Multigrid Preconditioned Augmented Lagrangians for the Stokes Problem

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Outline

- Setting of the saddle–point problem
- Optimal Uzawa–type methods
- Semi–monotonic augmented Lagrangian method
- Multigrid preconditioning
- An application to the Stokes problem, numerical results
- Outlook: Applications in topology optimization

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Setting of the Saddle–Point Problem

Saddle–point problem

V, Z ... Hilbert spaces,
$(.,.)_V$, $(.,.)_Z$ ... inner products, $\langle .,. \rangle_V$, $\langle .,. \rangle_Z$ ... duality pairings:

$$\min_{u \in V} h(u) \quad \text{s.t.} \;\; Bu = 0,$$

where $h(u) := \frac{1}{2}\langle Au, u\rangle_V - \langle b, u\rangle_V$, $A : V \longrightarrow V'$ linear bounded self–adjoint and positive definite, $B : V \longrightarrow Z'$ linear and bounded, $b \in V'$.

Lagrange formalism

$p \in Z$ ... Lagrange multiplier

$$\min_{u \in V} \max_{p \in Z} \{h(u) + \langle Bu, p\rangle_Z\} \quad \text{equivalent to} \quad \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix},$$

where $B^* : Z \longrightarrow V'$ adjoint to B, i.e. $\langle B^* z, v\rangle_V = \langle Bv, z\rangle_Z$.

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Optimal Uzawa–Type Methods

Symmetric Uzawa methods

$\widehat{A^{-1}}$ ... preconditioner to A, $\widehat{C^{-1}}$ ... preconditioner to $C := BA^{-1}B^*$,
α_V, α_Z ... small enough positive damping parameters:

$$
\begin{aligned}
u^{(k+1/2)} &:= u^{(k)} + \alpha_V \widehat{A^{-1}} \left[b - \left(Au^{(k)} + B^* p^{(k)} \right) \right], \\
p^{(k+1)} &:= p^{(k)} + \alpha_Z \widehat{C^{-1}} B u^{(k+1/2)}, \\
u^{(k+1)} &:= u^{(k)} + \alpha_V \widehat{A^{-1}} \left[b - \left(Au^{(k)} + B^* p^{(k+1)} \right) \right].
\end{aligned}
$$

α_V and α_Z often difficult to choose!

Optimal Uzawa–Type Methods

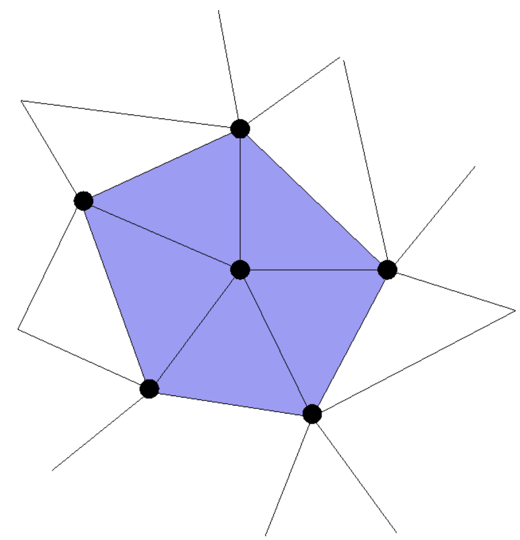
Schöberl & Zulehner, 2003: Additive multigrid smoother

(V_i), (Z_i) ... sequences of FE-subspaces of V and Z, respectively
(P_i), (Q_i) ... the corresponding prolongations, i.e. $P_i : V_i \to V$, $Q_i : Z_i \to Z$

$$u^{(k+1)} := u^{(k)} + \alpha_V \sum_{i=1}^{N} P_i w_i^{(k)},$$

$$p^{(k+1)} := p^{(k)} + \alpha_Z \sum_{i=1}^{N} Q_i r_i^{(k)},$$

where $(w_i^{(k)}, r_i^{(k)})$ is a solution to a local saddle–point subproblem, typically defined around a discret. node:



Equivalent to a symmetric Uzawa method.

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Augmented Lagrangians Methods

Formalism

ρ ... positive augmented Lagrange parameter,
$M : Z \to Z'$ ... inner product on Z, easily invertible!
The k-th iteration:

Step 1. Update $u^{(k)}$ by solving $\min_{u \in V} \left\{ h(u) + \langle Bu, p^{(k)} \rangle_Z + \frac{\rho^{(k)}}{2} \langle B^* M^{-1} Bu, u \rangle_V \right\}$,

Steps 2.,3. Update $p^{(k)}, \rho^{(k)}$.

Denote further:
$L(u, p, \rho) := h(u) + \langle Bu, p \rangle_Z + \frac{\rho}{2} \langle B^* M^{-1} Bu, u \rangle_V$ ... Lagrange functional,
$g(u, p, \rho) := \nabla_u L(u, p, \rho) = Au - b + B^* p + \rho B^* M^{-1} Bu$ ... its Fréchet derivative.

Then, Step 1 is equivalent to solution of the system

$$\left[A + \rho^{(k)} B^* M^{-1} B \right] u^{(k+1)} = b - B^* p^{(k)}.$$

Semi–Monotonic Augmented Lagrangians Method

Dostál, 2005: The algorithm

The k-th iteration:

Find $u^{(k+1)}$: $\|g(u^{(k+1)}, p^{(k)}, \rho^{(k)})\|_{V'} \leq \min \left\{ \nu \sqrt{\langle B^* M^{-1} B u^{(k+1)}, u^{(k+1)} \rangle_V}, \eta \right\}$

if $\|g(u^{(k+1)}, p^{(k)}, \rho^{(k)})\|_{V'} \leq \varepsilon$ and $\sqrt{\langle B^* M^{-1} B u^{(k+1)}, u^{(k+1)} \rangle_V} \leq \varepsilon_{\text{feas}}$ **then**

break

end if

$p^{(k+1)} := p^{(k)} + \rho^{(k)} M^{-1} B u^{(k+1)}$

if $L(u^{(k+1)}, p^{(k+1)}, \rho^{(k)}) < L(u^{(k)}, p^{(k)}, \rho^{(k)}) + \frac{\rho^{(k)}}{2} \langle B^* M^{-1} B u^{(k+1)}, u^{(k+1)} \rangle_V$ **then**

$\rho^{(k+1)} := \beta \rho^{(k)}$

else

$\rho^{(k+1)} := \rho^{(k)}$

end if

$\nu > 0$, $\eta > 0$, $\beta > 1$, $\varepsilon > 0$, $\varepsilon_{\text{feas}} > 0$

Semi–Monotonic Augmented Lagrangians Method

Dostál, 2005: Analysis

1. The number of the outer iterations is bounded:

$$k \leq \frac{(1+s)\eta^2 + \|b\|_{V'}^2}{\lambda \rho^{(0)} \min^2\{\nu\varepsilon, \varepsilon_{\text{feas}}\}} =: k_{\max},$$

where $\lambda > 0 \ldots$ a lower spectral bound of A,
$s > 0 \ldots$ the smallest integer so that $\beta^s \rho^{(0)} \geq \frac{\nu^2}{\lambda}$.

2. The augmented Lagrange parameter is bounded

$$\rho^{(k)} \leq \beta^s \rho^{(0)} =: \rho_{\max}$$

with the same β.

Consequently, we have the uniform spectral equivalence

$$\forall k : A + \rho^{(k)} B^* M^{-1} B \approx A,$$

since $\forall v \in V : \lambda(v,v)_V \leq \left(\|A\|_{V'}^2 + \rho_{\max}\|B^* M^{-1} B\|_{V'}^2\|\right)(v,v)_V$ and $\|v\|_A \approx \|v\|_V$.

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Multigrid Preconditioned Augmented Lagrangians Method

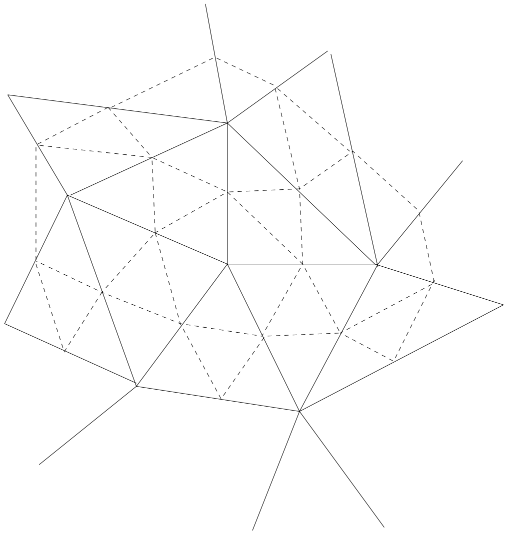
Optimality

Provided multigrid preconditioners M^{-1} and $\widehat{A^{-1}}$ to M (an inner product on Z) and A (an inner product on V), respectively,

the method is of linear complexity!

Multigrid Preconditioned Augmented Lagrangians Method

$\widehat{\mathbf{A}_l^{-1}} \cdot \mathbf{b}_l \equiv \mathbf{u}_l := \mathrm{MG}(l, \mathbf{b}_l)$



if $l = 1$ **then**
  $\mathbf{w}_1 := \mathbf{A}_1^{-1} \cdot \mathbf{b}_1$
else
  $\mathbf{u}_l := \mathbf{0}$
  $\mathbf{u}_l := \mathrm{Presmooth}(\mathbf{b}_l - \mathbf{A}_l \cdot \mathbf{u}_l)$
  $\mathbf{r}_l := \mathbf{b}_l - \mathbf{A}_l \cdot \mathbf{u}_l$
  $\mathbf{r}_{l-1} := \mathrm{Restrict}(\mathbf{r}_l)$
  $\mathbf{w}_{l-1} := \mathrm{MG}(l-1, \mathbf{r}_{l-1})$
  $\mathbf{u}_l := \mathbf{u}_l + \mathrm{Prolong}(\mathbf{w}_{l-1})$
  $\mathbf{u}_l := \mathrm{Postsmooth}(\mathbf{b}_l - \mathbf{A}_l \cdot \mathbf{u}_l)$
end if

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An Application to the Stokes Problem

The Stokes problem

$\Omega \subset \mathbb{R}^2 \ldots$ bounded polygonal computational domain, $\mathbf{f} \in [L^2(\Omega)]^2$

$$\begin{aligned} -\triangle \mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} &\quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} &\ \text{ on } \partial\Omega. \end{aligned}$$

Fitting to our framework

$$V := [H_0^1(\Omega)]^2, \quad Z := L^2(\Omega), \quad \langle A\mathbf{u}, \mathbf{v}\rangle_V := \int_\Omega \sum_{i=1}^{2} \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v_i \, d\mathbf{x},$$

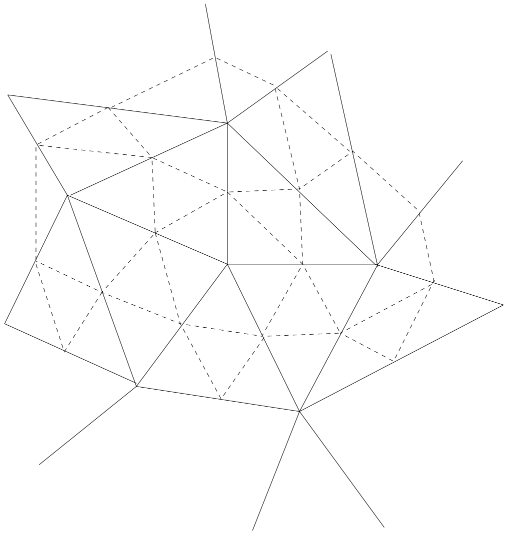
$$\langle b, \mathbf{v}\rangle_V := \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x}, \quad \langle B\mathbf{u}, z\rangle_Z := \int_\Omega \operatorname{div} \mathbf{u} \, z \, d\mathbf{x}.$$

Thus, A is a tensor product Laplacian and M is the L^2–inner product.

An Application to the Stokes Problem

FE–multigrid discretization

$(\mathcal{T}_l)$ . . . a nested sequence of triangulations of Ω



Crouzeix–Raviart elements

A nonnested sequence of FE–spaces:

$$V_l := \Big\{ v \in \big[L^2(\Omega)\big]^2 : v|_T \text{ is linear for all } T \in \mathcal{T}_l, \\ v \text{ is continuous at the midpoints of interelement boundaries} \\ \text{and } v = 0 \text{ along } \partial\Omega \Big\},$$

$$Z_l := \left\{ z \in L^2(\Omega) : z|_T \text{ is constant for all } T \in \mathcal{T}_l \right\}.$$

An Application to the Stokes Problem

Brenner, 1993: Multigrid preconditioning

Smoothers: multilevel diagonal scaling for $\widehat{A_l^{-1}}$, $M_l^{-1} = [\mathrm{diag}(M_l)]^{-1}$

Prolongations: $I_{l-1}^l : V_{l-1} \times Z_{l-1} \longrightarrow V_l \times Z_l$ given by

$$I_{l-1}^l(v, z) := (J_{l-1}^l v, z)$$

with

$$J_{l-1}^l v(m_e) := \begin{cases} v(m_e) & \text{if } m_e \in \mathrm{int}\, T \text{ for some } T \in \mathcal{T}_{l-1} \\ \frac{1}{2}\left[v|_{T_1} + v|_{T_2}\right] & \text{if } e \subset T_1 \cap T_2 \text{ for some } T_1, T_2 \in \mathcal{T}_{l-1} \end{cases}$$

at midpoints m_e of internal edges e in $\mathcal{T}_l$.

An Application to the Stokes Problem

Numerical results

$\Omega := (-1,1) \times (-1,1)$, $\mathbf{f}(x_1, x_2) := \text{sign}(x_1)\,\text{sign}(x_2)\,(1,1)$, random (from [0,1]) initial values of $\mathbf{u}^{(0)}$, $p^{(0)}$

3 pre– and 3 post–smoothing steps, $\varepsilon/\varepsilon^{(0)} = \varepsilon_{\text{feas}}/\varepsilon_{\text{feas}}^{(0)} = 10^{-3}$:

level l	$\dim V_l$	$\dim Z_l$	point additive smoother		block multiplicative smoother	
			outer/PCG **iterations**	**total PCG iterations**	**outer**/PCG **iterations**	**total PCG iterations**
1	56	32	**5**/1,0,1,2,6	**10**	**5**/1,0,1,2,6	**10**
2	208	128	**4**/4,1,5,18	**28**	**5**/1,0,1,2,5	**9**
3	800	512	**4**/5,1,7,21	**34**	**5**/1,0,1,2,6	**10**
4	3136	2048	**4**/6,1,1,7	**15**	**4**/1,0,1,2	**4**
5	12416	8192	**4**/7,1,1,7	**16**	**4**/1,0,1,2	**4**
6	49408	32768	**4**/8,1,1,7	**17**	**3**/1,0,1	**2**

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Outlook: Applications in Topology Optimization

$$\min_{q\in Q} J(q,u) \quad \text{s.t.} \quad A(q,u) = b \text{ on } V'$$

Nested approach

$$q^* := \arg\min_{q\in Q} J(q,u(q)), \text{ where } A(q,u(q)) = b \text{ on } V'$$

The Hessian is dense!

All–at–once approach

$$\min_{(q,u)\in Q\times V} \max_{\lambda\in V} \{J(q,u) + \langle\lambda, A(q,u) - b\rangle\}$$

The Hessian is bigger, but sparse and well–structured!

Outlook: Applications in Topology Optimization

Burger & Mühlhuber, 2002: All–at–once parameter identification

$$\min_{q\in L^2(\Omega)} \frac{1}{2}\|\nabla u - \mathbf{B}_{\text{given}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|q\|^2_{L^2(\Omega)} \qquad \text{s.t.} \qquad -\operatorname{div}(q\nabla u) = f \text{ in } H^1_0(\Omega)'$$

Leads to a sequence of

$$\begin{pmatrix} I & 0 & \text{sym.} \\ 0 & -\triangle & \text{sym.} \\ -\operatorname{div}(\cdot\nabla u^{(k)}) & -\operatorname{div}(q^{(k)}\nabla\cdot) & 0 \end{pmatrix} \begin{pmatrix} \delta q^{(k)} \\ \delta u^{(k)} \\ \delta\lambda^{(k)} \end{pmatrix} = \begin{pmatrix} . \\ . \\ . \end{pmatrix}$$

$$\begin{aligned} q^{(k+1)} &:= q^{(k)} + \delta q^{(k)} \\ u^{(k+1)} &:= u^{(k)} + \delta u^{(k)} \\ \lambda^{(k+1)} &:= \lambda^{(k)} + \delta\lambda^{(k)} \end{aligned}$$

Outlook: Applications in Topology Optimization

All–at–once topology optimization

$$\min_{q \in L^2(\Omega)} \frac{1}{2}\|\nabla u - \mathbf{B}_{\text{given}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|q\|^2_{L^2(\Omega)} \quad \text{s.t.} \quad -\operatorname{div}(\nu(q)\nabla u) = f \text{ in } H^1_0(\Omega)'$$

Leads to a sequence of

$$\begin{pmatrix} I + \langle \lambda^{(k)}, A''_{qq}(q^{(k)}, u^{(k)})(\cdot)\rangle & \text{sym.} & \text{sym.} \\ \langle \lambda^{(k)}, A''_{uq}(q^{(k)}, u^{(k)})(\cdot)\rangle & -\triangle + \langle \lambda^{(k)}, A''_{uu}(q^{(k)}, u^{(k)})(\cdot)\rangle & \text{sym.} \\ -\operatorname{div}(\nu'_q(q^{(k)})(\cdot)\nabla u^{(k)}) & -\operatorname{div}(\nu(q^{(k)})\nabla \cdot) & 0 \end{pmatrix} \begin{pmatrix} \delta q^{(k)} \\ \delta u^{(k)} \\ \delta \lambda^{(k)} \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$\begin{aligned} q^{(k+1)} &:= q^{(k)} + \delta q^{(k)} \\ u^{(k+1)} &:= u^{(k)} + \delta u^{(k)} \\ \lambda^{(k+1)} &:= \lambda^{(k)} + \delta \lambda^{(k)} \end{aligned}$$