrt{\frac{3}{4}}, \qquad \gamma_{DA} \leq \sqrt{\frac{3}{4}}$$

for any element size and shape and general anisotropic Laplacian.

Theorem (Margenov, Neytcheva, RB, NLAA 2005): There is block tridiagonal optimal robust solver for the pivot block and it is possible to construct robust optimal AMLI preconditioners for CR FEM systems.

Conclusions

We talk about AMLI type hierarchical preconditioners

- main ingredients - decomposition, local block solver, global block solver - polynomial stabilization
- construction - multiplicative/addititive, geometric/algebraic hierarchy, global/local refinement
- various local block solvers
- implementation including parallelization
- robustness

References

[1] R. Blaheta, O. Axelsson, Two simple derivations of universal bounds for the C.B.S. inequality constant, KUN Report 2001, Appl. Math. 2004.

[2] R. Blaheta, Nested tetrahedral grids and strengthened CBS inequality, NLAA 2003

[3] R. Blaheta, M. Neytcheva, S. Margenov: Uniform estimate of the constant in the strengthened CBS inequality for anisotropic non-conforming FEM systems , Numerical Linear Algebra with Applications 11(2004), issue 4, 309-326

[4] R. Blaheta, M. Neytcheva, S. Margenov, Robust optimal multilevel preconditioners for nonconforming FE systems. Numerical Linear Algebra with Applications, 12(2005), pp. 495-514

[5] R. Blaheta, S. Margenov, M. Neytcheva, Aggregation-based multilevel preconditioning of non-conforming FEM elasticity problems, PARA 2004, LNCS 3732, Springer 2005, pp. 847-856