



# Conferences 2007

---

## **A SMOOTH VARIANT OF THE FICTITIOUS DOMAIN APPROACH**

Jaroslav Haslinger

Tomáš Kozubek

Radek Kučera

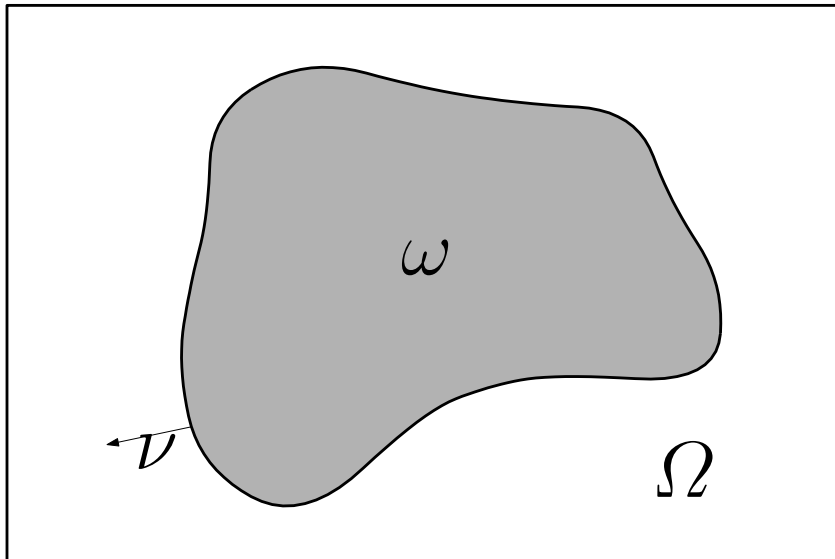


# Outline

---

- Classical fictitious domain approach
- A new smooth fictitious domain approach
- Numerical examples

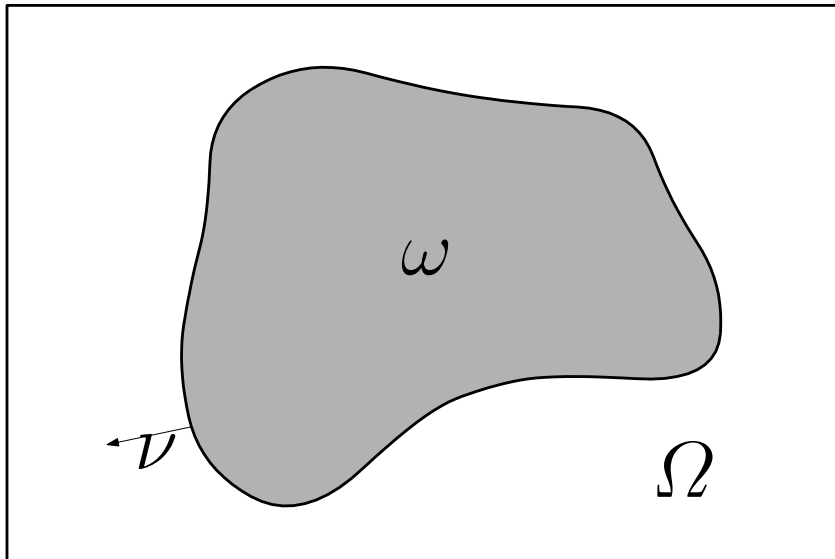
# Fictitious domain method - idea



**Problem  $(\mathcal{P}(\omega))$**

$$\begin{aligned} Au &= f & \text{in } \omega, \\ &+ b.c. & \text{on } \gamma := \partial\omega. \end{aligned}$$

# Fictitious domain method - idea



**Problem  $(\mathcal{P}(\omega))$**

$$\begin{aligned} \mathcal{A}u &= f & \text{in } \omega, \\ +b.c. & & \text{on } \gamma := \partial\omega. \end{aligned}$$

**Problem  $(\hat{\mathcal{P}}(\Omega))$**

$$\begin{aligned} \hat{\mathcal{A}}\hat{u} &= \hat{f} & \text{in } \Omega, \\ +b.c. & & \text{on } \partial\Omega. \end{aligned}$$



# Fictitious domain method

---

**Requirement:**  $\hat{u}|_{\omega}$  solves  $(\mathcal{P}(\omega))$ .



# Fictitious domain method

---

**Requirement:**  $\hat{u}|_{\omega}$  solves  $(\mathcal{P}(\omega))$ .

**Advantages:** use of structured meshes in  $\Omega \implies$   
efficient numerical realization of  $(\hat{\mathcal{P}}(\Omega))$ .



# Fictitious domain method

**Requirement:**  $\hat{u}|_{\omega}$  solves  $(\mathcal{P}(\omega))$ .

**Advantages:** use of structured meshes in  $\Omega \implies$   
efficient numerical realization of  $(\hat{\mathcal{P}}(\Omega))$ .

## Techniques of FDM

- algebraic approach (Marchuk, Kuznetsov, . . .);
- optimal control approach (Glowinski, Periaux, . . .);
- Lagrange multiplier methods (Glowinski, Girault, . . .).



# Fictitious domain method

$$\begin{aligned} -\Delta u &= f & \text{in } \omega \subseteq \mathbb{R}^2, \\ u &= g & \text{on } \gamma. \end{aligned}$$

## Interpretation

membrane prolongation,  
stationary heat equation,  
diffusion



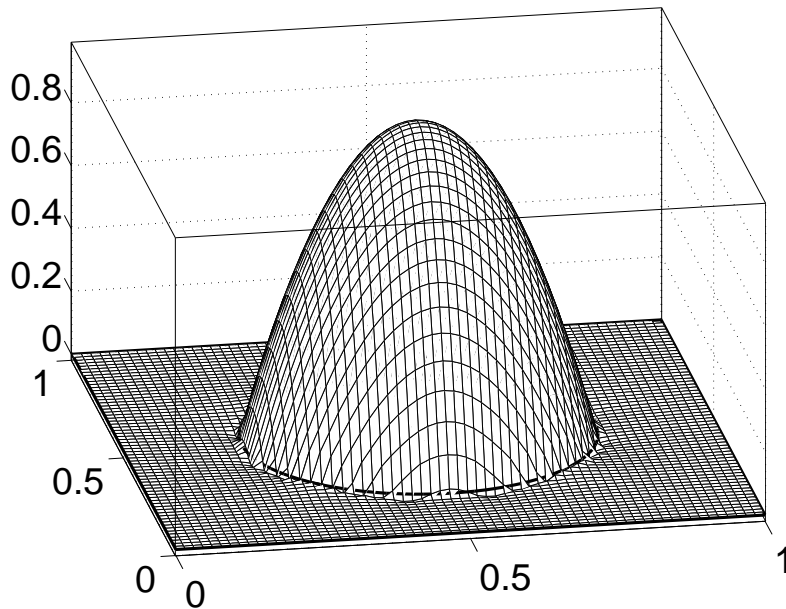


# Fictitious domain method

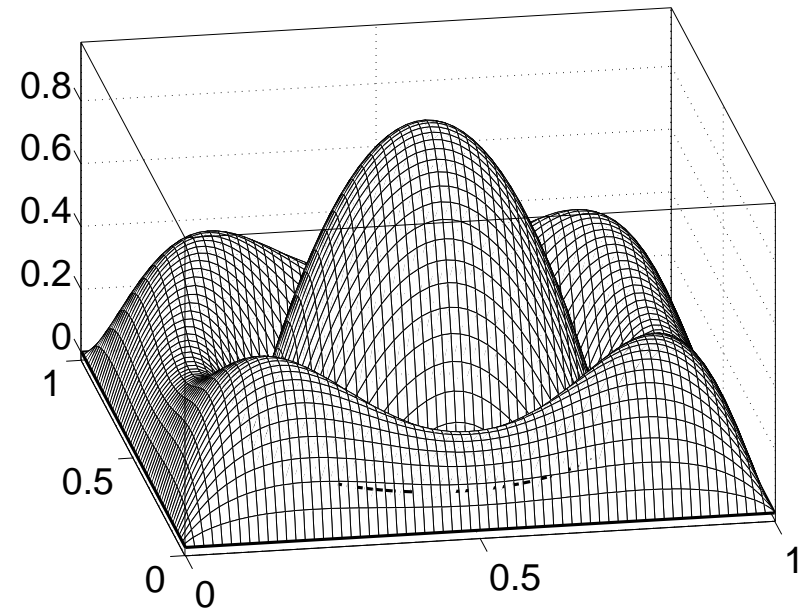
$$\begin{aligned} -\Delta u &= f & \text{in } \omega \subseteq \mathbb{R}^2, \\ u &= g & \text{on } \gamma. \end{aligned}$$

## Interpretation

membrane prolongation,  
stationary heat equation,  
diffusion



DLM method



BLM method



# Model example

**Problem  $(\mathcal{P}(\omega))'$**

$$\begin{aligned} -\Delta u &= f & \text{in } \omega \subseteq \mathbb{R}^2, \\ u &= g & \text{on } \gamma. \end{aligned}$$



# Model example

## Problem $(\mathcal{P}(\omega))'$

$$\begin{aligned} -\Delta u &= f & \text{in } \omega \subseteq \mathbb{R}^2, \\ u &= g & \text{on } \gamma. \end{aligned}$$

## Problem $(\mathcal{P}(\omega))$

$$\begin{aligned} &\text{Find } u \in H^1(\omega), \text{ } u = g \text{ on } \gamma \text{ such that} \\ &\int_{\omega} \nabla u \cdot \nabla v \, dx = \int_{\omega} f v \, dx \quad \forall v \in H_0^1(\omega). \end{aligned}$$



# Boundary Lagrange multipliers method

## Problem $(\hat{\mathcal{P}}(\Omega))$

*Find  $(\hat{u}, \lambda) \in H_0^1(\Omega) \times H^{-1/2}(\gamma)$  such that*

$$\int_{\Omega} \nabla \hat{u} \cdot \nabla \hat{v} \, dx = \int_{\Omega} \hat{f} \hat{v} \, dx + \langle \lambda, \hat{v} \rangle_{\gamma} \quad \forall \hat{v} \in H_0^1(\Omega),$$

$$\langle \mu, \hat{u} \rangle_{\gamma} = \langle \mu, g \rangle_{\gamma} \quad \forall \mu \in H^{-1/2}(\gamma).$$



# Boundary Lagrange multipliers method

## Problem $(\hat{\mathcal{P}}(\Omega))$

*Find  $(\hat{u}, \lambda) \in H_0^1(\Omega) \times H^{-1/2}(\gamma)$  such that*

$$\int_{\Omega} \nabla \hat{u} \cdot \nabla \hat{v} \, dx = \int_{\Omega} \hat{f} \hat{v} \, dx + \langle \lambda, \hat{v} \rangle_{\gamma} \quad \forall \hat{v} \in H_0^1(\Omega),$$

$$\langle \mu, \hat{u} \rangle_{\gamma} = \langle \mu, g \rangle_{\gamma} \quad \forall \mu \in H^{-1/2}(\gamma).$$

$\hat{f} \dots$  an extension of  $f$  from  $\omega$  to  $\Omega$ ,

$\langle \cdot, \cdot \rangle_{\gamma} \dots$  duality pairing between  $H^{-1/2}(\gamma)$  and  $H^{1/2}(\gamma)$ .



# Boundary Lagrange multipliers method

**It holds:**

$(\hat{\mathcal{P}}(\Omega))$  has a unique solution  $(\hat{u}, \lambda)$ ;

$$\hat{u}|_{\omega} \text{ solves } (\mathcal{P}(\omega)) \text{ and } \lambda = \left[ \frac{\partial \hat{u}}{\partial n} \right]_{\gamma}.$$



# Discretization of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

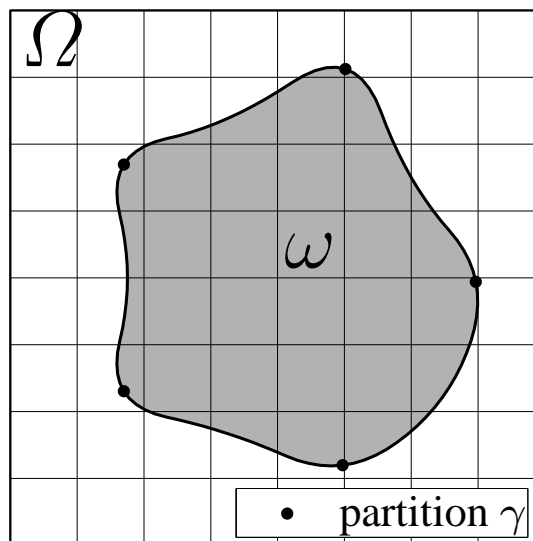
$$H_0^1(\Omega) \sim V_h, \quad H^{-1/2}(\gamma) \sim L_H^\gamma.$$



# Discretization of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

$$H_0^1(\Omega) \sim V_h, \quad H^{-1/2}(\gamma) \sim L_H^\gamma.$$

$$V_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_R \in Q_1(R) \quad \forall R \in \mathcal{R}_h, \\ v_h = 0 \text{ on } \partial\Omega\},$$







# Discretization of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

$$H_0^1(\Omega) \sim V_h, \quad H^{-1/2}(\gamma) \sim L_H^\gamma.$$

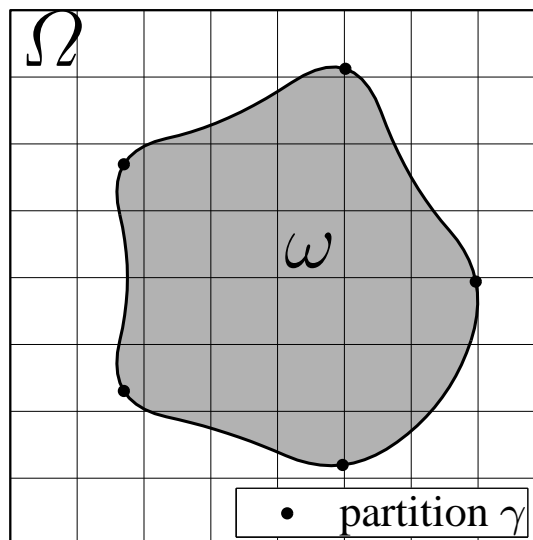


# Discretization of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

$$H_0^1(\Omega) \sim V_h, \quad H^{-1/2}(\gamma) \sim L_H^\gamma.$$

$$L_H^\gamma = \{\mu_H \in L^2(\gamma) \mid \mu_H|_{S_k} \in P_0(S_k) \quad \forall S_k \in \mathcal{R}_H\},$$

$$\mathcal{R}_H := \bigcup_{k=1}^m S_k.$$





# Discretization of $(\hat{\mathcal{P}}(\Omega))$

$$H_0^1(\Omega) \sim V_h, \quad H^{-1/2}(\gamma) \sim L_H^\gamma.$$



# Discretization of $(\hat{\mathcal{P}}(\Omega))$

$$H_0^1(\Omega) \sim V_h, \quad H^{-1/2}(\gamma) \sim L_H^\gamma.$$

**Problem  $(\hat{\mathcal{P}}(\Omega))_h^H$**

*Find  $(\hat{u}_h, \lambda_H) \in V_h \times L_H^\gamma$  such that*

$$\int_{\Omega} \nabla \hat{u}_h \cdot \nabla \hat{v}_h \, dx = \int_{\Omega} \hat{f} \hat{v}_h \, dx + \int_{\gamma} \lambda_H \hat{v}_h \, ds \quad \forall \hat{v}_h \in V_h,$$

$$\int_{\gamma} \mu_H \hat{u}_h \, ds = \int_{\gamma} \mu_H g \, ds \quad \forall \mu_H \in L_H^\gamma.$$



# Matrix form of $(\hat{\mathcal{P}}(\Omega))_h^H$

## Problem $(\vec{\mathcal{P}}(\Omega))$

*Find  $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$  such that*

$$\mathbb{A}\mathbf{u} + \mathbb{B}^T \boldsymbol{\lambda} = \mathbf{f},$$

$$\mathbb{B}\mathbf{u} = \mathbf{g}.$$



# Matrix form of $(\hat{\mathcal{P}}(\Omega))_h^H$

## Problem $(\vec{\mathcal{P}}(\Omega))$

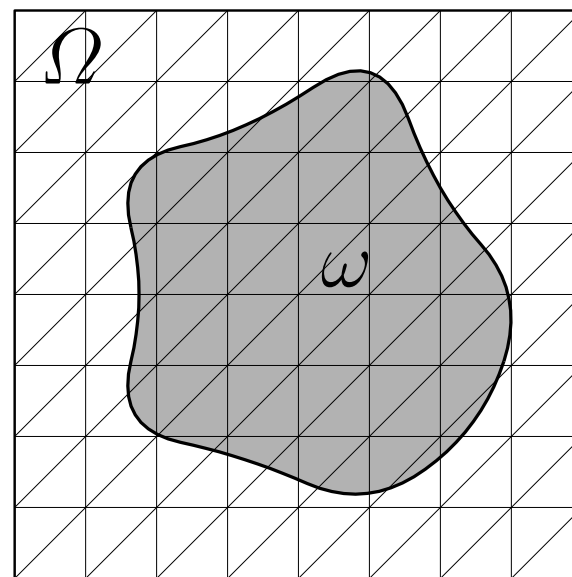
*Find  $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$  such that*

$$\mathbb{A}\mathbf{u} + \mathbb{B}^T \boldsymbol{\lambda} = \mathbf{f},$$

$$\mathbb{B}\mathbf{u} = \mathbf{g}.$$

## Remark

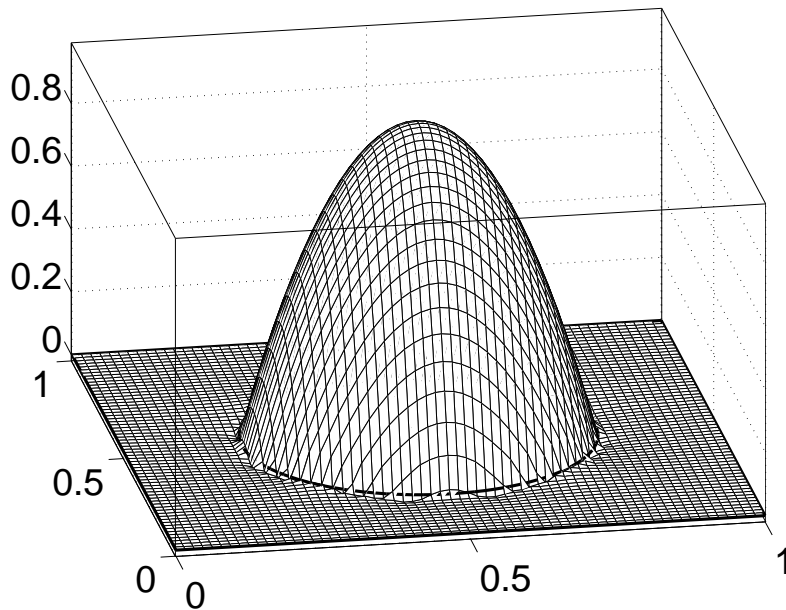
If non-fitted meshes are used then  $\mathbb{A}$  **does not depend** on the geometry of  $\omega$ .



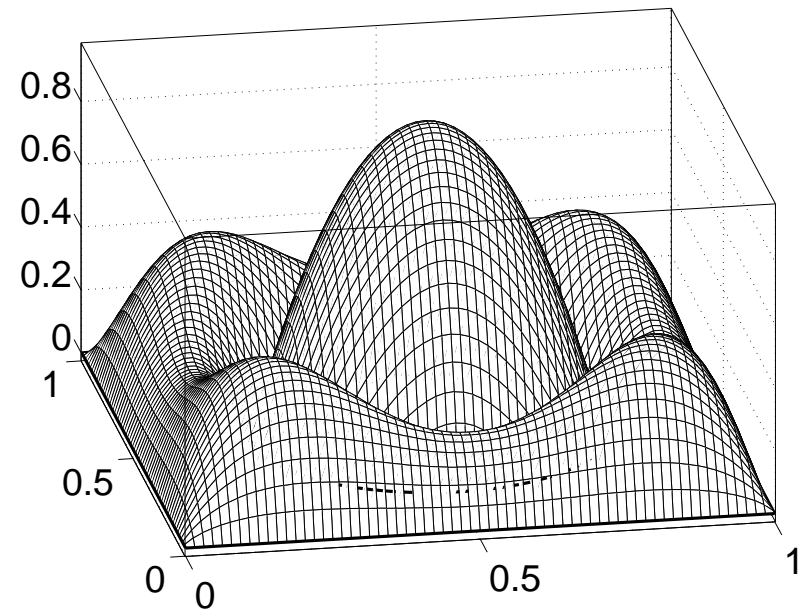
# FDM disadvantages

More unknowns, problems with local refinements and low regularity of  $\hat{u}$ :  $\hat{u} \in H^{3/2-\varepsilon}(\Omega) \quad \forall \varepsilon > 0$ .

## Typical behavior



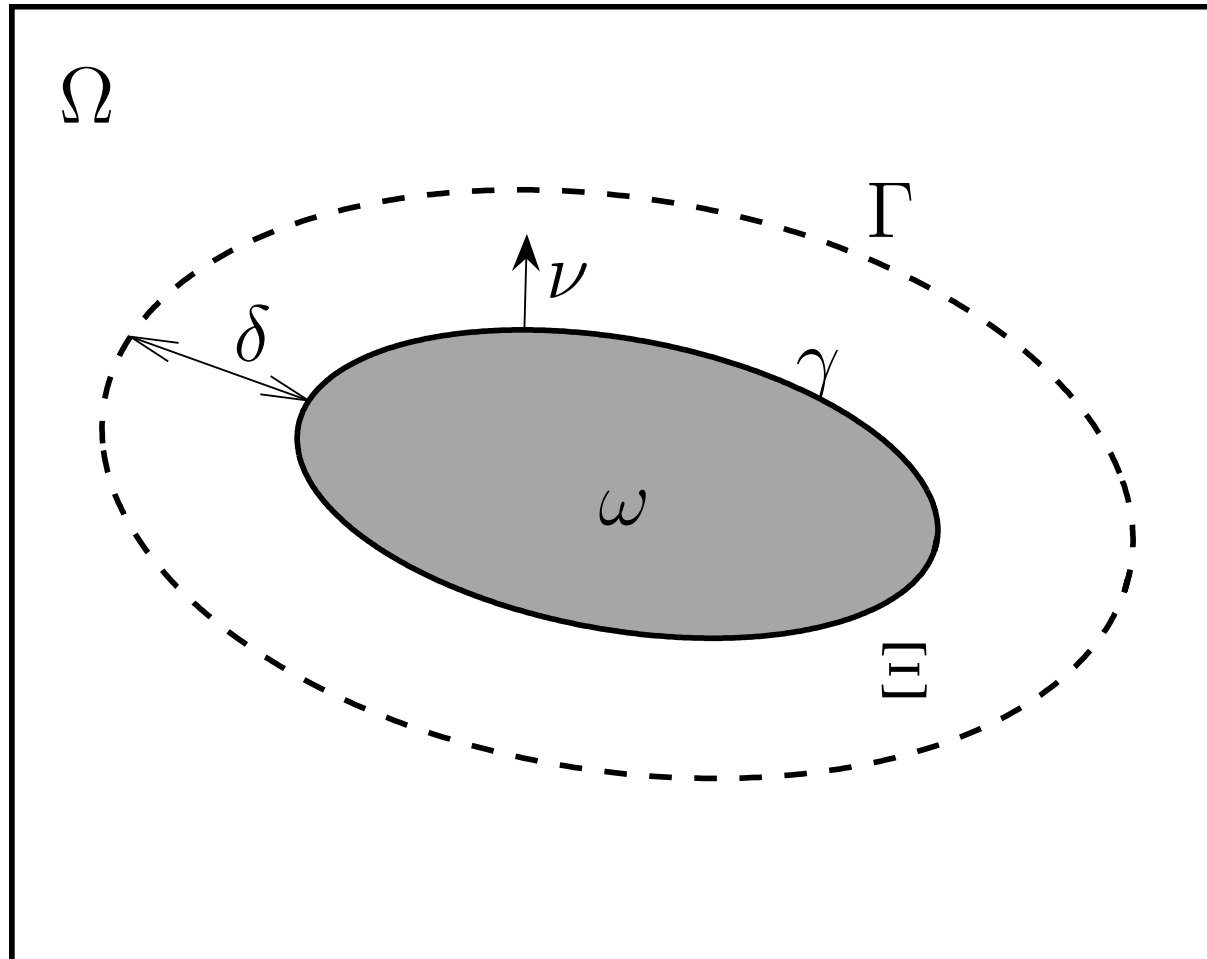
Zero extension of  $f$



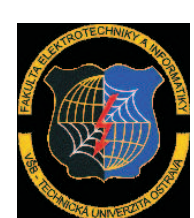
Natural extension of  $f$



# New variant of the FDM - idea







# New variant of the FDM

## Problem $(\hat{\mathcal{P}}(\gamma, \Gamma))$

*Find  $(\hat{u}, \lambda) \in H_0^1(\Omega) \times H^{-1/2}(\Gamma)$  such that*

$$\int_{\Omega} \nabla \hat{u} \cdot \nabla \hat{v} \, dx = \int_{\Omega} \hat{f} \hat{v} \, dx + \langle \lambda, \hat{v} \rangle_{\Gamma} \quad \forall \hat{v} \in H_0^1(\Omega),$$

$$\langle \mu, \hat{u} \rangle_{\gamma} = \langle \mu, g \rangle_{\gamma} \quad \forall \mu \in H^{-1/2}(\gamma).$$



# New variant of the FDM

## Problem $(\hat{\mathcal{P}}(\gamma, \Gamma))$

*Find  $(\hat{u}, \lambda) \in H_0^1(\Omega) \times H^{-1/2}(\Gamma)$  such that*

$$\int_{\Omega} \nabla \hat{u} \cdot \nabla \hat{v} \, dx = \int_{\Omega} \hat{f} \hat{v} \, dx + \langle \lambda, \hat{v} \rangle_{\Gamma} \quad \forall \hat{v} \in H_0^1(\Omega),$$

$$\langle \mu, \hat{u} \rangle_{\gamma} = \langle \mu, g \rangle_{\gamma} \quad \forall \mu \in H^{-1/2}(\gamma).$$

$\langle \cdot, \cdot \rangle_{\Gamma} \dots$  duality pairing between  $H^{-1/2}(\Gamma)$  and  $H^{1/2}(\Gamma)$ .



# Discretization of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

$$\begin{aligned} H_0^1(\Omega) &\sim V_h, & H^{-1/2}(\gamma) &\sim L_H^\gamma, \\ H^{-1/2}(\Gamma) &\sim L_H^\Gamma, & \dim L_H^\gamma &= \dim L_H^\Gamma. \end{aligned}$$



# Discretization of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

$$\begin{aligned} H_0^1(\Omega) &\sim V_h, & H^{-1/2}(\gamma) &\sim L_H^\gamma, \\ H^{-1/2}(\Gamma) &\sim L_H^\Gamma, & \dim L_H^\gamma &= \dim L_H^\Gamma. \end{aligned}$$

**Problem  $(\hat{\mathcal{P}}(\gamma, \Gamma))_h^H$**

*Find  $(\hat{u}_h, \lambda_H) \in V_h \times L_H^\Gamma$  such that*

$$\int_{\Omega} \nabla \hat{u}_h \cdot \nabla \hat{v}_h \, dx = \int_{\Omega} \hat{f} \hat{v}_h \, dx + \int_{\Gamma} \lambda_H \hat{v}_h \, ds \quad \forall \hat{v}_h \in V_h,$$
$$\int_{\gamma} \mu_H \hat{u}_h \, ds = \int_{\gamma} \mu_H g \, ds \quad \forall \mu_H \in L_H^\gamma.$$



# Matrix form of $(\hat{\mathcal{P}}(\gamma, \Gamma))_h^H$

## New FD approach

$$\begin{aligned} & \text{Find } (\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m \\ & \mathbf{A}\mathbf{u} + \mathbf{C}^T(\Gamma)\boldsymbol{\lambda} = \mathbf{f}, \\ & \mathbf{B}(\gamma)\mathbf{u} = \mathbf{g}(\gamma). \end{aligned}$$

$$\boldsymbol{\lambda} := \boldsymbol{\lambda}(\Gamma)$$

Projected BiCG method

## Classical FD approach

$$\begin{aligned} & \text{Find } (\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m \\ & \mathbf{A}\mathbf{u} + \mathbf{B}^T(\gamma)\boldsymbol{\lambda} = \mathbf{f}, \\ & \mathbf{B}(\gamma)\mathbf{u} = \mathbf{g}(\gamma). \end{aligned}$$

$$\boldsymbol{\lambda} := \boldsymbol{\lambda}(\gamma)$$

Projected CG method



# Solvability of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

---

There is a dense subset  $\mathcal{V} \subseteq H^{1/2}(\gamma)$  such that if  $g \in \mathcal{V}$  then  $(\hat{\mathcal{P}}(\gamma, \Gamma))$  has a solution. Moreover,  $\hat{u}|_{\omega}$  is determined uniquely and solves  $(\mathcal{P}(\omega))$ .



# Solvability of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

There is a dense subset  $\mathcal{V} \subseteq H^{1/2}(\gamma)$  such that if  $g \in \mathcal{V}$  then  $(\hat{\mathcal{P}}(\gamma, \Gamma))$  has a solution. Moreover,  $\hat{u}|_{\omega}$  is determined uniquely and solves  $(\mathcal{P}(\omega))$ .

If  $g \notin \mathcal{V}$  then  $\forall \varepsilon > 0 \exists \tilde{g} \in \mathcal{V}$  such that  $\|g - \tilde{g}\|_{1/2, \gamma} < \varepsilon$ .



# Solvability of $(\hat{\mathcal{P}}(\gamma, \Gamma))$

There is a dense subset  $\mathcal{V} \subseteq H^{1/2}(\gamma)$  such that if  $g \in \mathcal{V}$  then  $(\hat{\mathcal{P}}(\gamma, \Gamma))$  has a solution. Moreover,  $\hat{u}|_{\omega}$  is determined uniquely and solves  $(\mathcal{P}(\omega))$ .

If  $g \notin \mathcal{V}$  then  $\forall \varepsilon > 0 \exists \tilde{g} \in \mathcal{V}$  such that  $\|g - \tilde{g}\|_{1/2, \gamma} < \varepsilon$ .

If  $\hat{w} \in H_0^1(\Omega)$  solves  $(\hat{\mathcal{P}}(\gamma, \Gamma))$  with  $g := \tilde{g}$  then

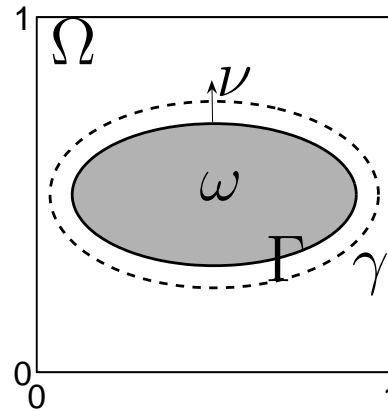
$$\|\hat{u} - \hat{w}\|_{1, \omega} \leq c \varepsilon, \quad c = \text{const.} > 0.$$



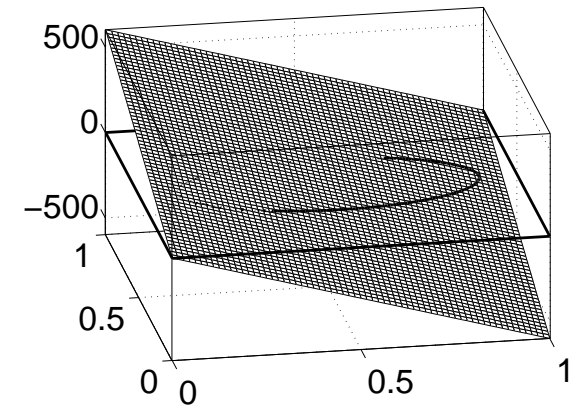
# Numerical experiments

## Problem $(\mathcal{P}(\omega))$

$$\begin{aligned} -\Delta u &= f \text{ in } \omega, \\ u &= g \text{ on } \gamma. \end{aligned}$$



Geometry



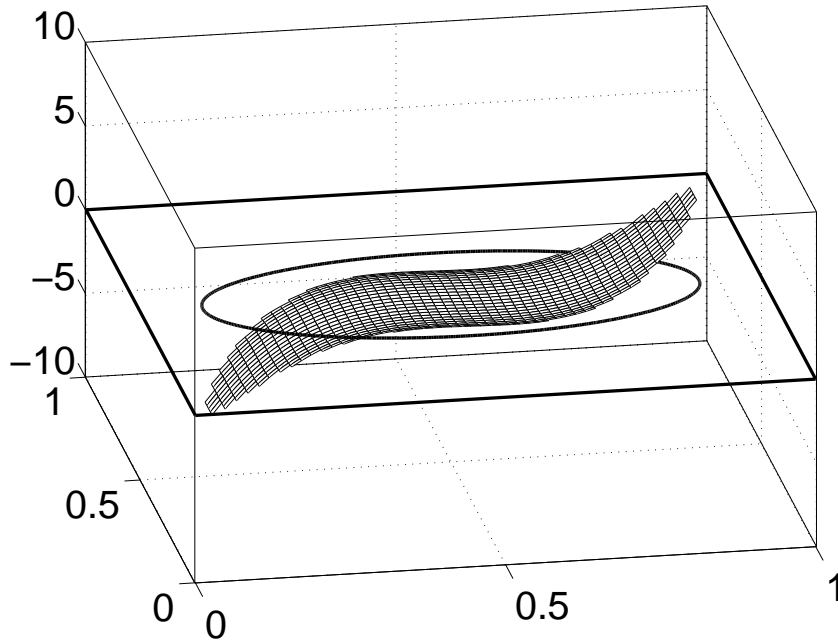
Right hand side  $f$

$$u_{ex}(x, y) = 100 \left( (x - 0.5)^3 - (y - 0.5)^3 \right) - x^2, \quad g(x, y) = u_{ex}|_{\gamma},$$

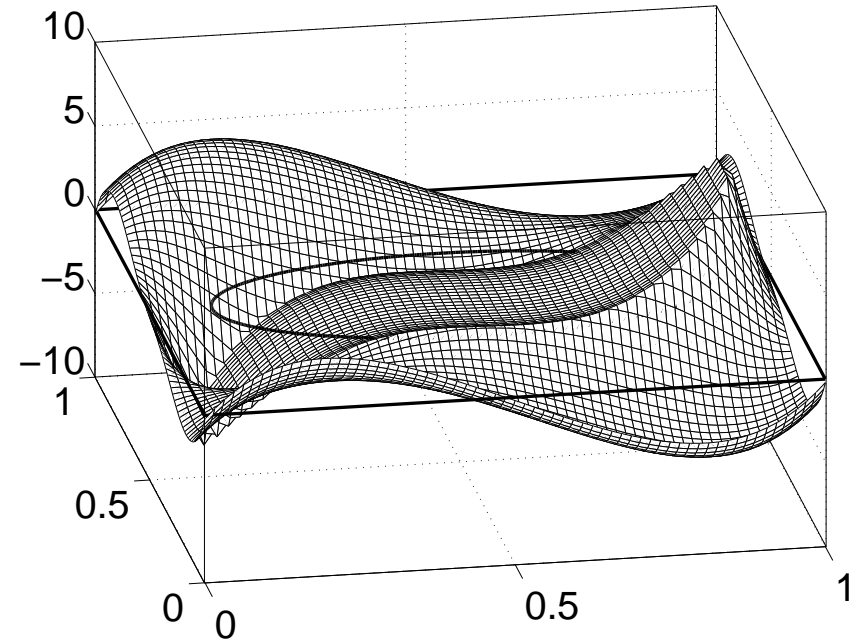
$$f(x, y) = -\Delta u_{ex} = -600 \left( (x - 0.5) - (y - 0.5) \right) + 2.$$



# Numerical experiments



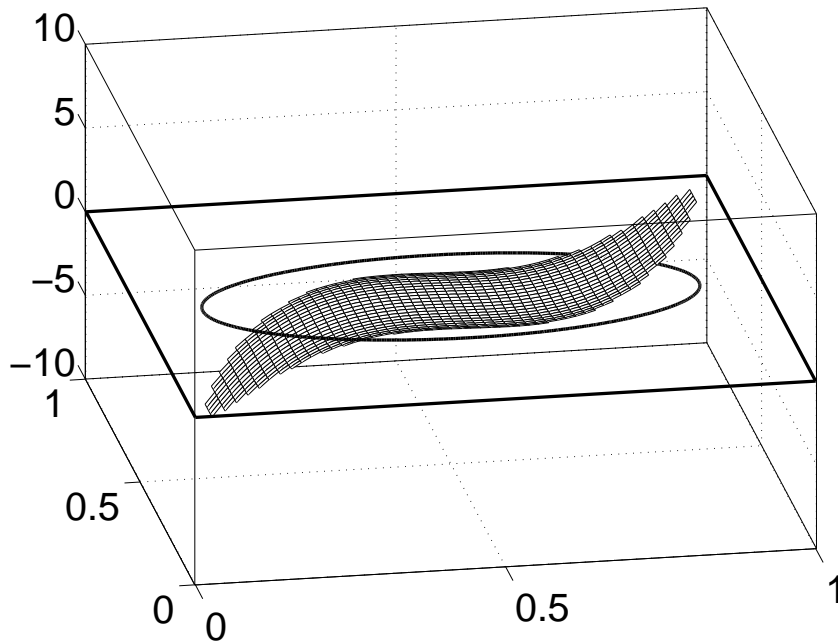
Exact solution  $u_{ex}$



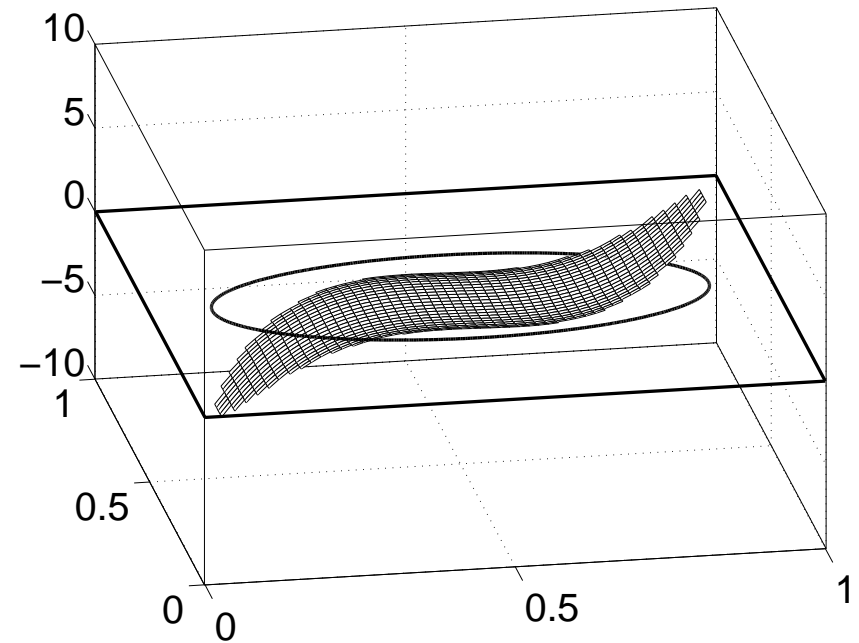
Computed solution  $\hat{u}_h$



# Numerical experiments



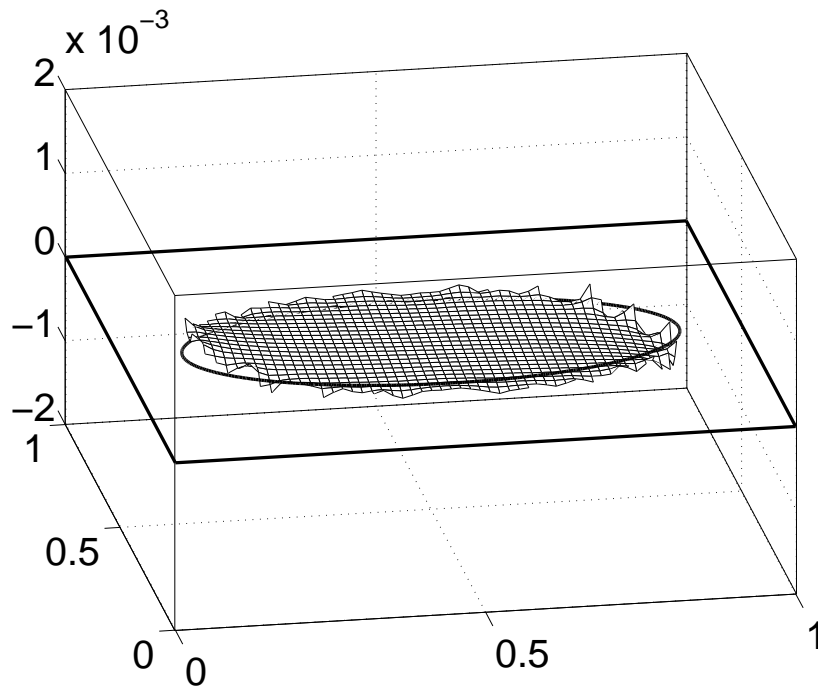
Exact solution  $u_{ex}$



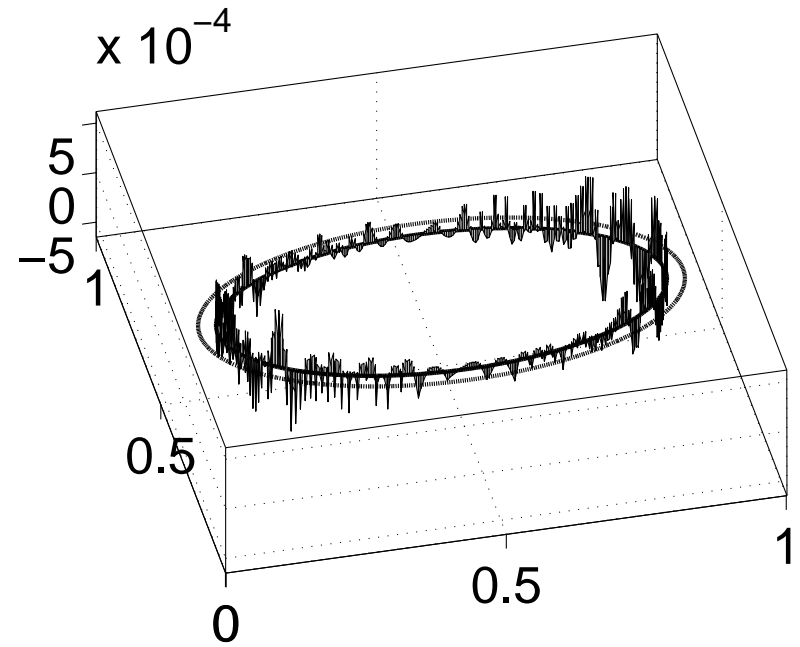
Computed solution  $\hat{u}_h$  in  $\omega$



# Numerical experiments



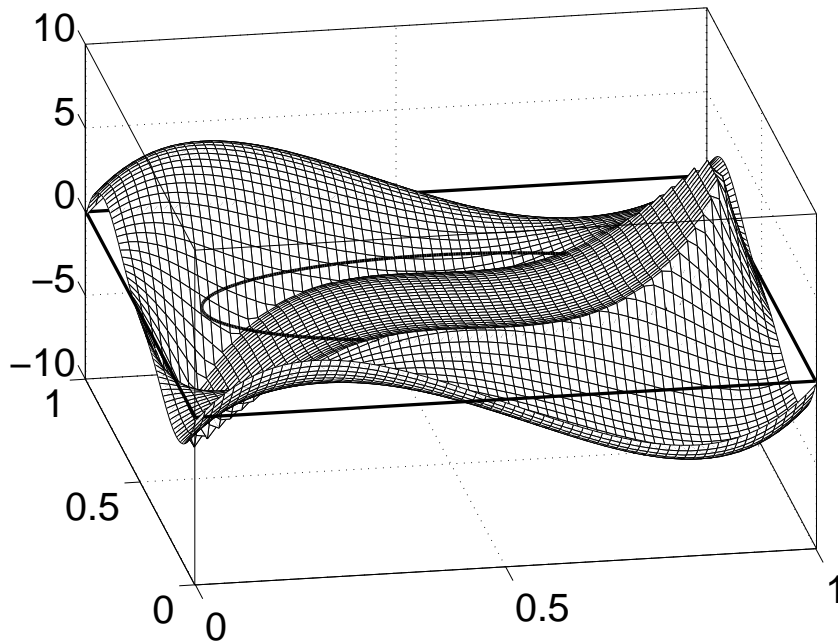
Difference  $\hat{u}_h - u_{ex}$  in  $\omega$



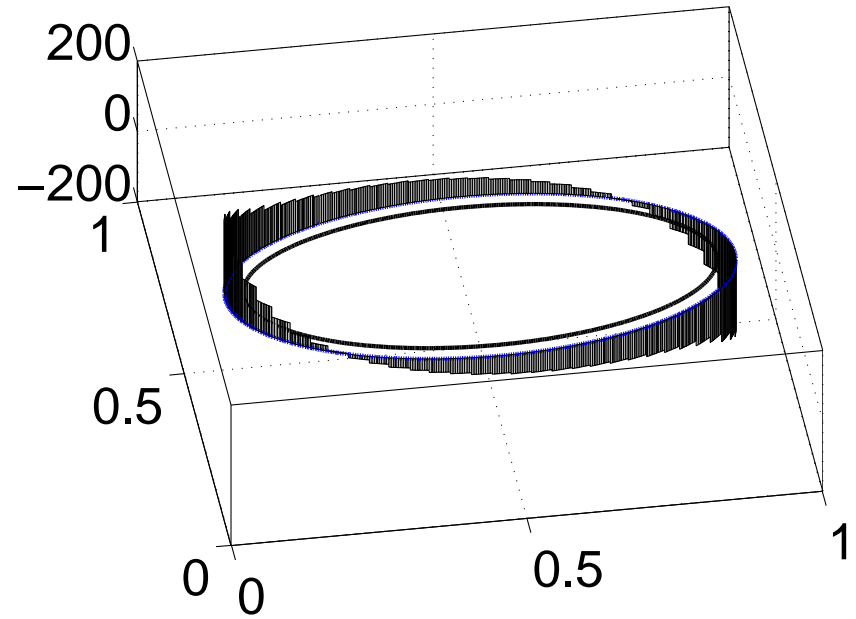
Difference  $\hat{u}_h - u_{ex}$  on  $\gamma$



# Numerical experiments



Computed solution  $\hat{u}_h$



Control variable  $\lambda_H$  on  $\Gamma$



# Numerical experiments

Ellipse, classical FD approach.

Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2}(\omega)$	$\text{Err}_{H^1}(\omega)$	$\text{Err}_{L^2}(\gamma)$
1/128	16641/35	8	0.14	2.0860e-2	1.9647e+0	6.6516e-2
1/256	66049/62	9	0.56	1.1092e-2	1.2884e+0	3.2175e-2
1/512	263169/110	12	5.19	5.3989e-3	8.6517e-1	1.5019e-2
1/1024	1050625/198	20	33.05	2.7453e-3	6.0511e-1	7.3265e-3
1/2048	4198401/360	26	167.00	1.3349e-3	4.4015e-1	3.6245e-3
Convergence rates:				0.995	0.541	1.053

Ellipse, new FD approach; ProjBiCGSTAB,  $\epsilon = h^2 \|\tilde{d}\|$ .

Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2}(\omega)$	$\text{Err}_{H^1}(\omega)$	$\text{Err}_{L^2}(\gamma)$
1/128	16641/35	13	0.17	2.2550e-4	1.6884e-2	1.1689e-3
1/256	66049/62	25	1.33	5.4869e-5	7.7891e-3	2.9342e-4
1/512	263169/110	40	14.97	1.4177e-5	4.0160e-3	1.1504e-4
1/1024	1050625/198	55	83.56	3.4507e-6	1.9028e-3	2.4769e-5
1/2048	4198401/360	94	571.50	9.0638e-7	9.9895e-4	1.2495e-5
Convergence rates:				1.991	1.019	1.666



# Numerical experiments

Ellipse, new FD approach; ProjBiCGSTAB,  $\epsilon = h^2 \|\tilde{d}\|$ .

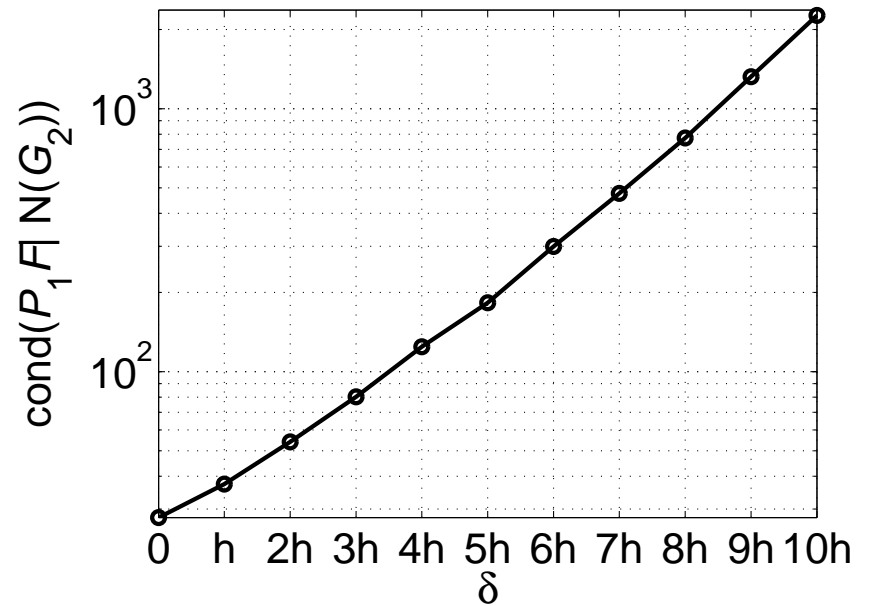
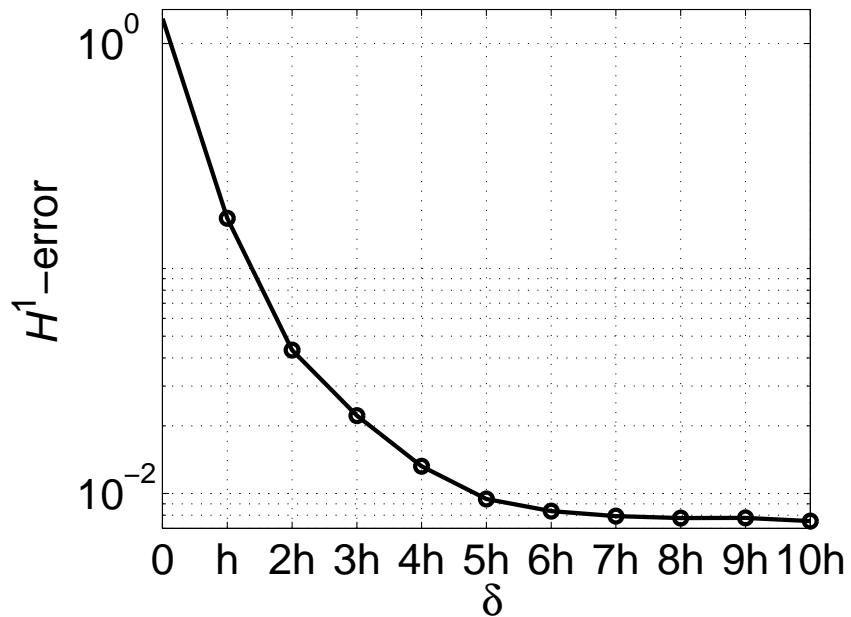
Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/35	13	0.17	2.2550e-4	1.6884e-2	1.1689e-3
1/256	66049/62	25	1.33	5.4869e-5	7.7891e-3	2.9342e-4
1/512	263169/110	40	14.97	1.4177e-5	4.0160e-3	1.1504e-4
1/1024	1050625/198	55	83.56	3.4507e-6	1.9028e-3	2.4769e-5
1/2048	4198401/360	94	571.50	9.0638e-7	9.9895e-4	1.2495e-5
Convergence rates:				1.991	1.019	1.666

Ellipse, new FD approach; Multigrid.

Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/34	11	0.22	2.4444e-4	1.8988e-2	1.4694e-3
1/256	66049/62	13	0.88	5.5030e-5	7.6303e-3	2.5171e-4
1/512	263169/110	19	8.41	1.3952e-5	3.8638e-3	8.3976e-5
1/1024	1050625/198	22	41.91	3.3209e-6	1.8681e-3	2.5253e-5
1/2048	4198401/360	31	243.50	8.5762e-7	9.6771e-4	1.1555e-5
Convergence rates:				2.036	1.062	1.730



# Numerical experiments

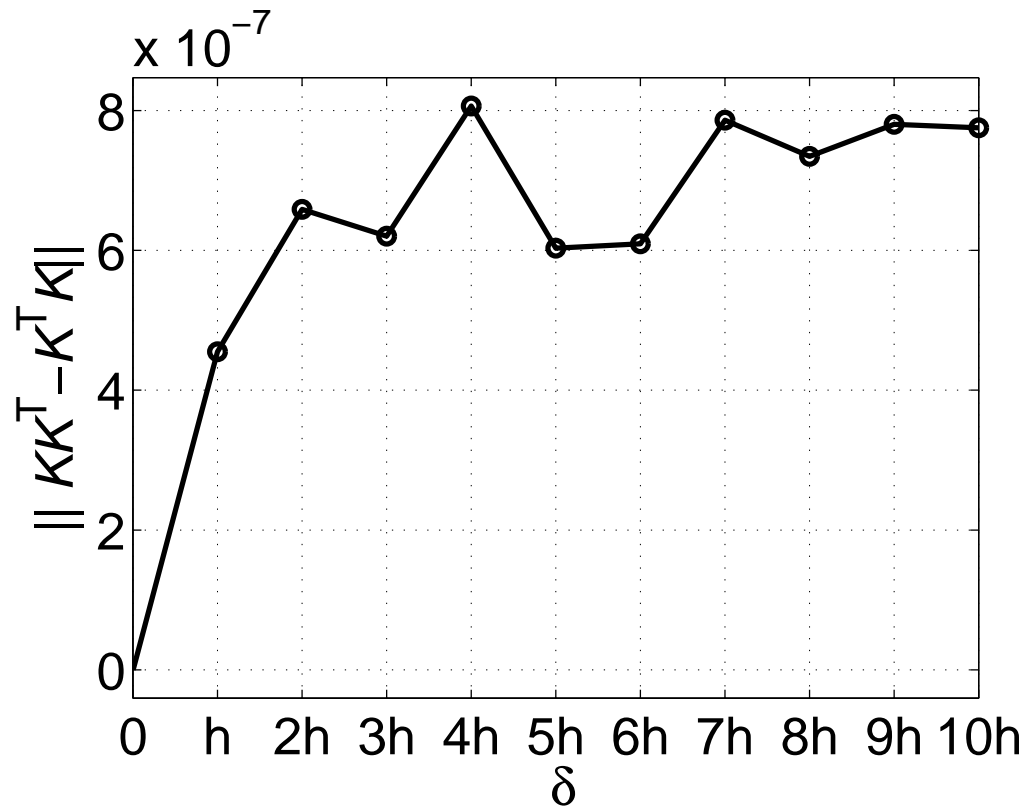


$H_1(\omega)$ -error sensitivity with  $\delta$ .  $\text{Cond}(P_1 F | \mathbb{N}(G_2))$  sensitivity with  $\delta$ .





# Numerical experiments

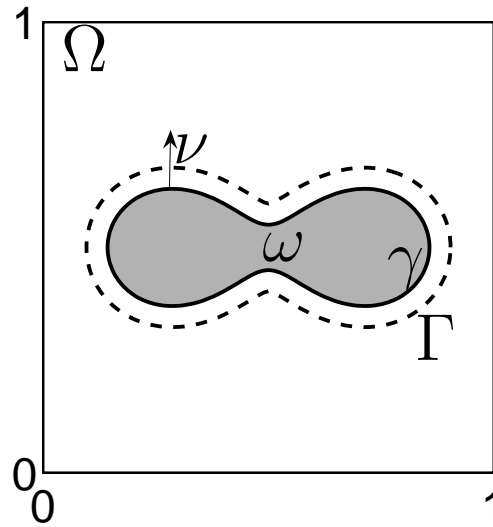


Violation of normality as a function of  $\delta$ , i.e.  $\|KK^T - K^T K\|$ ,  $K = P_1 F$ .

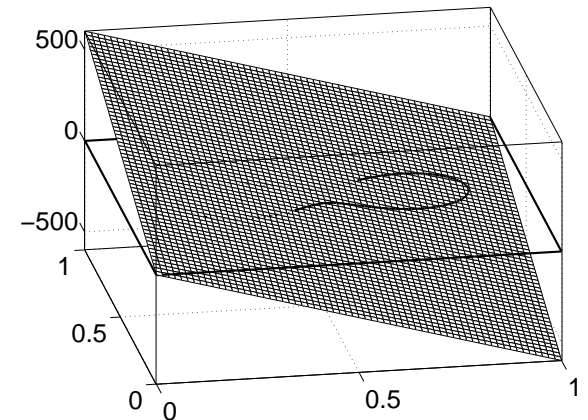
# Numerical experiments

**Problem  $(\mathcal{P}(\omega))$**

$$\begin{aligned} -\Delta u &= f \text{ in } \omega, \\ u &= g \text{ on } \gamma. \end{aligned}$$



Geometry

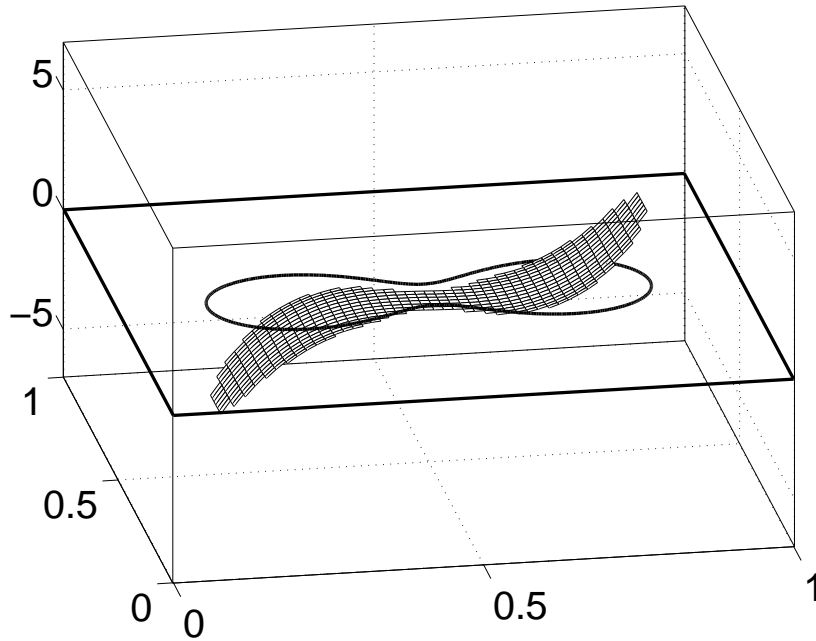


Right hand side  $f$

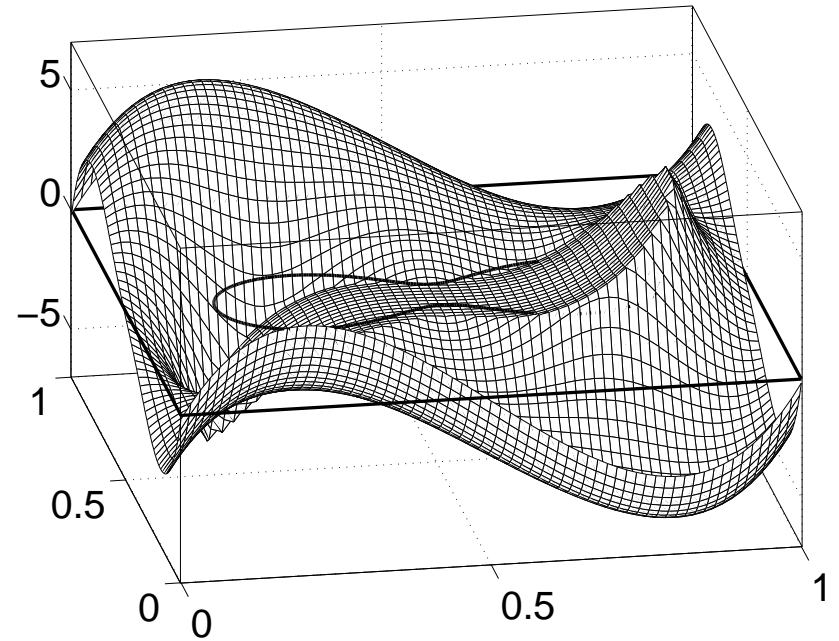
$$u_{ex}(x, y) = 100 \left( (x - 0.5)^3 - (y - 0.5)^3 \right) - x^2, \quad g(x, y) = u_{ex}|_{\gamma},$$

$$f(x, y) = -\Delta u_{ex} = -600 \left( (x - 0.5) - (y - 0.5) \right) + 2.$$

# Numerical experiments



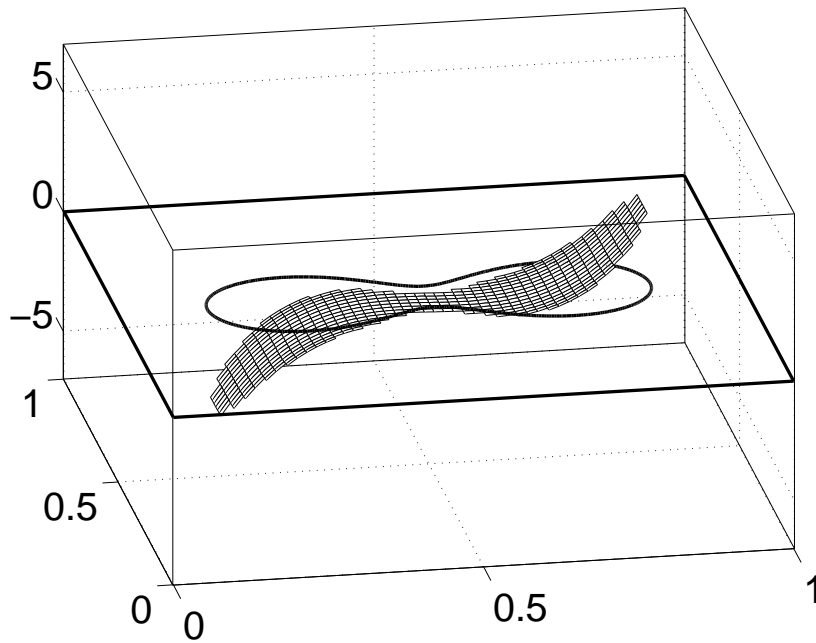
Exact solution  $u_{ex}$



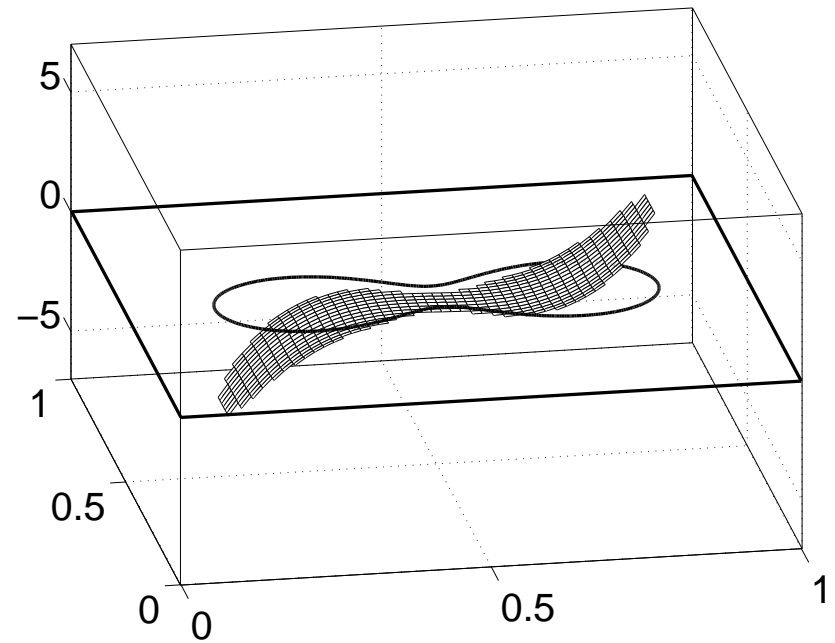
Computed solution  $\hat{u}_h$



# Numerical experiments



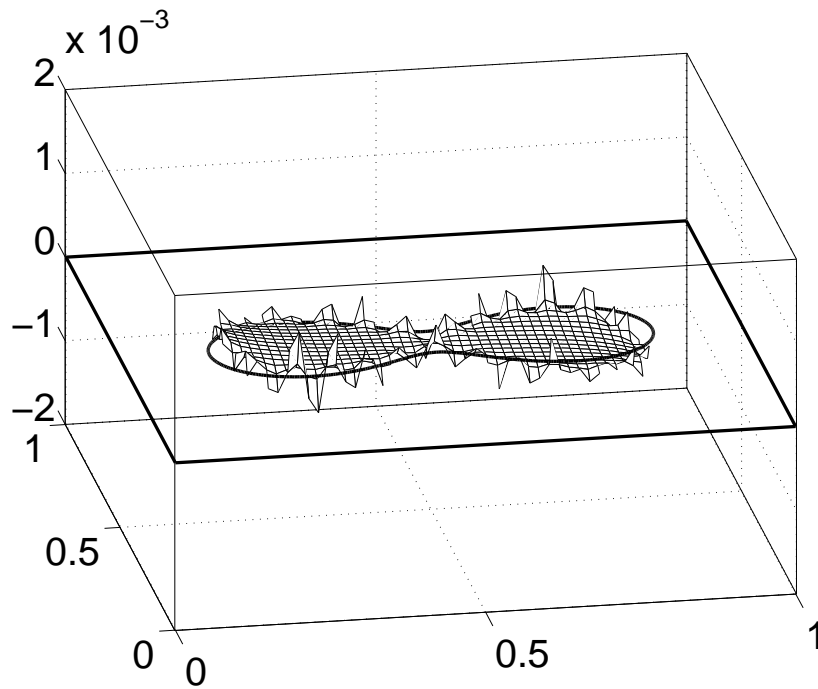
Exact solution  $u_{ex}$



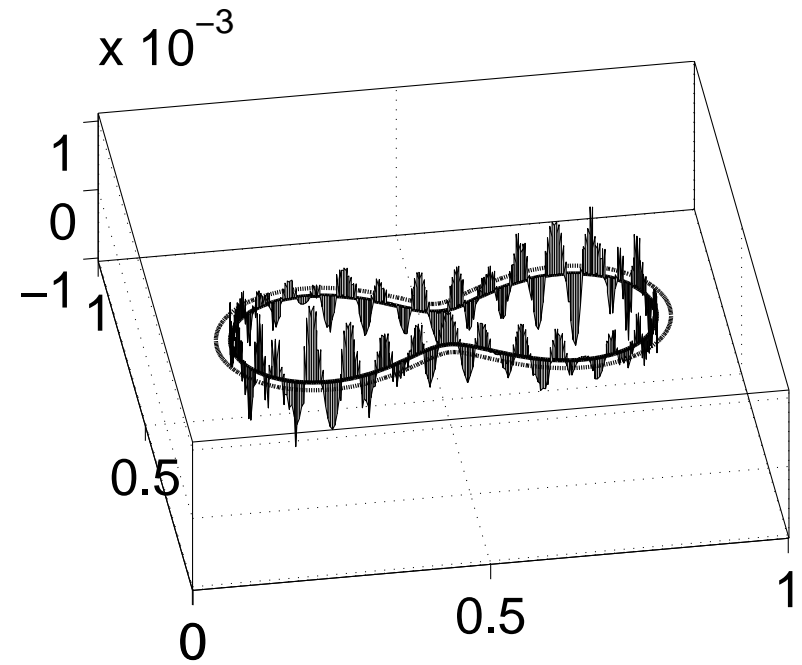
Computed solution  $\hat{u}_h$  in  $\omega$



# Numerical experiments



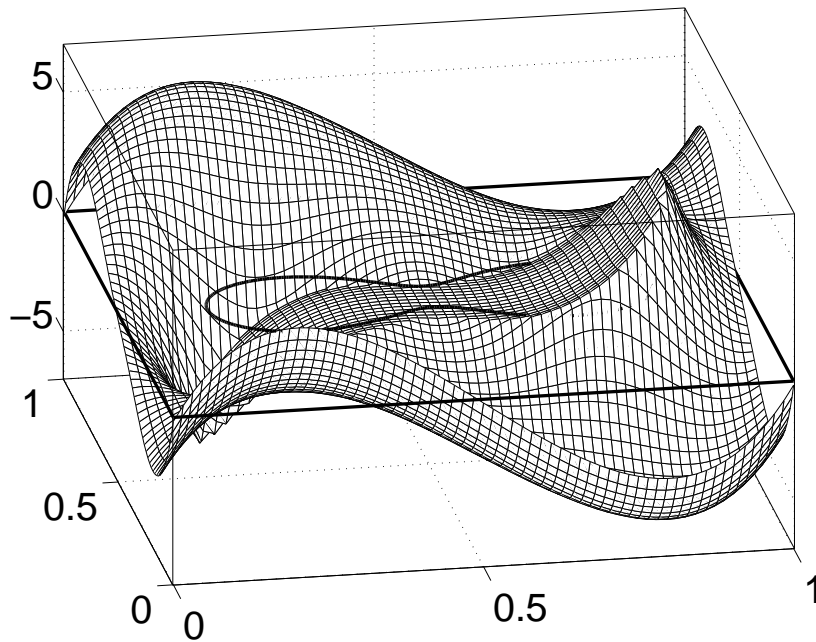
Difference  $\hat{u}_h - u_{ex}$  in  $\omega$



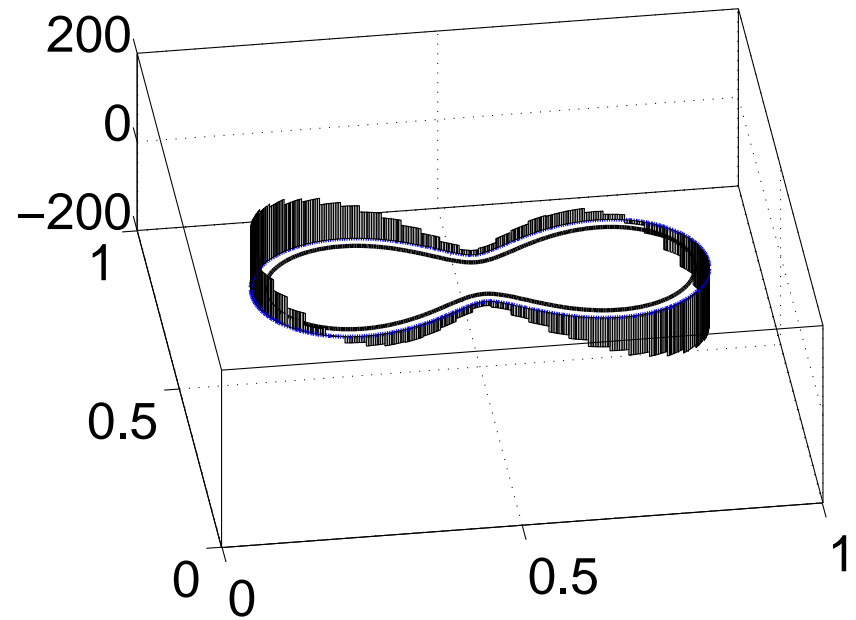
Difference  $\hat{u}_h - u_{ex}$  on  $\gamma$



# Numerical experiments



Computed solution  $\hat{u}_h$



Control variable  $\lambda_H$  on  $\Gamma$



# Numerical experiments

## Cassini oval, classical FD approach.

Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/32	7	0.11	1.8469e-2	1.4316e+0	4.9200e-2
1/256	66049/57	9	0.56	8.5003e-3	9.9211e-1	2.5737e-2
1/512	263169/101	13	5.27	4.0795e-3	6.9436e-1	1.1640e-2
1/1024	1050625/182	17	27.77	1.9695e-3	4.9840e-1	5.9052e-3
1/2048	4198401/332	21	136.80	9.9397e-4	3.5127e-1	3.0024e-3
Convergence rates:				1.054	0.505	1.019

## Cassini oval, new FD approach; ProjBiCGSTAB, $\epsilon = h^2 \|\tilde{d}\|$ .

Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/32	16	0.20	4.8818e-4	4.3430e-2	5.2433e-3
1/256	66049/57	30	1.59	5.8574e-5	1.0141e-2	7.0059e-4
1/512	263169/101	51	18.86	1.3846e-5	4.6618e-3	2.1672e-4
1/1024	1050625/182	100	149.70	2.7136e-6	1.8784e-3	4.6878e-5
1/2048	4198401/332	186	1135.00	7.5260e-7	1.0081e-3	1.9824e-5
Convergence rates:				2.311	1.329	2.000



# Numerical experiments

Cassini oval, new FD approach; ProjBiCGSTAB,  $\epsilon = h^2 \|\tilde{d}\|$ .

Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/32	16	0.20	4.8818e-4	4.3430e-2	5.2433e-3
1/256	66049/57	30	1.59	5.8574e-5	1.0141e-2	7.0059e-4
1/512	263169/101	51	18.86	1.3846e-5	4.6618e-3	2.1672e-4
1/1024	1050625/182	100	149.70	2.7136e-6	1.8784e-3	4.6878e-5
1/2048	4198401/332	186	1135.00	7.5260e-7	1.0081e-3	1.9824e-5
Convergence rates:				2.311	1.329	2.000

Cassini oval, new FD approach; Multigrid.

Step $h$	$n/m$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/32	15	0.2031	4.2930e-4	3.8333e-2	4.5419e-3
1/256	66049/56	29	1.672	4.6345e-5	8.1012e-3	4.5772e-4
1/512	263169/100	30	12.33	1.0902e-5	3.7576e-3	1.3216e-4
1/1024	1050625/182	44	76	2.6887e-6	1.8829e-3	4.8691e-5
1/2048	4198401/332	63	439.3	7.3218e-7	9.8655e-4	1.8763e-5
Convergence rates:				2.250	1.267	1.907





# Problems defined in domains with variable boundary

---

## Free boundary problems (Bernoulli, Stefan)

- J. Haslinger, T. Kozubek, K. Kunisch and G. Peichl (2003): **Shape Optimization and Fictitious Domain Approach for Solving Free Boundary Problems of Bernoulli Type**, COAP, **26**, 231–251.



# Problems defined in domains with variable boundary

## Free boundary problems (Bernoulli, Stefan)

- J. Haslinger, T. Kozubek, K. Kunisch and G. Peichl (2003): **Shape Optimization and Fictitious Domain Approach for Solving Free Boundary Problems of Bernoulli Type**, COAP, **26**, 231–251.

## Shape optimization

- J. Haslinger, D. Jedelský, T. Kozubek, J. Tvrdík: **Genetic and random search methods in optimal shape design problems**, JOGO **16** (2000), pp. 109–131, **IF** 0.750.



# Problems defined in domains with variable boundary

## Free boundary problems (Bernoulli, Stefan)

- J. Haslinger, T. Kozubek, K. Kunisch and G. Peichl (2003): **Shape Optimization and Fictitious Domain Approach for Solving Free Boundary Problems of Bernoulli Type**, COAP, **26**, 231–251.

## Shape optimization

- J. Haslinger, D. Jedelský, T. Kozubek, J. Tvrdlík: **Genetic and random search methods in optimal shape design problems**, JOGO **16** (2000), pp. 109–131, **IF** 0.750.

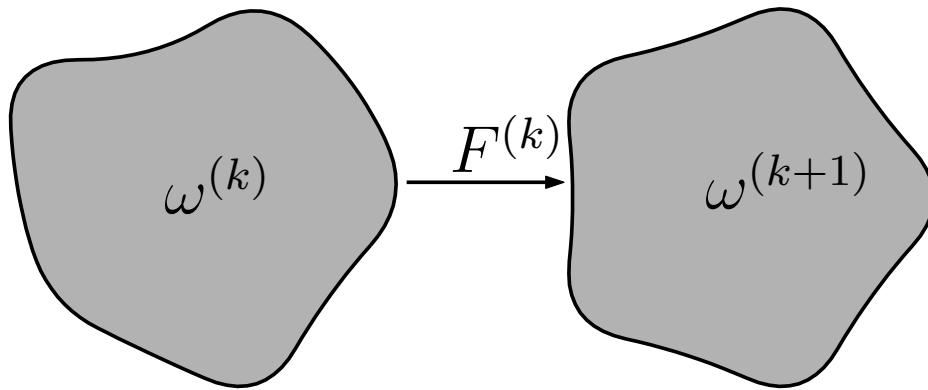
## Problems with uncertain geometry

- C. Canuto, T. Kozubek: **A fictitious domain approach to the numerical solution of PDEs in stochastic domains**, accepted for publishing in Numerische Mathematik, 2007.



# FDM in shape optimization

**Standard approach:** boundary variation technique



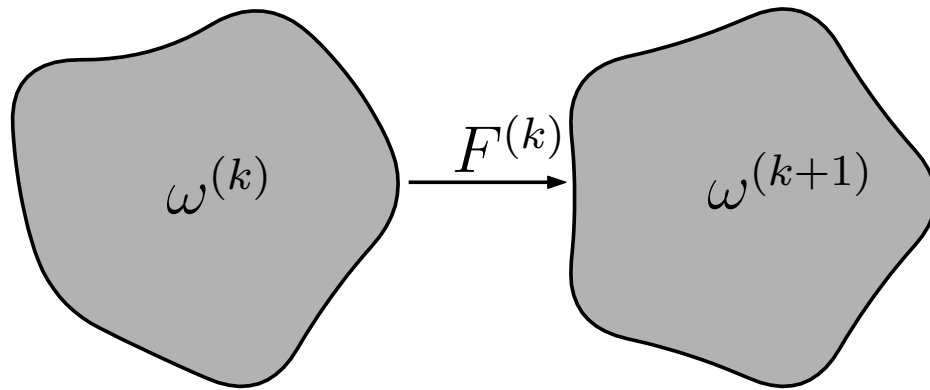
**Problem**  $(\vec{\mathcal{P}}(\omega))$

$$\begin{aligned} &\text{Find } \mathbf{u} \in \mathbb{R}^n \\ &\mathbb{A}(\omega)\mathbf{u} = \mathbf{f}(\omega). \end{aligned}$$



# FDM in shape optimization

**Standard approach:** boundary variation technique



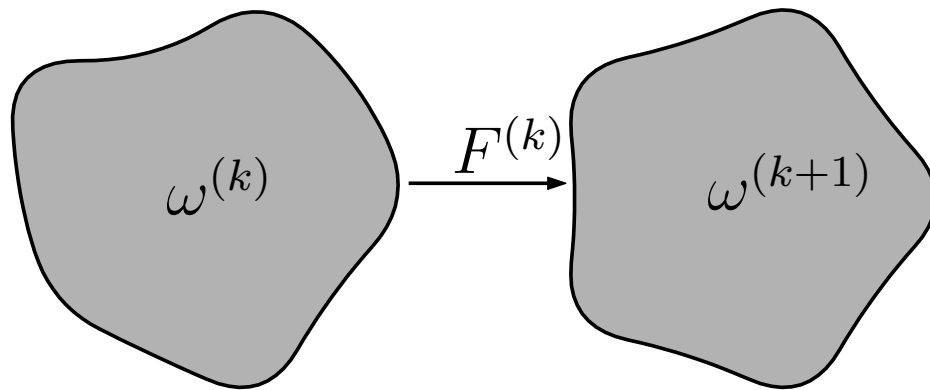
**Problem**  $(\vec{\mathcal{P}}(\omega))$

$$\begin{aligned} &\text{Find } \mathbf{u} \in \mathbb{R}^n \\ &\mathbb{A}(\omega)\mathbf{u} = \mathbf{f}(\omega). \end{aligned}$$

$$\omega^{(k+1)} = F^{(k)}(\omega^{(k)}), \quad J(\omega^{(k+1)}) \leq J(\omega^{(k)})$$

# FDM in shape optimization

**Standard approach:** boundary variation technique



**Problem**  $(\vec{\mathcal{P}}(\omega))$

$$\begin{aligned} &\text{Find } \mathbf{u} \in \mathbb{R}^n \\ &\mathbb{A}(\omega)\mathbf{u} = \mathbf{f}(\omega). \end{aligned}$$

$$\omega^{(k+1)} = F^{(k)}(\omega^{(k)}), \quad J(\omega^{(k+1)}) \leq J(\omega^{(k)})$$

**The following steps have to be performed:**

- (i) remeshing of the new configuration;
- (ii) recomputing of  $\mathbb{A}$  and  $\mathbf{f}$ ;
- (iii) solving the discrete state problem.



# The use of the BLM method

**Problem  $(\vec{\mathcal{P}}(\omega))$**

$$\begin{aligned} \text{Find } \mathbf{u} &\in \mathbb{R}^n \\ \mathbb{A}(\omega)\mathbf{u} &= \mathbf{f}(\omega). \end{aligned}$$



**Problem  $(\vec{\mathcal{P}}(\Omega))$**

$$\begin{aligned} \text{Find } (\mathbf{u}, \boldsymbol{\lambda}) &\in \mathbb{R}^n \times \mathbb{R}^m \\ \mathbb{A}\mathbf{u} + \mathbb{B}^T(\omega)\boldsymbol{\lambda} &= \mathbf{f}, \\ \mathbb{B}(\omega)\mathbf{u} &= \mathbf{g}(\omega). \end{aligned}$$

# The use of the BLM method

**Problem  $(\vec{\mathcal{P}}(\omega))$**

$$\begin{aligned} \text{Find } \mathbf{u} &\in \mathbb{R}^n \\ \mathbb{A}(\omega)\mathbf{u} &= \mathbf{f}(\omega). \end{aligned}$$



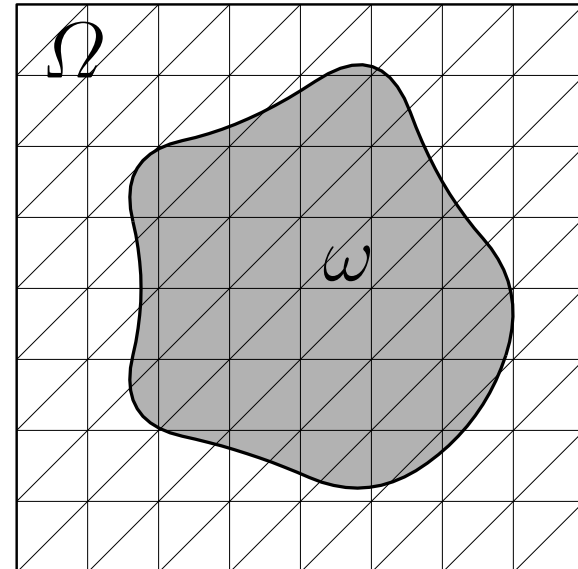
**Problem  $(\vec{\mathcal{P}}(\Omega))$**

$$\begin{aligned} \text{Find } (\mathbf{u}, \boldsymbol{\lambda}) &\in \mathbb{R}^n \times \mathbb{R}^m \\ \mathbb{A}\mathbf{u} + \mathbb{B}^T(\omega)\boldsymbol{\lambda} &= \mathbf{f}, \\ \mathbb{B}(\omega)\mathbf{u} &= \mathbf{g}(\omega). \end{aligned}$$

## Remark

If non-fitted meshes are used then  $\mathbb{A}$  **does not depend** on the geometry of  $\omega$ .

**We avoid step (i) and partially step (ii).**







# Numerical realization of free boundary problems of Bernoulli type

## Exterior Bernoulli problem

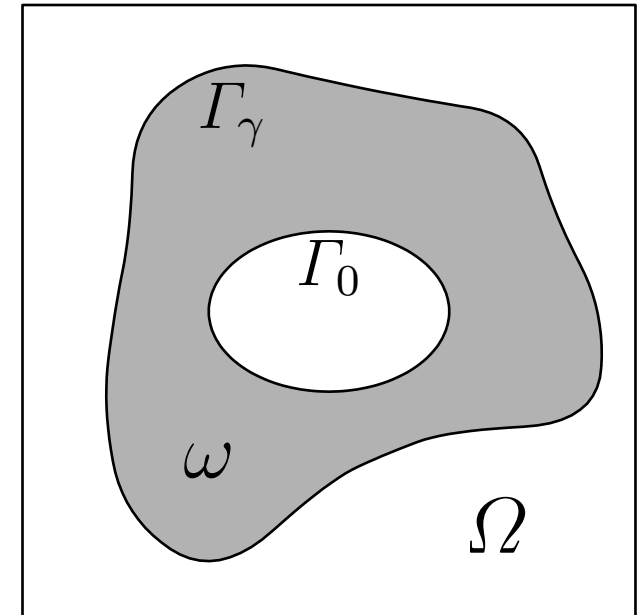
*Find  $\omega$  and  $u : \omega \mapsto \mathbb{R}^1$  such that*

$$\Delta u = 0 \text{ in } \omega,$$

$$u = 1 \text{ on } \Gamma_0,$$

$$u = 0 \text{ on } \Gamma_\gamma,$$

$$\frac{\partial u}{\partial n} = Q \text{ on } \Gamma_\gamma, \quad Q < 0.$$



## Interior Bernoulli problem

Role of the interior and exterior boundary is interchanged.



# Transform to a shape optimization problem (Bégis, Glowinski, ...)

## Problem $(\mathcal{P}(\omega))$

$$\begin{aligned}\Delta u(\omega) &= 0 \quad \text{in } \omega \in \mathcal{O}, \\ u(\omega) &= 1 \quad \text{on } \Gamma_0, \\ u(\omega) &= 0 \quad \text{on } \Gamma_\gamma,\end{aligned}$$

$\mathcal{O}$  ... a set of admissible domains.



# Transform to a shape optimization problem (Bégis, Glowinski, ...)

## Problem $(\mathcal{P}(\omega))$

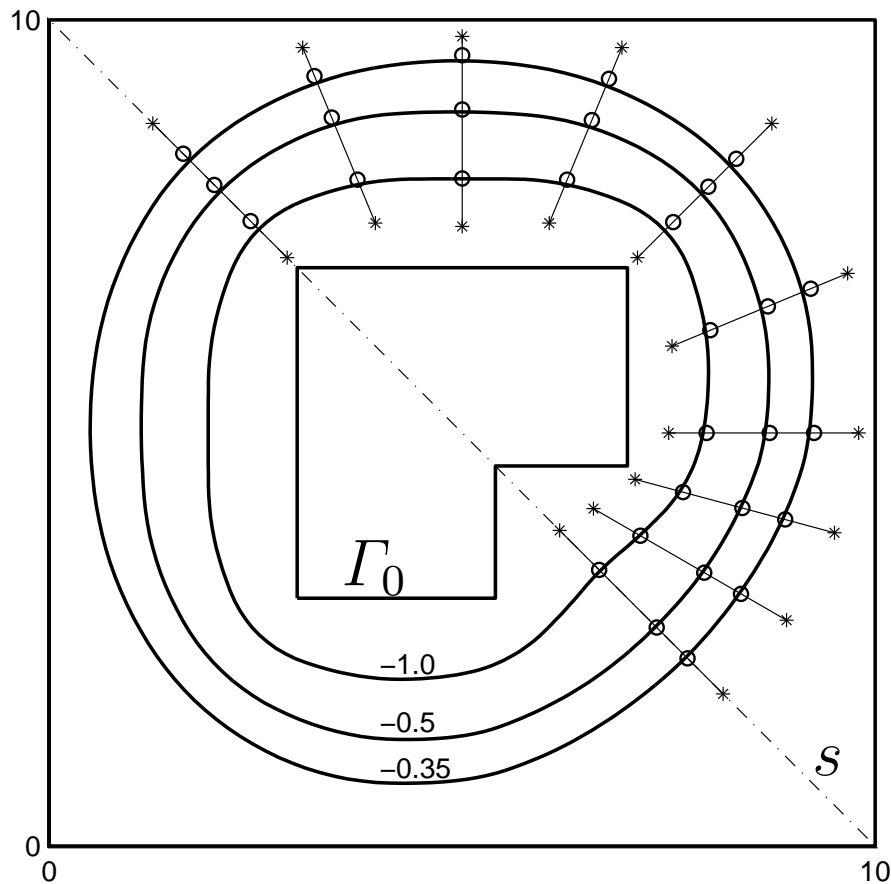
$$\begin{aligned}\Delta u(\omega) &= 0 \quad \text{in } \omega \in \mathcal{O}, \\ u(\omega) &= 1 \quad \text{on } \Gamma_0, \\ u(\omega) &= 0 \quad \text{on } \Gamma_\gamma,\end{aligned}$$

$\mathcal{O}$  ... a set of admissible domains.

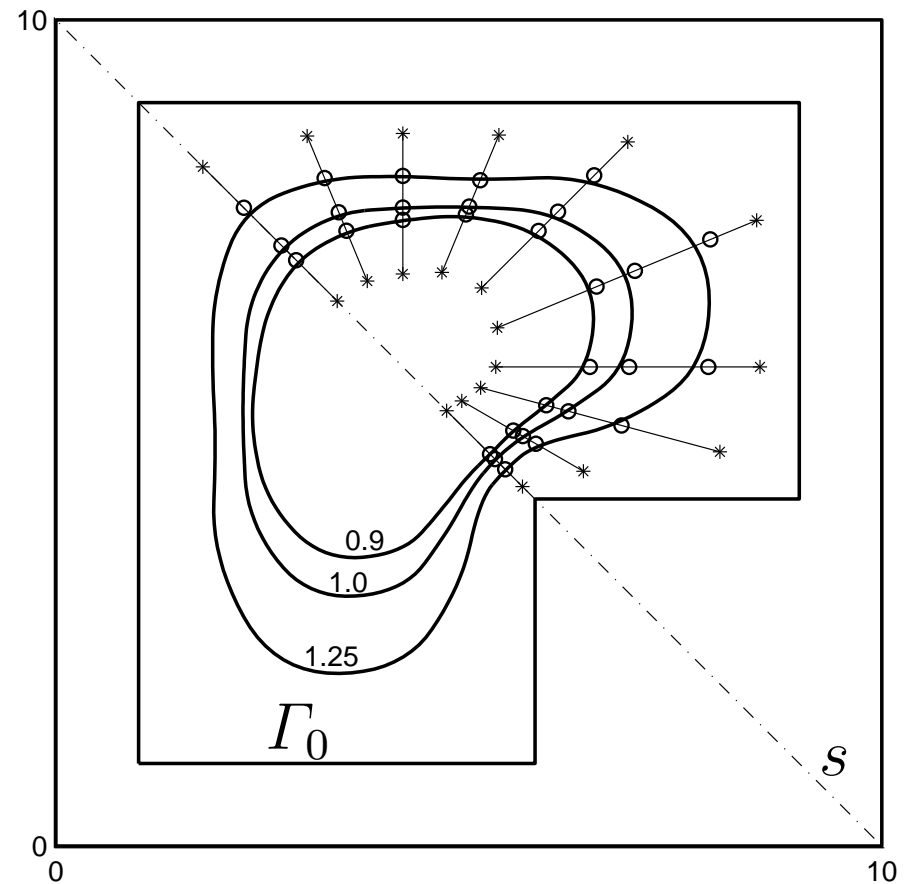
## Optimality criterion (cost functional)

$$J(u(\omega)) = \frac{1}{2} \left\| \frac{\partial u(\omega)}{\partial n} - Q \right\|_{0, \Gamma_\gamma}^2, \quad \frac{\partial u(\omega)}{\partial n} = \lambda$$

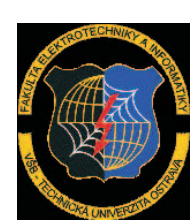
# Numerical experiments



Exterior Bernoulli problem.



Interior Bernoulli problem.



# For more details

---

- J. Haslinger, T. Kozubek, K. Kunisch and G. Peichl (2003): **Shape Optimization and Fictitious Domain Approach for Solving Free Boundary Problems of Bernoulli Type**, COAP, **26**, 231–251.



# CONCLUSIONS

---

- Introduction of the new, more regular fictitious domain approach, in which the singularity of a solution is shifted away from the original domain.



# CONCLUSIONS

---

- Introduction of the new, more regular fictitious domain approach, in which the singularity of a solution is shifted away from the original domain.
- The same rate of convergence as a classical finite element method.



# CONCLUSIONS

---

- Introduction of the new, more regular fictitious domain approach, in which the singularity of a solution is shifted away from the original domain.
- The same rate of convergence as a classical finite element method.
- Presentation of the efficient method for solving non-symmetric singular systems of saddle point type arising from its discretization.





# CONCLUSIONS

---

- Introduction of the new, more regular fictitious domain approach, in which the singularity of a solution is shifted away from the original domain.
- The same rate of convergence as a classical finite element method.
- Presentation of the efficient method for solving non-symmetric singular systems of saddle point type arising from its discretization.
- This procedure is easy to implement and converges fast as demonstrated by numerical examples.



# CONCLUSIONS

---

- Introduction of the new, more regular fictitious domain approach, in which the singularity of a solution is shifted away from the original domain.
- The same rate of convergence as a classical finite element method.
- Presentation of the efficient method for solving non-symmetric singular systems of saddle point type arising from its discretization.
- This procedure is easy to implement and converges fast as demonstrated by numerical examples.
- Accelerations of the BiCGSTAB iterations using a hierarchical multigrid scheme.



---

Thanks for your attention.