Fictitious domain approach for the numerical realization of PDEs with stochastic data and geometry

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Outline

1. Problem setting
2. Fictitious domain method
3. Stochastic FD formulation
4. Polynomial chaos
5. Deterministic formulation of the stochastic FD problem
6. Discretization of the deterministic formulation
7. Numerical examples
8. Conclusions

Key ideas

1. suitable combination of fictitious domain approach and polynomial chaos;
2. adding one or more stochastic variables to the deterministic PDE, as an alternative to sampling techniques;
3. stochastic data in PDE are decomposed using fully orthonormal polynomial basis into separable deterministic and stochastic components.

Setting of the problem

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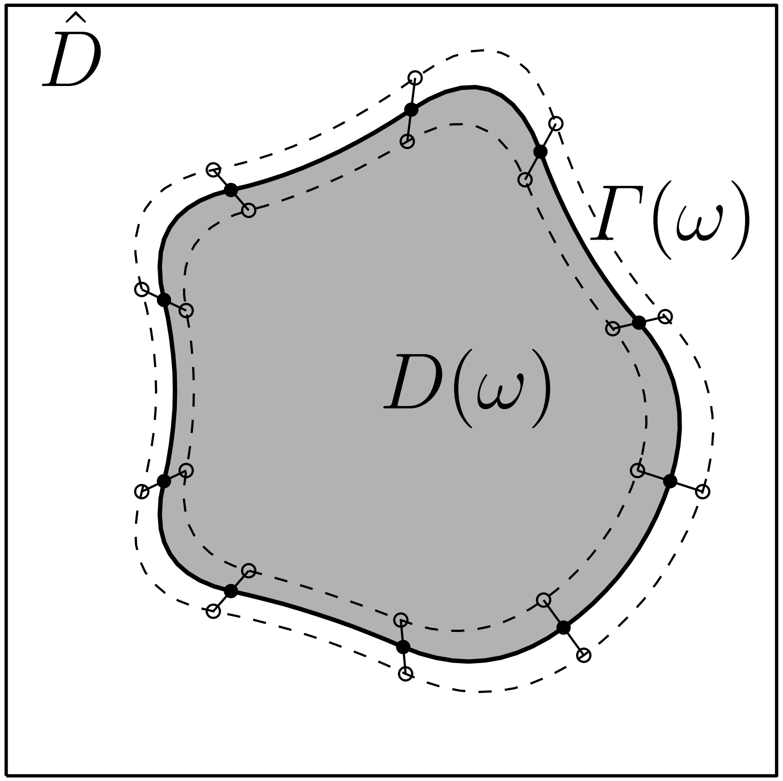
$\hat{D}$ ... fictitious domain containing all domains with their boundaries.

Geometry of the problem



Model problem in $D(\omega)$

Find $u : \overline{D(\omega)} \times \Omega \to \mathbb{R}$ such that almost certainly (a.c.) in Ω we have

$$\left\{ \begin{array}{rcl} -\triangle u(\,\cdot\,,\omega) & = & f \;\text{ in } D(\omega), \\ u(\,\cdot\,,\omega) & = & 0 \;\text{ on } \Gamma(\omega), \end{array} \right. \qquad (\mathcal{P}(\omega))$$

f ... a given function in $L^2(\hat{D})$.

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A fictitious domain method with nonfitted meshes avoids step (i) and partially step (ii).

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$$\left\{\begin{array}{l} \text{Find } (\hat{u}, \lambda) \in V \times M \text{ such that} \\ \displaystyle\int_{\hat{D}} \nabla \hat{u} \cdot \nabla v \, dx + \langle \lambda, \tau v \rangle_\Gamma = \int_{\hat{D}} f v \, dx \quad \forall v \in V, \\ \langle \mu, \tau \hat{u} \rangle_\Gamma = 0 \quad \forall \mu \in M. \end{array}\right. \qquad (\hat{\mathcal{P}})$$

$\langle ., . \rangle$ ... the duality between $M := H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$,

$\tau : \; H_0^1(\hat{D}) \to H^{1/2}(\Gamma)$ ... the trace mapping,

V ... a closed subspace of $H^1(\hat{D})$.

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$\hat{u}\big|_D$ **solves** $(\mathcal{P})$

by Lagrange multipliers.

Theoretical background of the BLM method

- F. Brezzi and M. Fortin: *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.
- V. Girault and R. Glowinski: *Error analysis of a fictitious domain method applied to a Dirichlet problem*, Japan J. Indust. Appl. Math., 12 (1995).
- J. Haslinger and R.A.E. Mäkinen: *Introduction to Shape Optimization, Theory, Approximation, and Computation*, SIAM, Philadelphia, 2003.

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Theorem.

$(\hat{\mathcal{P}})$ has a unique solution $(\hat{u}, \lambda) \in V \times M$, $\hat{u}_{|_D} = u$, $\lambda = \left[\frac{\partial \hat{u}}{\partial n}\right]$.

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$V, M \implies V_h, M_H$ **by mixed FEM**

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With any $\mathcal{R}_h$ we associate

$$V_h = \{\hat{v}_h \in C(\overline{\hat{D}}) \mid \quad \hat{v}_h\big|_R \in Q_1(R) \;\; \forall R \in \mathcal{R}_h,$$
$$+b.c. \text{ on } \partial\hat{D}\},$$

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$$\hat{u}_h = \sum_{i=1}^{N_h} u_i \varphi_i .$$

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The approximation of $(\hat{\mathcal{P}})$

$$
\left\{
\begin{array}{l}
\text{Find } (\hat{u}_h, \lambda_H) \in V_h \times M_H \text{ such that} \\
\displaystyle\int_{\hat{D}} \nabla \hat{u}_h \cdot \nabla \hat{v}_h \, dx + \int_{\Gamma} \lambda_H \hat{v}_h \, ds = \int_{\hat{D}} f \hat{v}_h \, dx \\
\hfill \forall \hat{v}_h \in V_h, \\
\displaystyle\int_{\Gamma} \mu_H \hat{u}_h \, ds = 0 \quad \forall \mu_H \in M_H.
\end{array}
\right.
\qquad (\hat{\mathcal{P}}_h^H)
$$

The convergence of solutions of $(\hat{\mathcal{P}}_h^H)$ to $(\hat{\mathcal{P}})$

Ladyzhenskaja-Babuška-Brezzi condition:

$$\sup_{\hat{v}_h \in V_h} \frac{\displaystyle\int_\Gamma \mu_H \hat{v}_h \, ds}{\|\hat{v}_h\|_{1,\hat{D}}} \geq \overline{\beta} \|\mu_H\|_{-1/2,\Gamma} \quad \forall \mu_H \in M_H,$$

$\overline{\beta} > 0$ independent of h and H.

For triangulation: $3 \leq H/h \leq L$, $L > 0$.

Theorem

Let the ratio H/h is sufficiently large and let $(\hat{u}, \lambda)$ be a solution to $(\hat{\mathcal{P}})$. Then there exists a unique solution $(\hat{u}_h, \lambda_H)$ to $(\hat{\mathcal{P}}_h^H)$ and

$$\|\hat{u} - \hat{u}_h\|_{1,\hat{D}} + \|\lambda - \lambda_H\|_{-1/2,\Gamma} = \mathcal{O}(h^{1/2-\varepsilon}), \quad h \to 0+,$$

holds for any $\varepsilon > 0$ provided that the restriction of $\hat{u}_h$ to D belongs to $H^2(D)$.

The matrix formulation of $\left(\hat{\mathcal{P}}_h^H\right)$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & \mathbb{O} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \qquad (\mathbf{P})$$

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$\mathbb{A}$ ... the stiffness matrix,

$\mathbb{B}$ ... the matrix coupling the primal and dual variables,

$\mathbf{u}, \boldsymbol{\lambda}$ ... the vectors of the nodal values of $\hat{u}_h$ and of the constant values of λ_H,

$\mathbf{f}$ ... the load vector.

Solving (P)

Eliminate the vector $\mathbf{u}$ and solve

$$\mathbb{B}\mathbb{A}^{-1}\mathbb{B}^T\boldsymbol{\lambda} = \mathbb{B}\mathbb{A}^{-1}\mathbf{f}$$

by *a conjugate gradient method*:

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- the multiplication by $\mathbb{A}^{-1}$: *Choleski factorization* with SYMAMD, *multigrid* approach, *domain decomposition* methods, *Fourier and cyclic reduction* methods;
- *efficient preconditioners* based on orthonormalization of $\mathbb{B}$ or on the pseudoinverse to the Schur complement.

Stochastic FD formulation

For any $\omega \in \Omega$, find $\hat{u}(\,\cdot\,,\omega) \in H_0^1(\hat{D})$ and $\lambda(\,\cdot\,,\omega) \in M(\omega) := H^{-1/2}(\Gamma(\omega))$ such that, a. c. in Ω,

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$\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathcal{P}(\omega)$ ... the expected value of a random variable X.

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$L^2(\Omega, d\mathcal{P};\ H_0^1(\hat{D}))$ and $L^2(\Omega, d\mathcal{P};\ H^{-1/2}(\Gamma_0))$ the spaces of random variables with finite second order moments.

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$$F(x,y) = \sum_{k=0}^{\infty} F_k(x)\psi_k(y) \simeq \sum_{k=0}^{N_{pc}} F_k(x)\psi_k(y) = F^{N_{pc}}(x,y),$$

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Mean value and variance

$$F_{mean}(x) := F_0(x), \quad F_{var}(x) := \sum_{k=1}^{\infty} F_k^2(x)$$

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The discrete random space

$$F^{N_{pc}}(x,y)\in\mathbb{P}_{N_{pc}}(I;H),\qquad \text{span}\,\{\psi_k(y)\}_{k=0}^{N_{pc}}$$

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Next step

Transforming the stochastic problem into a purely deterministic one. This will be accomplished by expanding the random variables into polynomial chaos.

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$$\hat{u}(x,y) \simeq \hat{u}^{N_{pc}}(x,y) = \sum_{k=0}^{N_{pc}} \hat{u}_k(x)\psi_k(y).$$

$$\lambda(s,y) \simeq \lambda^{N_{pc}}(s,y) = \sum_{k=0}^{N_{pc}} \lambda_k(s)\psi_k(y).$$

Discretization of the stochastic space

$$\left\{\begin{array}{l} \text{Find } (\hat{u},\lambda) \in L^2_\rho(I;H^1_0(\hat{D})) \times L^2_\rho(I;H^{-1/2}(\gamma(y))) \text{ s.t.} \\ \mathbb{E}\left[\int_{\hat{D}} \nabla\hat{u}\cdot\nabla v\,dx\right] + \mathbb{E}\left[\langle\lambda,\tau v\rangle_{\gamma(y)}\right] = \mathbb{E}\left[\int_{\hat{D}} f v\,dx\right] \\ \hfill \forall v \in L^2_\rho(I;H^1_0(\hat{D})), \\ \mathbb{E}\left[\langle\tau\hat{u},\mu\rangle_{\gamma(y)}\right] = 0 \quad \forall\mu \in L^2_\rho(I;H^{-1/2}(\gamma(y))). \end{array}\right. \qquad \left[\,\overline{\mathcal{P}}\,\right]$$

- Substitute the expansions $\hat{u}^{N_{pc}}$ and $\lambda^{N_{pc}}$ to $\left[\,\overline{\mathcal{P}}\,\right]$,

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- Substitute the expansions $\hat{u}^{N_{pc}}$ and $\lambda^{N_{pc}}$ to $\left[\,\overline{\mathcal{P}}\,\right]$,
- choose successively for $l = 0, \dots, N_{pc}$ the test functions v, μ:

$$v(x,y) = \overline{v}(x)\psi_l(y), \quad \overline{v} \in H^1_0(\hat{D}),$$

$$\mu(s,y) = \overline{\mu}(s)\psi_l(y), \quad \overline{\mu} \in H^{-1/2}(\gamma(y)).$$

Discretization of the stochastic space ($\psi_0(y) = 1$)

$$\mathbb{E}\left[\int_{\hat{D}} \nabla \hat{u}^{N_{pc}} \cdot \nabla v \, dx\right] = \sum_{k=0}^{N_{pc}} \int_{\hat{D}} \nabla u_k \cdot \nabla \overline{v} \, dx \int_I \psi_k(y)\psi_l(y)\rho(y) \, dy$$

$$= \int_{\hat{D}} \nabla u_l \cdot \nabla \overline{v} \, dx,$$

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$$= \int_{\hat{D}} \nabla u_l \cdot \nabla \overline{v} \, dx,$$

$$\mathbb{E}\left[\int_{\hat{D}} f v \, dx\right] = \int_{\hat{D}} f \overline{v} \, dx \int_I 1 \psi_l(y)\rho(y)\, dy = \delta_{0\,l} \int_{\hat{D}} f \overline{v} \, dx,$$

Discretization of the stochastic space ($\psi_0(y) = 1$)

$$\mathbb{E}\left[\int_{\hat{D}} \nabla \hat{u}^{N_{pc}} \cdot \nabla v \, dx\right] = \sum_{k=0}^{N_{pc}} \int_{\hat{D}} \nabla u_k \cdot \nabla \overline{v} \, dx \int_I \psi_k(y)\psi_l(y)\rho(y)\, dy$$
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$$\mathbb{E}\left[\langle \hat{u}^{N_{pc}}, \mu \rangle_{\gamma(y)}\right] = \sum_{k=0}^{N_{pc}} \int_I \langle \tau u_k, \overline{\mu} \rangle_{\gamma(y)} \psi_k(y)\psi_l(y)\rho(y)\, dy.$$

The approximation to $\left[\,\overline{\mathcal{P}}\,\right]$

Problem $\left[\,\overline{\mathcal{P}}^{N}\,\right]$

$$
\left\{
\begin{array}{l}
\text{Find } (\hat{u}^{N_{pc}}, \lambda^{N_{pc}}) \in \mathbb{P}_{N_{pc}}(I; H_0^1(\hat{D})) \times \mathbb{P}_{N_{pc}}(I; H^{-1/2}(\gamma(y))) \text{ s.t.} \\[2ex]
\displaystyle\int_{\hat{D}} \nabla u_l \cdot \nabla \overline{v}\, dx + \sum_{k=0}^{N_{pc}} \int_I \langle \lambda_k, \tau \overline{v} \rangle_{\gamma(y)} \psi_k(y) \psi_l(y) \rho(y)\, dy \\[2ex]
\displaystyle\qquad\qquad = \delta_{0\,l} \int_{\hat{D}} f \overline{v}\, dx \quad \forall \overline{v} \in H_0^1(\hat{D}),\ l = 0, \ldots, N_{pc}, \\[2ex]
\displaystyle\sum_{k=0}^{N_{pc}} \int_I \langle \tau u_k, \overline{\mu} \rangle_{\gamma(y)} \psi_k(y) \psi_l(y) \rho(y)\, dy = 0 \\[2ex]
\displaystyle\qquad\qquad\qquad\qquad \forall \overline{\mu} \in H^{-1/2}(\gamma(y)),\ l = 0, \ldots, N_{pc}.
\end{array}
\right.
$$

The matrix formulation of $[\,\overline{\mathcal{P}}^{N}_{h,H}\,]$

$$\begin{pmatrix} \tilde{\mathbb{A}} & \tilde{\mathbb{B}}^T \\ \tilde{\mathbb{B}} & \tilde{\mathbb{O}} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}} \\ \tilde{\boldsymbol{\lambda}} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}} \\ \tilde{\mathbf{o}} \end{pmatrix}, \qquad (\tilde{\mathbf{P}})$$

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$$\tilde{\mathbb{A}} = \begin{pmatrix} \mathbb{A} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \mathbb{A} & \cdots & \mathbb{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{A} \end{pmatrix}, \ \tilde{\mathbb{B}} = \begin{pmatrix} \mathbb{B}_{00} & \mathbb{B}_{01} & \cdots & \mathbb{B}_{0N_{pc}} \\ \mathbb{B}_{10} & \mathbb{B}_{11} & \cdots & \mathbb{B}_{1N_{pc}} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbb{B}_{N_{pc}0} & \mathbb{B}_{N_{pc}1} & \cdots & \mathbb{B}_{N_{pc}N_{pc}} \end{pmatrix},$$

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$$\tilde{\mathbf{u}} = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \dots \\ \mathbf{u}_{N_{pc}} \end{pmatrix}, \quad \tilde{\boldsymbol{\lambda}} = \begin{pmatrix} \boldsymbol{\lambda}_0 \\ \boldsymbol{\lambda}_1 \\ \dots \\ \boldsymbol{\lambda}_{N_{pc}} \end{pmatrix}, \quad \tilde{\mathbf{f}} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \\ \dots \\ \mathbf{0} \end{pmatrix}.$$

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Computing the elements of $\mathbb{A}$, f and $\tilde{\mathbb{B}}$

$$a_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx, \quad f_j = \int_\Omega f\varphi_j \, dx, \quad i, j = 1, \dots, N_h,$$

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$$\tilde{\mathbb{B}} = [\mathbb{B}_{kl}]_{rj} = \int\limits_{I} \Big[\int\limits_{\gamma_r(y)} \varphi_j \, ds \Big] \psi_k(y)\psi_l(y)\rho(y) \, dy, \quad r = 1,\dots,N_H.$$

Remarks

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- the same methods to realize $(\tilde{\mathbf{P}})$,
- $\text{cond}\,(\tilde{\mathbb{A}})$ does not depend on m and n,
- the multiplication by $\tilde{\mathbb{B}}$, $\tilde{\mathbb{B}}^T$ and $\tilde{\mathbb{A}}^{-1}$ can be done *in parallel*.

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$\hat{D} := (0,1) \times (0,1)$ ... the fictitious domain,

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- $x = (x_1, x_2)$ is the deterministic variable,
- the radius $y \in I$ is the stochastic variable with the normal distribution $\rho(y)$ on I.

Example 1

$$\left\{ \begin{aligned} -\triangle u(x,y) &= f(x) \ \text{ in } D(y), \\ u(x,y) &= 0 \ \text{ on } \gamma(y), \end{aligned} \right. \qquad \big(\mathcal{P}(y)\big)$$

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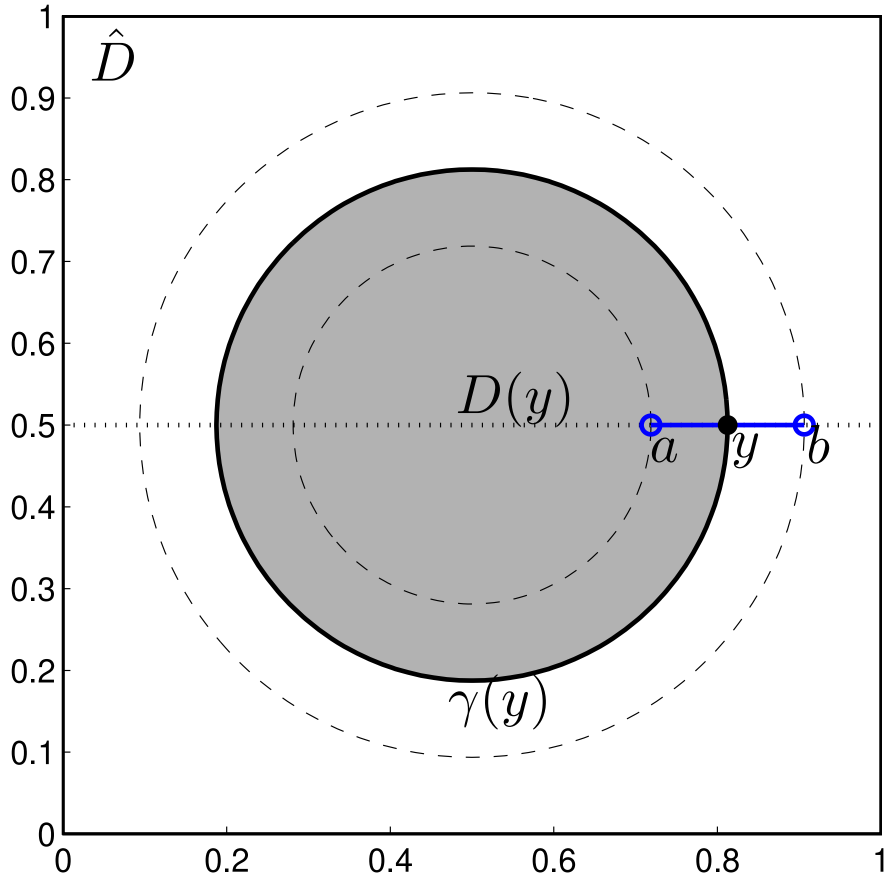
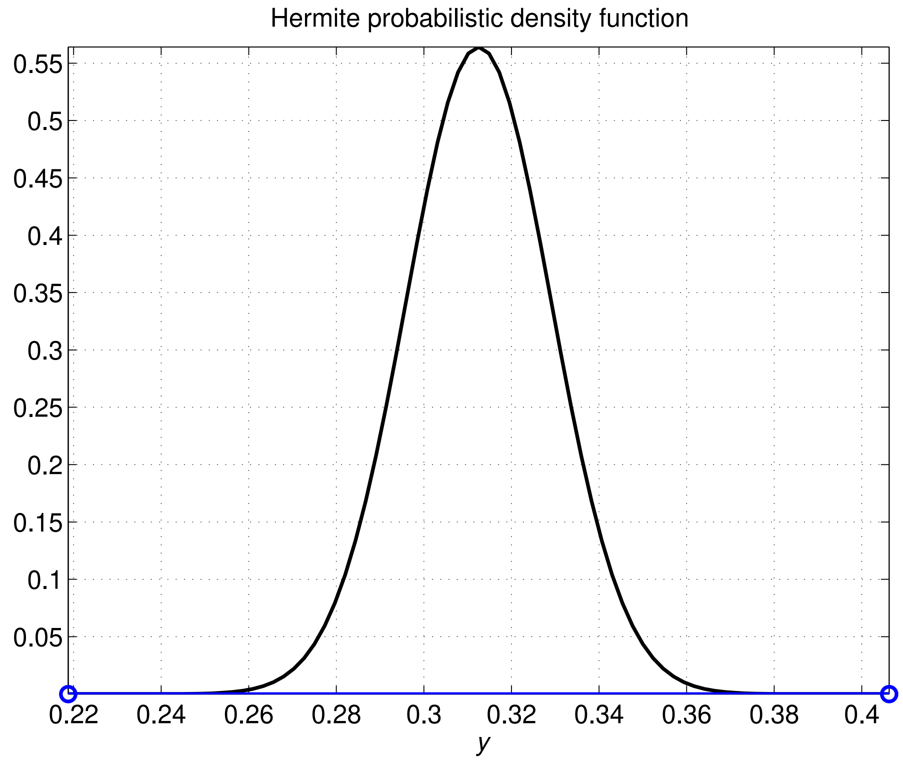
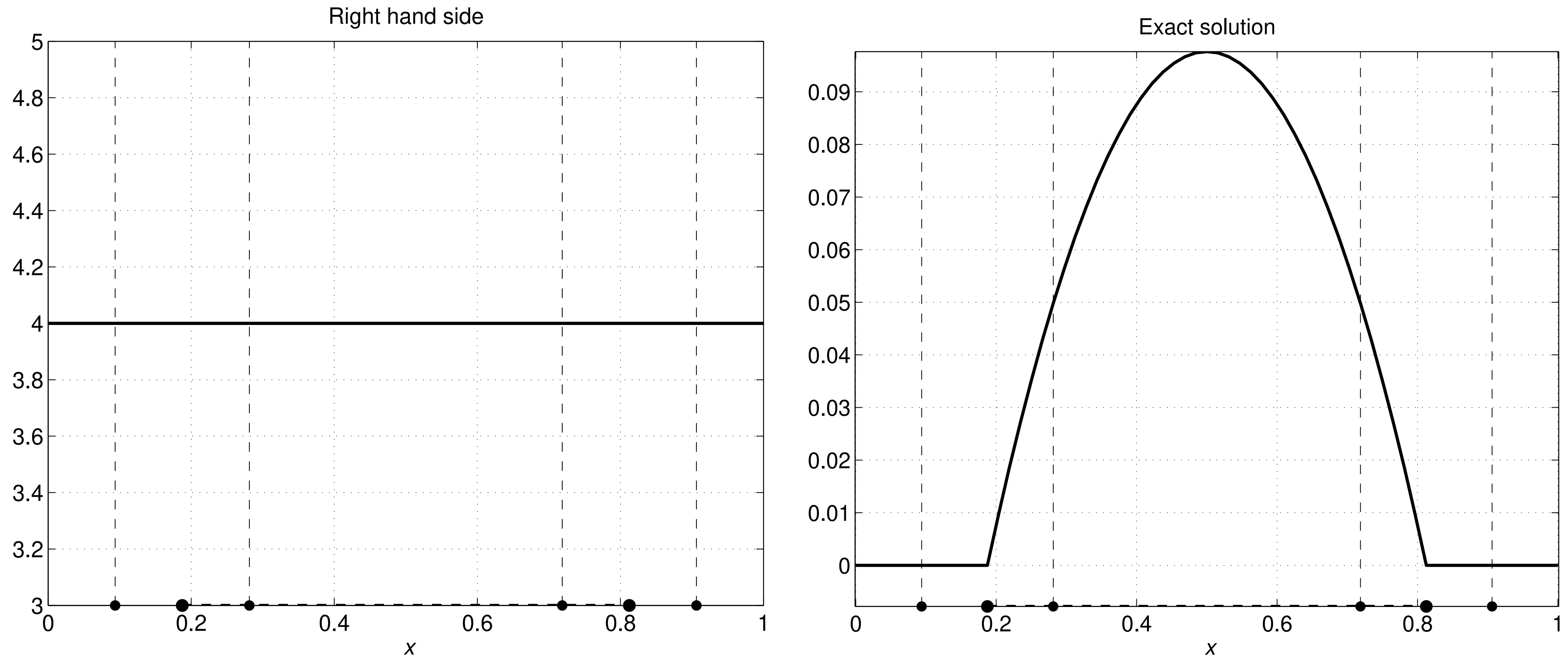
$$f = -\triangle u_{ex} \text{ and } u_{ex}(x,y) = y^2 - (x_1 - \frac{1}{2})^2 - (x_2 - \frac{1}{2})^2.$$

Geometry of $D(y)$ and density function $\rho(y)$

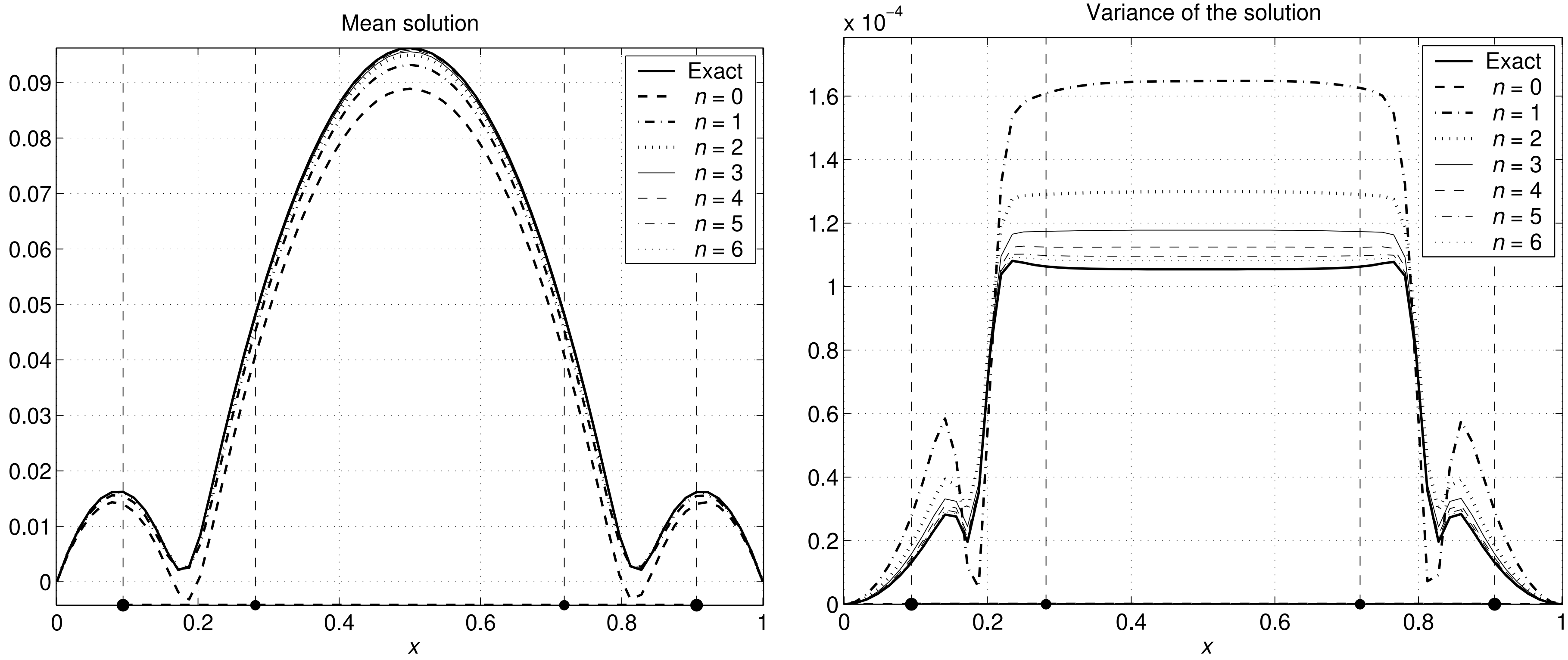




Right hand side f and exact solution $u_{ex}(x, \overline{y})$



Mean solution and its variance

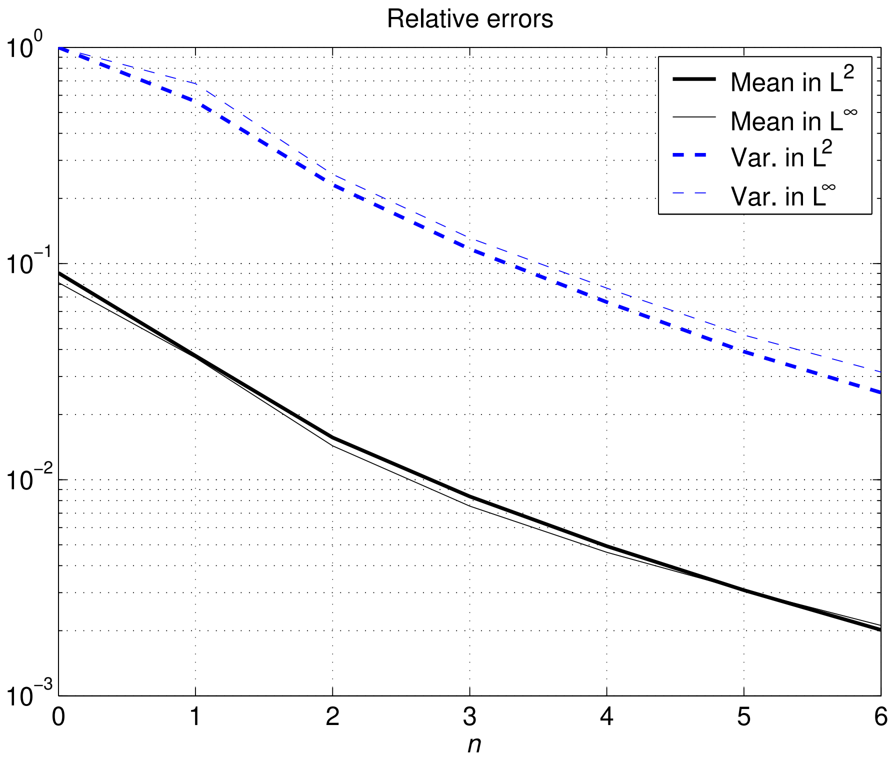


Relative errors with respect to n

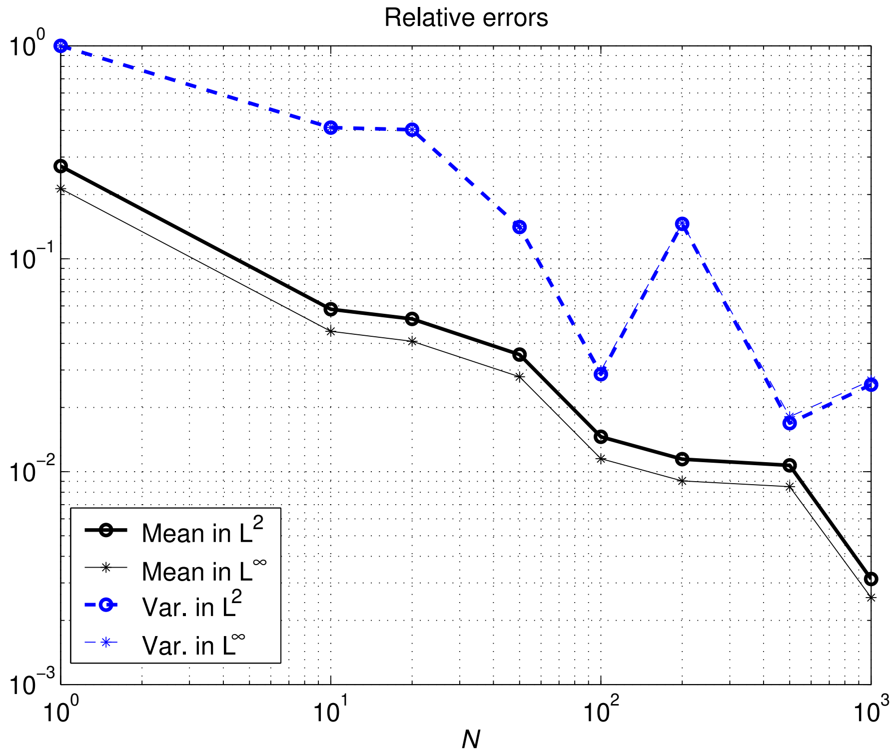
$$\delta^{mean}_{L^2(\tilde{D})} = \frac{\|\overline{u} - \overline{u}_{ex}\|_{L^2(\tilde{D})}}{\|\overline{u}_{ex}\|_{L^2(\tilde{D})}}, \qquad \delta^{mean}_{L^\infty(\tilde{D})} = \frac{\|\overline{u} - \overline{u}_{ex}\|_{L^\infty(\tilde{D})}}{\|\overline{u}_{ex}\|_{L^\infty(\tilde{D})}}$$

$$\delta^{var}_{L^2(\tilde{D})} = \frac{\|\sigma^2_u - \sigma^2_{u_{ex}}\|_{L^2(\tilde{D})}}{\|\sigma^2_{u_{ex}}\|_{L^2(\tilde{D})}}, \qquad \delta^{var}_{L^\infty(\tilde{D})} = \frac{\|\sigma^2_u - \sigma^2_{u_{ex}}\|_{L^\infty(\tilde{D})}}{\|\sigma^2_{u_{ex}}\|_{L^\infty(\tilde{D})}}$$

Relative errors behavior (comparison)



Polynomial Chaos



Monte Carlo

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$$\left\{ \begin{array}{rcl} -\triangle u(x,y) & = & 60 \;\text{ in } D(y), \\ u(x,y) & = & g \;\text{ on } \gamma(y), \end{array} \right. \qquad (\mathcal{P}(y))$$

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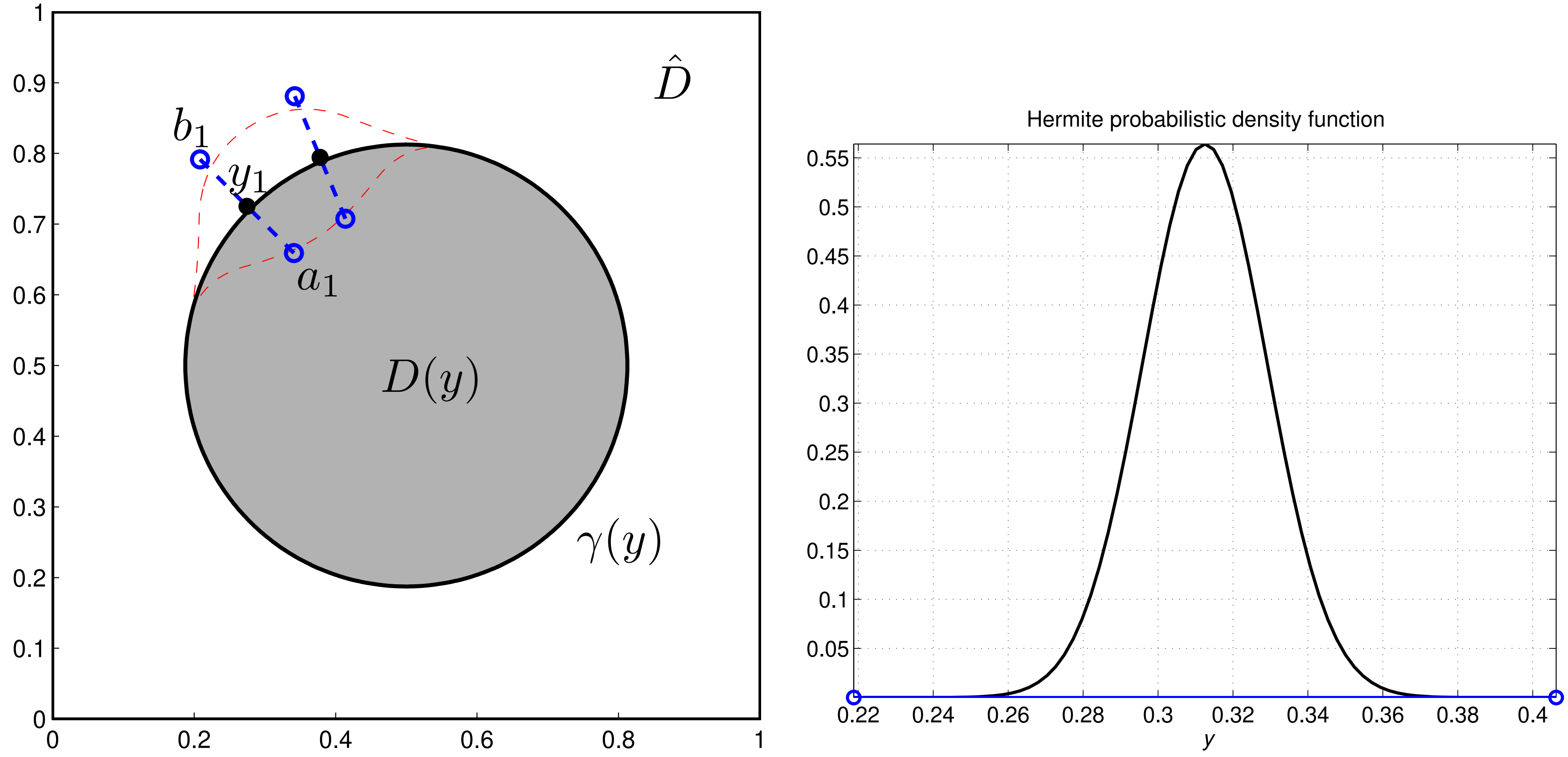
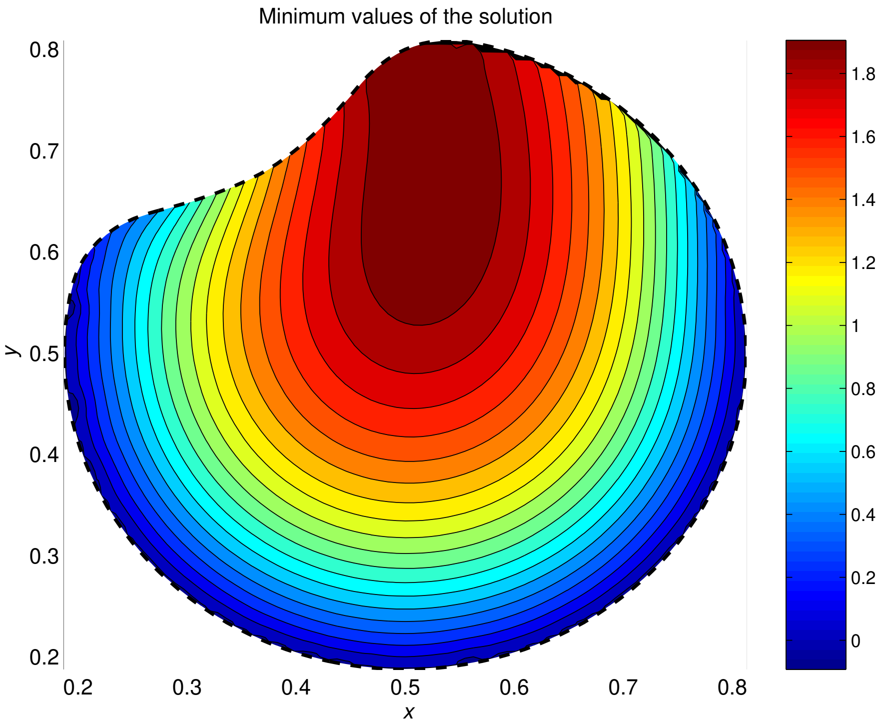
$$\begin{cases} -\triangle u(x,y) &= 60 \ \text{ in } D(y), \\ u(x,y) &= g \ \text{ on } \gamma(y), \end{cases} \qquad (\mathcal{P}(y))$$

$$g(\varphi) = \begin{cases} 0, & \varphi \in < -\pi, 0 >, \\ 1 - \cos(2\varphi), & \varphi \in (0,\pi). \end{cases}$$

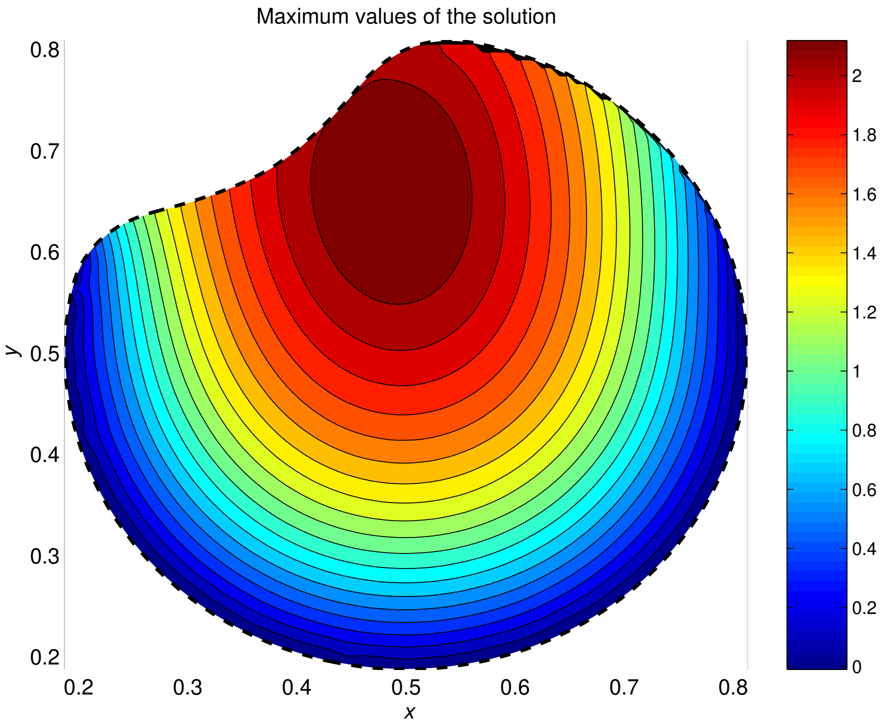
Geometry of $D(y)$ and density function $\rho(y)$



Minimum and maximum values of the solution

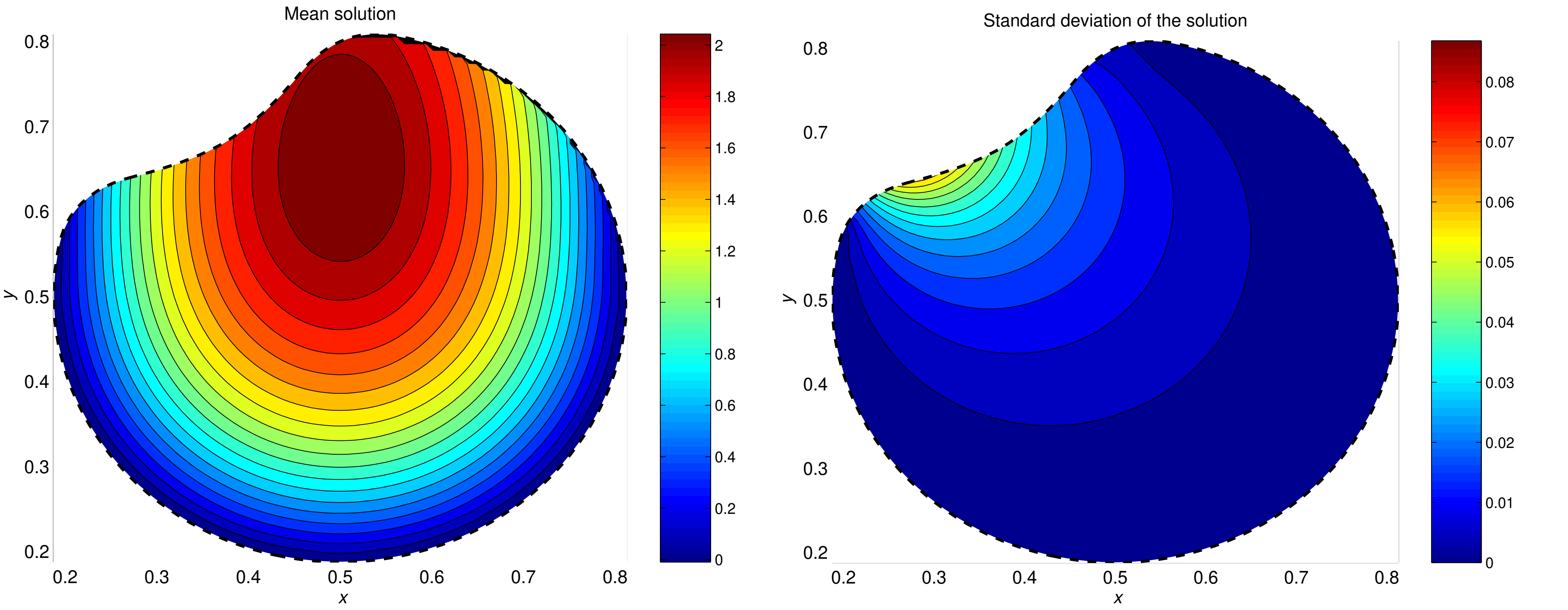


Minimum values



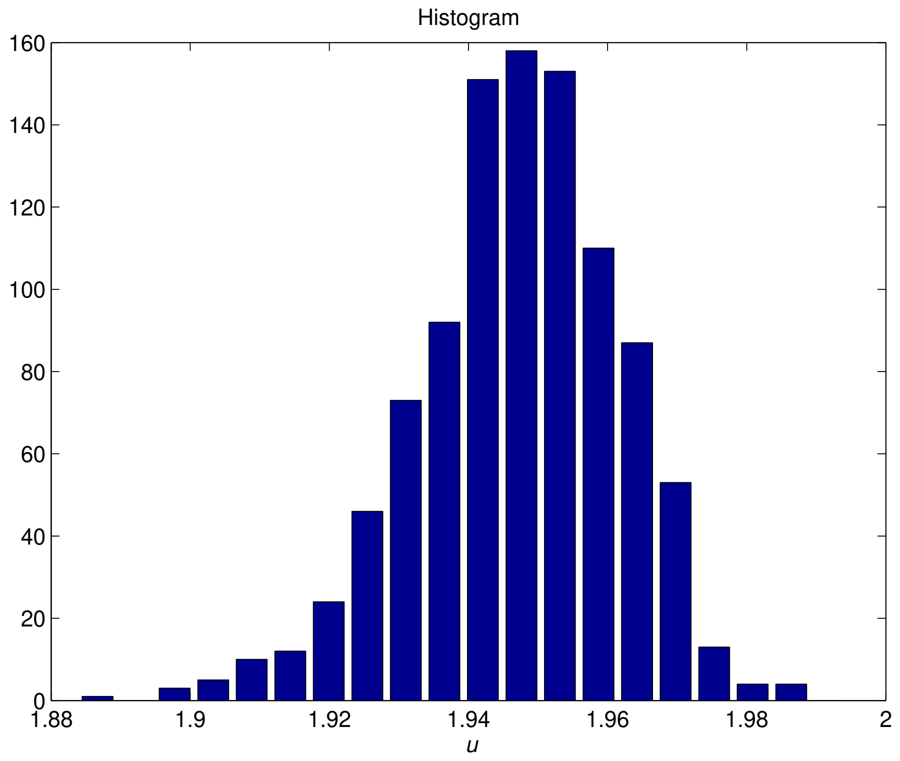
Maximum values

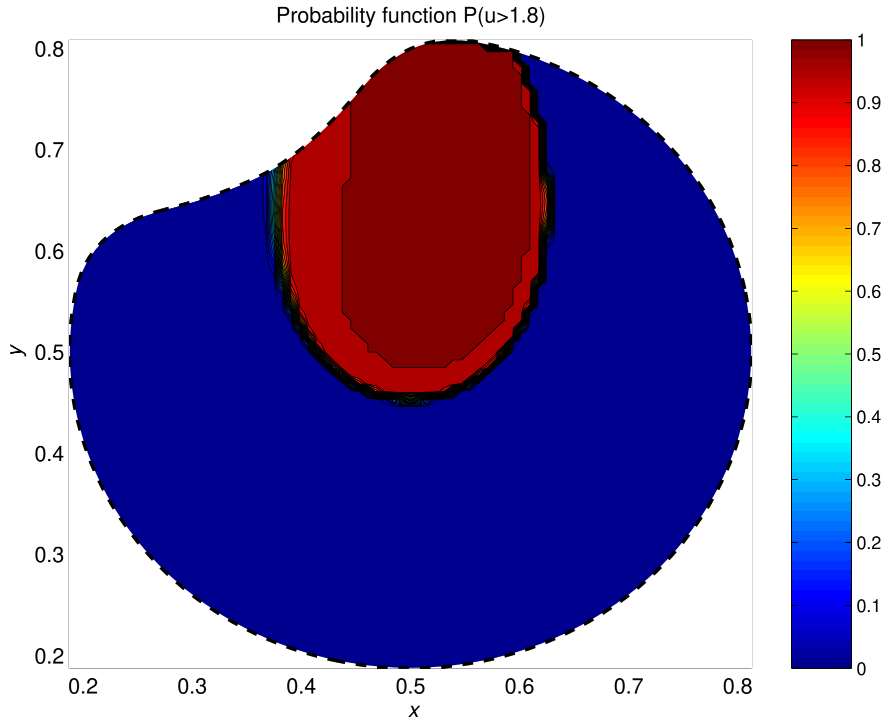
Minimum and maximum values of the solution



Mean solution Standard deviation

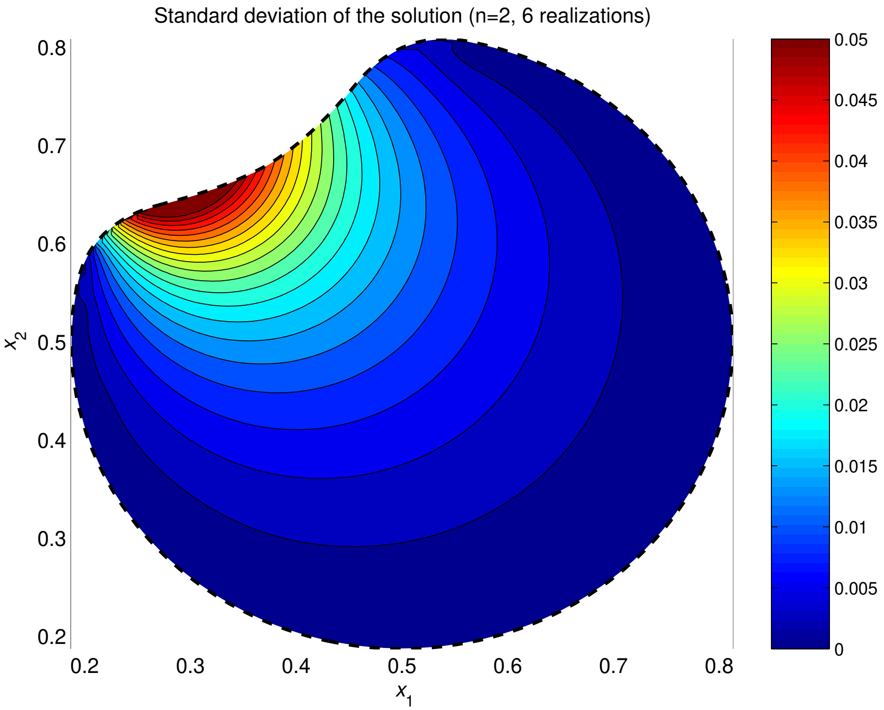
Other statistics

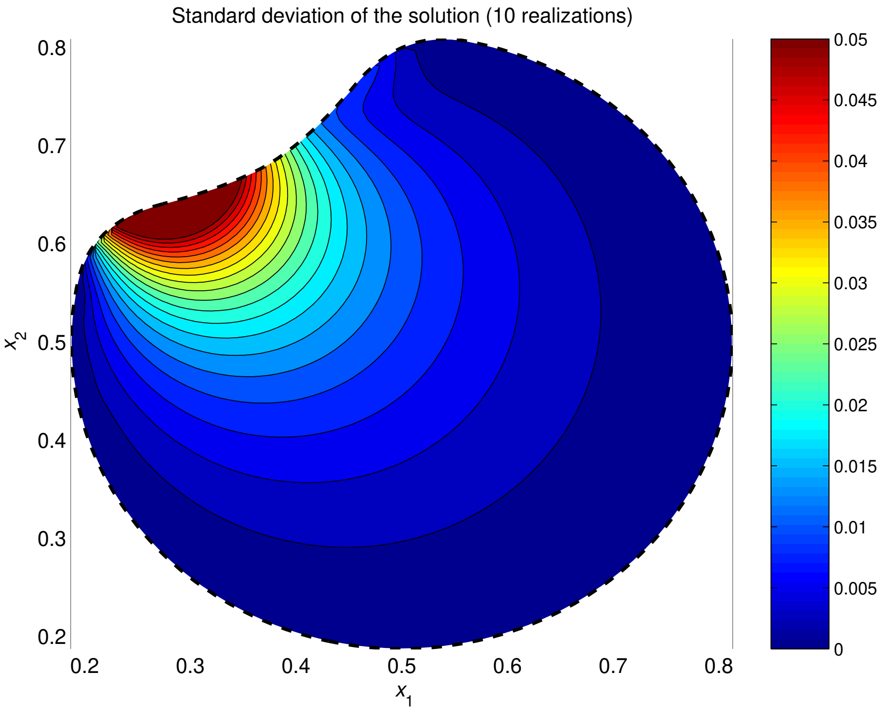




Histogram of the solution in the center   Probability function $P_{u>1.8}$

Standard deviation of the solution (comparison)

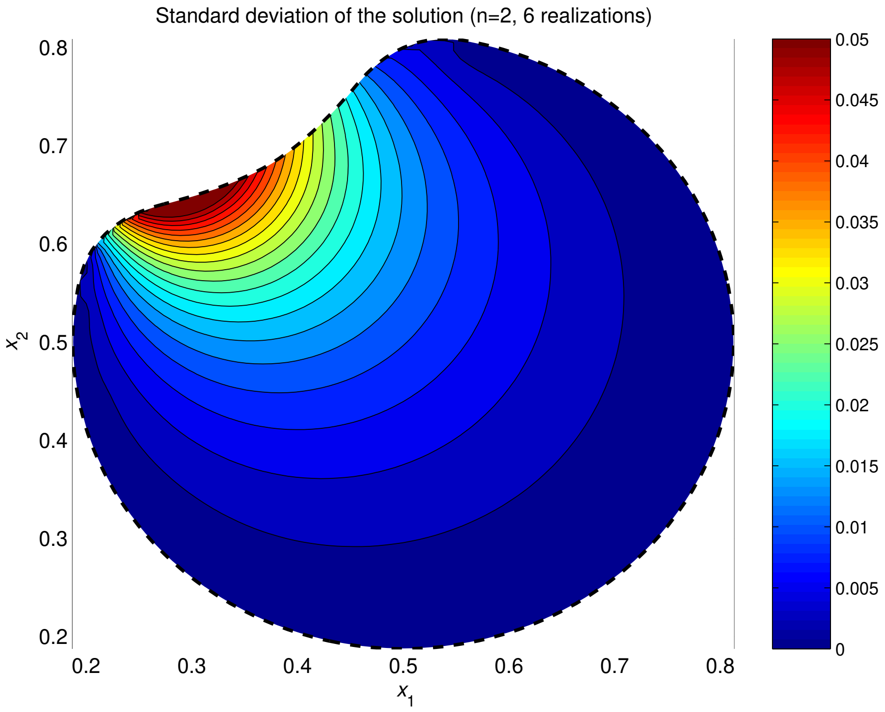


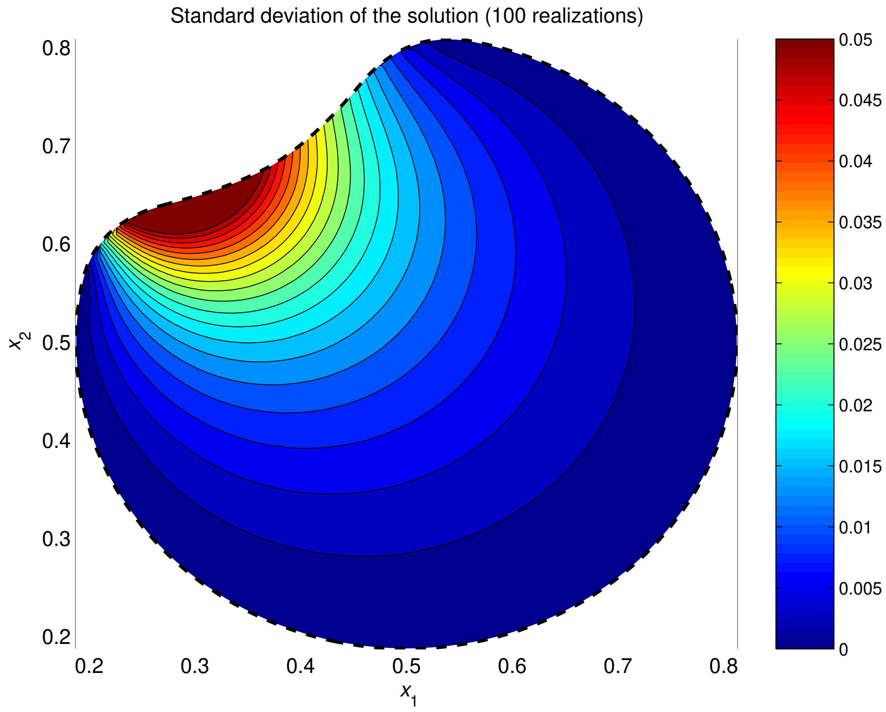


PC order = 2 $\sim$ 6 realizations

10 MC realizations

Standard deviation of the solution (comparison)





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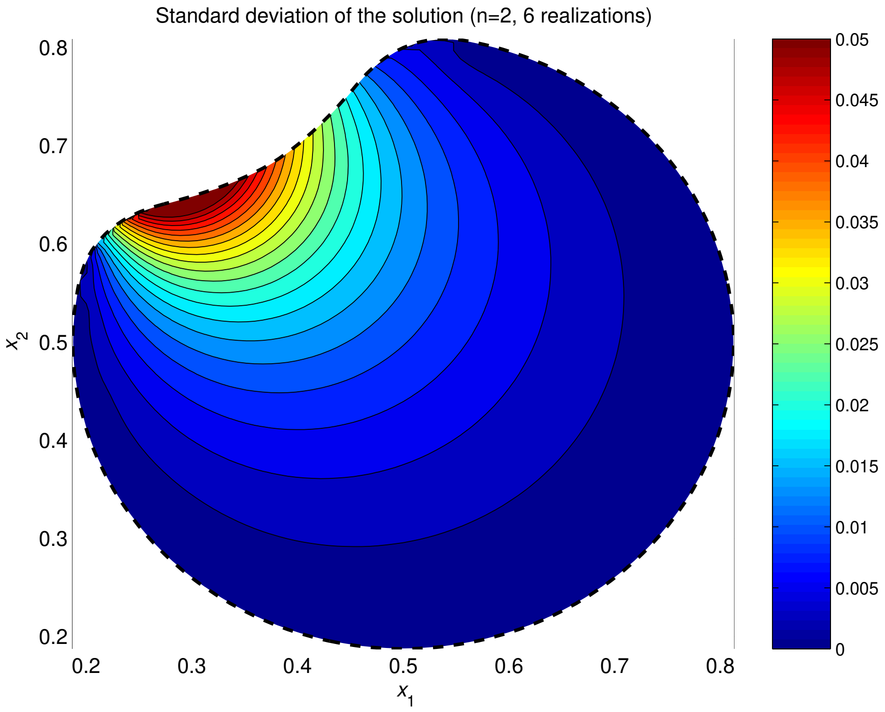
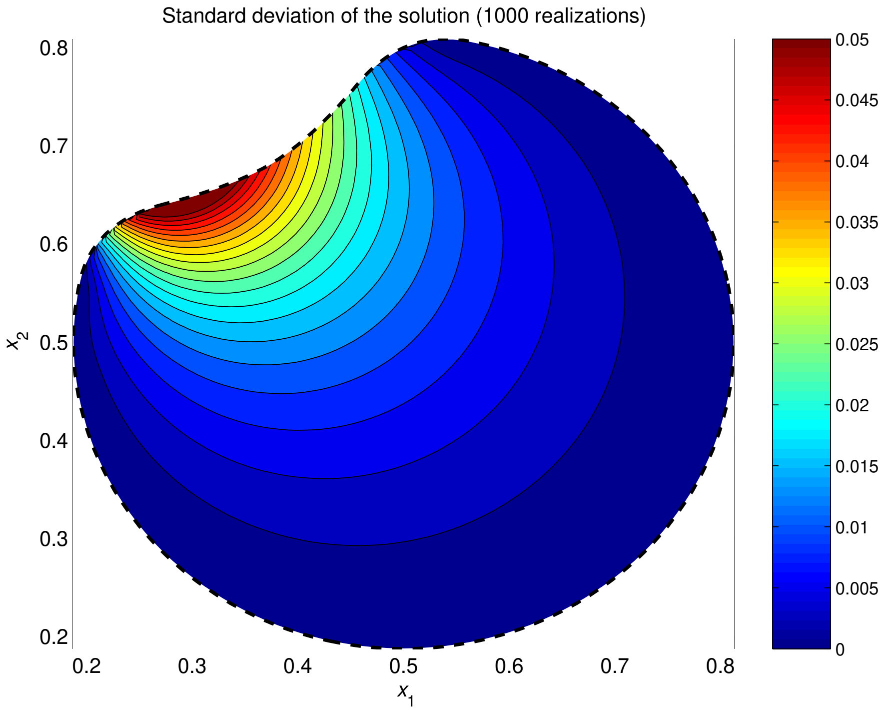
100 MC realizations

Standard deviation of the solution (comparison)





PC order = 2 $\sim$ 6 realizations   1000 MC realizations

More details

1. T. Kozubek and C. Canuto: *Fictitious domain approach for the numerical realization of PDEs with stochastic data and geometry*, will be sent to Numerische Mathematik.