A Coupling of Finite Elements and a Boundary Integral Collocation Method for the 3–Dimensional Axisymmetric Magnetostatics

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Outline

- 3–dimensional axisymmetric magnetostatics
- Finite element approach
- Boundary integral equation approach
- Coupling scheme
- Outlook: Galerkin BEM, nonlinearities, shape optimization, hierarchical matrices

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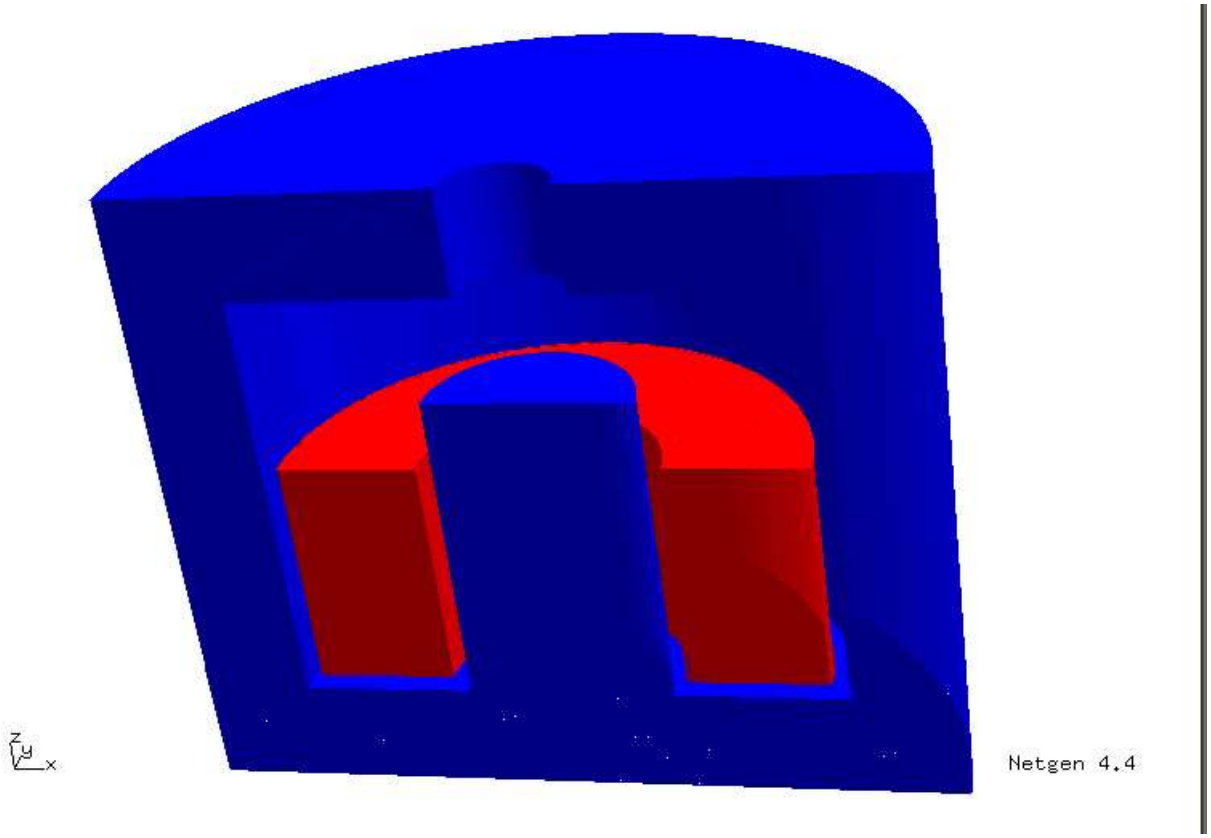
3–Dimensional Axisymmetric Magnetostatics

An electromagnet benchmark problem



3–Dimensional Axisymmetric Magnetostatics

Maxwell's equations: the nonlinear magnetostatic case

$\Omega \subset \mathbb{R}^3$, $\Omega^{\mathrm{e}} := \mathbb{R}^3 \setminus \overline{\Omega}$ ... domains occupied by ferromagnetics and air, respectively,
$\mathbf{n}$ ... the unit outer normal to $\partial\Omega$,
$\mathbf{u}^{\mathrm{i}}$, $\mathbf{u}^{e}$ ... magnetic vector potentials in Ω and Ω^{e}, resp.,
$\nu_0 := 4\pi 10^{-7}$, $\nu_{\mathrm{r}} = \nu_{\mathrm{r}}(\|\mathbf{curl}(\mathbf{u}^{\mathrm{i}})\|^2)$ ... reluctivities of air and ferromagnetics, resp.,
$\mathbf{J}$ ... a divergence–free current density compactly supported in Ω^{e}

$$
\begin{aligned}
\mathbf{curl}\left(\nu_{\mathrm{r}}(\|\mathbf{curl}(\mathbf{u}^{\mathrm{i}})\|^2)\mathbf{curl}(\mathbf{u}^{\mathrm{i}}(\mathbf{x}))\right) &= \mathbf{0} && \text{in } \Omega,\\
\operatorname{div}(\mathbf{u}^{\mathrm{i}}(\mathbf{x})) &= 0 && \text{in } \Omega,\\
\nu_0\,\mathbf{curl}\left(\mathbf{curl}(\mathbf{u}^{\mathrm{e}}(\mathbf{x}))\right) &= \mathbf{J}(\mathbf{x}) && \text{in } \Omega^{\mathrm{e}},\\
\operatorname{div}(\mathbf{u}^{\mathrm{e}}(\mathbf{x})) &= 0 && \text{in } \Omega^{\mathrm{e}},\\
\mathbf{n}(\mathbf{x}) \times \left((\mathbf{u}^{\mathrm{i}}(\mathbf{x}) - \mathbf{u}^{\mathrm{e}}(\mathbf{x})) \times \mathbf{n}(\mathbf{x})\right) &= \mathbf{0} && \text{on } \partial\Omega,\\
\left(\nu_{\mathrm{r}}(\|\mathbf{curl}(\mathbf{u}^{\mathrm{i}})\|^2)\,\mathbf{curl}(\mathbf{u}^{\mathrm{i}}(\mathbf{x})) - \mathbf{curl}(\mathbf{u}^{\mathrm{e}}(\mathbf{x}))\right) \times \mathbf{n}(\mathbf{x}) &= \mathbf{0} && \text{on } \partial\Omega,\\
\mathbf{u}^{\mathrm{e}}(\mathbf{x}) &= O(\|\mathbf{x}\|^{-1}) \ \text{ for } \|\mathbf{x}\| \to \infty,
\end{aligned}
$$

3–Dimensional Axisymmetric Magnetostatics

Axisymmetric ansatz

Assume that
$\Gamma := \partial\Omega := \big\{(r(p)\cos(t), r(p)\sin(t), z(p)) \in \mathbb{R}^3 \ : \ p \in (0,1), \ t \in [-\pi,\pi]\big\}$,
$\Omega_{\mathbf{J}} := \operatorname{supp}\mathbf{J} := \big\{(r\cos(t), r\sin(t), z) \in \mathbb{R}^3 \ : \ r \in (\underline{r},\overline{r}), \ t \in [-\pi,\pi], \ z \in (\underline{z},\overline{z})\big\}$,
$\mathbf{x} := \mathbf{x}(r,t,z) := (r\cos(t), r\sin(t), z)$,
$\mathbf{J}(\mathbf{x}) := J(r,z)\,(-\sin(t), \cos(t), 0)$ in $\Omega_{\mathbf{J}}$, where $r(p) \geq 0$, $\underline{r} > 0$, $r \geq 0$.

This gives rise to $\mathbf{u}^{\mathrm{i/e}}(\mathbf{x}) = u^{\mathrm{i/e}}(r,z)\,(-\sin(t), \cos(t), 0)$,

$$\mathbf{curl}\left(\mathbf{u}^{\mathrm{i/e}}(\mathbf{x})\right) = \left(-\frac{\partial u^{\mathrm{i/e}}(r,z)}{\partial z}\cos(t), -\frac{\partial u^{\mathrm{i/e}}(r,z)}{\partial z}\sin(t), \frac{1}{r}\frac{\partial(r u^{\mathrm{i/e}}(r,z))}{\partial r}\right),$$

$$\mathbf{curl}\left(\mathbf{curl}\left(\mathbf{u}^{\mathrm{i/e}}(\mathbf{x})\right)\right) = -\triangle_{(r,z)} u^{\mathrm{i/e}}(r,z) - \frac{1}{r}\frac{\partial u^{\mathrm{i/e}}(r,z)}{\partial r} + \frac{1}{r^2}u^{\mathrm{i/e}}(r,z)$$

and

$$\operatorname{div}\left(\mathbf{u}^{\mathrm{i/e}}(\mathbf{x})\right) = 0.$$

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Finite Element Approach

Domain truncation, linear case, variational formulation

Assume
$\partial\widehat{\Omega} := \left\{(\widehat{r}(p)\cos(t), \widehat{r}(p)\sin(t), \widehat{z}(p)) \in \mathbb{R}^3 \ : \ p \in (0,1), \ t \in [-\pi, \pi]\right\}$, $\overline{\Omega} \cup \overline{\Omega_{\mathbf{J}}} \subset \widehat{\Omega}$,
denote $\widehat{D} := \{(r,z) \in \mathbb{R}^2 \ : \ (r,0,z) \in \widehat{\Omega}\}$, $D := \{(r,z) \in \mathbb{R}^2 \ : \ (r,0,z) \in \Omega\}$,
$D_{\mathbf{J}} := \{(r,z) \in \mathbb{R}^2 \ : \ (r,0,z) \in \Omega_{\mathbf{J}}\}$,
$\nu(r,z) := \nu_0\nu_{\mathrm{r}}$ in D, $\nu(r,z) := \nu_0$ in $\widehat{D} \setminus \overline{D}$, where $\nu_{\mathrm{r}} \in (0,1]$,
and assume $u^e(r,z) = 0$ on $\partial\hat{D}$.

Find $u(r,z) \in H_0^1(\widehat{D})$ such that

$$\forall v(r,z) \in H_0^1(\widehat{D}) : \int\limits_{\widehat{D}} \nu \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \frac{1}{r^2}\frac{\partial (ru)}{\partial r}\frac{\partial (rv)}{\partial r} \right) r \, dr dz = \int\limits_{D_{\mathbf{J}}} J v r \, dr dz$$

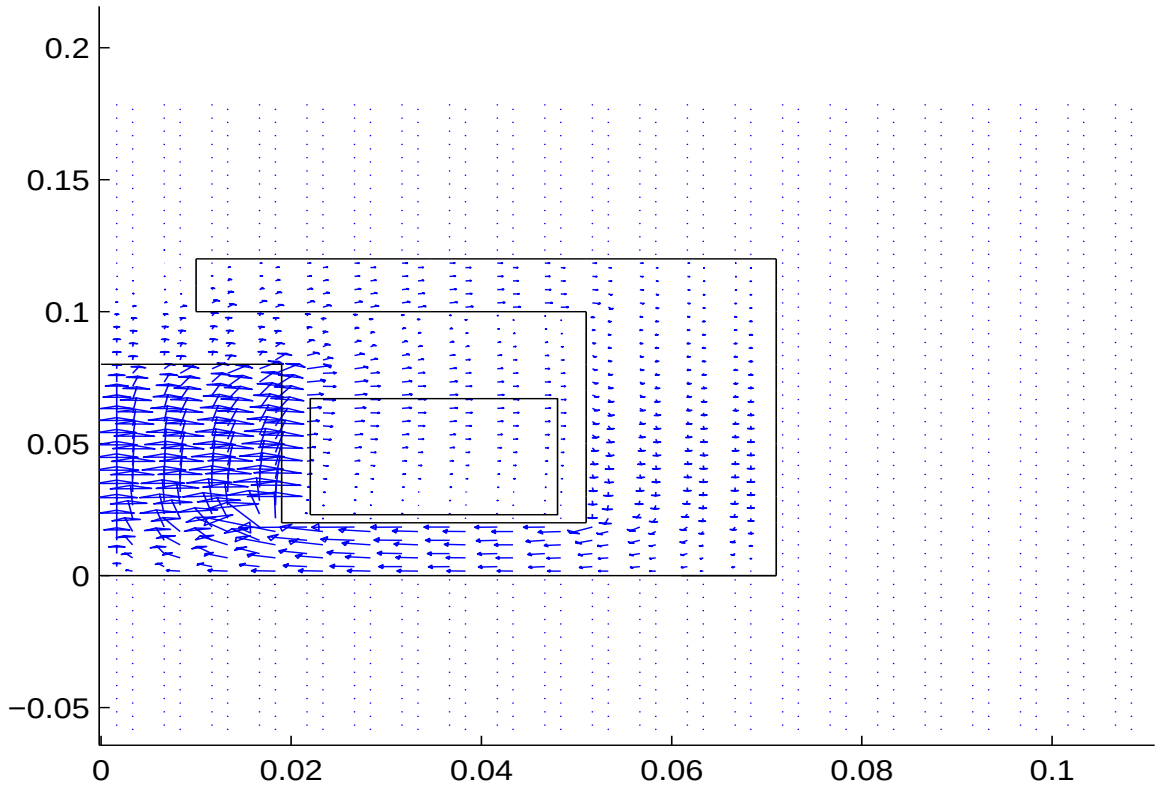
Then

$$u(r,z)|_D \to u^{\mathrm{i}}(r,z)|_D \quad \text{and} \quad u(r,z)|_{\widehat{D}\setminus\overline{D}} \to u^{\mathrm{e}}(r,z)|_{\widehat{D}\setminus\overline{D}},$$

as $\operatorname{diam}\widehat{\Omega} \to \infty$.

Finite Element Approach

FEM solution



Finite Element Approach

Discretization

Assume $\widehat{D} := (0, R) \times (z_0, z_1)$,
employ tensor product grids, bilinear nodal elements.

Multigrid solver [Börm & Hiptmair, 2002]

Smoothing: r–line relaxation,
subspace correction: semi–coarsening in z–direction.

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Boundary Integral Equation Approach

Transmission formula [Hiptmair, 2002] for eddy current problem

Denote by $E(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi\|\mathbf{x}-\mathbf{y}\|}$ the Green's function for the Laplacian in $\mathbb{R}^3$ and define the following scalar and vector single–layer and volume potentials, respectively:

$$\Psi(\Phi)(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{y})E(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}), \quad \mathbf{\Psi}(\boldsymbol{\lambda})(\mathbf{x}) := \int_\Gamma \boldsymbol{\lambda}(\mathbf{y})E(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}),$$

$$\mathrm{G}(\Phi)(\mathbf{x}) := \int_{\mathbb{R}^3} \Phi(\mathbf{y})E(\mathbf{x},\mathbf{y})\,dV(\mathbf{y}), \quad \mathbf{G}(\boldsymbol{\lambda})(\mathbf{x}) := \int_{\mathbb{R}^3} \boldsymbol{\lambda}(\mathbf{y})E(\mathbf{x},\mathbf{y})\,dV(\mathbf{y}).$$

Then for $\mathbf{x} \in \Gamma$:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) =& \mathbf{G}(\mathbf{curl}(\mathbf{curl}(\mathbf{u}))) + \nabla\mathrm{G}(\mathrm{div}(\mathbf{u})) + \mathbf{curl}\left(\mathbf{\Psi}\left([\boldsymbol{\gamma}_\mathrm{D}\mathbf{u}]_\Gamma\right)\right)\\ &- \mathbf{\Psi}\left([\boldsymbol{\gamma}_\mathrm{N}\mathbf{u}]_\Gamma\right) - \nabla\Psi\left([\boldsymbol{\gamma}_\mathrm{n}\mathbf{u}]_\Gamma\right),\end{aligned}$$

where $[\boldsymbol{\gamma}.]_\Gamma := \boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-$, where γ^+, γ^- denote some trace from exterior and interior of Ω, respectively, $\boldsymbol{\gamma}_\mathrm{D}\mathbf{u} := \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$, $\boldsymbol{\gamma}_\mathrm{N}\mathbf{u} := \mathbf{curl}(\mathbf{u})\times\mathbf{n}$, $\boldsymbol{\gamma}_\mathrm{n}\mathbf{u} := \mathbf{u}\cdot\mathbf{n}$.

Boundary Integral Equation Approach

Transmission formula [Hiptmair, 2002] for axisym. magnetostatics?

Denote by $E(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi\|\mathbf{x}-\mathbf{y}\|}$ the Green's function for the Laplacian in $\mathbb{R}^3$ and define the following scalar and vector single–layer and volume potentials, respectively:

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Boundary Integral Equation Approach

Representation formula for axisymmetric magnetostatics?

Assume the axisymmetric ansatz, assume $\boldsymbol{\lambda}(\mathbf{x}) := \lambda(r,z)(-\sin(t),\cos(t),0)$, where $\mathbf{x} := (r\cos(t), r\sin(t), z)$, $r \geq 0$, and assume

$$\mathbf{u}^{\mathrm{i}}(\mathbf{x}) := -\boldsymbol{\Psi}(\boldsymbol{\lambda})(\mathbf{x}) \text{ in } \Omega, \quad \mathbf{u}^{\mathrm{e}}(\mathbf{x}) := -\boldsymbol{\Psi}(\boldsymbol{\lambda})(\mathbf{x}) + \mathbf{G}(\mathbf{J}/\nu_0)(\mathbf{x}) \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Boundary integral equation

Then $\left(\nu_{\mathrm{r}}(\|\mathbf{curl}(\mathbf{u}^{\mathrm{i}})\|^2)\,\mathbf{curl}(\mathbf{u}^{\mathrm{i}}(\mathbf{x})) - \mathbf{curl}(\mathbf{u}^{\mathrm{e}}(\mathbf{x}))\right) \times \mathbf{n}(\mathbf{x}) = \mathbf{0}$ on $\partial\Omega$, leads to

$$-\frac{1}{2}\boldsymbol{\lambda}(\mathbf{x}) + \frac{1-\nu_{\mathrm{r}}}{1+\nu_{\mathrm{r}}}\mathbf{V}(\boldsymbol{\lambda})(\mathbf{x}) = \frac{1}{1+\nu_{\mathrm{r}}}\mathbf{N}(\mathbf{J}/\nu_0)(\mathbf{x}) \text{ on } \partial\Omega,$$

where $\mathbf{V}(\boldsymbol{\lambda})(\mathbf{x}) = \int_{\Gamma} \mathbf{curl}_{\mathbf{x}}\left(\frac{\boldsymbol{\lambda}(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|}\right) \times \mathbf{n}(\mathbf{x})\, dS(\mathbf{y})$,

$\mathbf{N}(\mathbf{J}/\nu_0)(\mathbf{x}) = \frac{1}{\nu_0}\int_{\Omega_{\mathbf{J}}} \mathbf{curl}_{\mathbf{x}}\left(\frac{\mathbf{J}(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|}\right) \times \mathbf{n}(\mathbf{x})\, dV(\mathbf{y})$.

Boundary Integral Equation Approach

Boundary discretization

Assume $\partial\Omega = \sum_{j=1}^{n} \overline{\sigma_j}$,
$\sigma_j = \{(r_j(p)\cos(t), r_j(p)\sin(t), z_j(p)) : p \in (0,1),\ t \in [-\pi,\pi]\}$, $r_j(p)$, $z_j(p)$ affine,
$\mathbf{y} = (r\cos(t), r\sin(t), z) \in \sigma^j : \boldsymbol{\lambda}(\mathbf{y}) = \lambda_j(-\sin(t), \cos(t), 0)$.

Operator discretizations

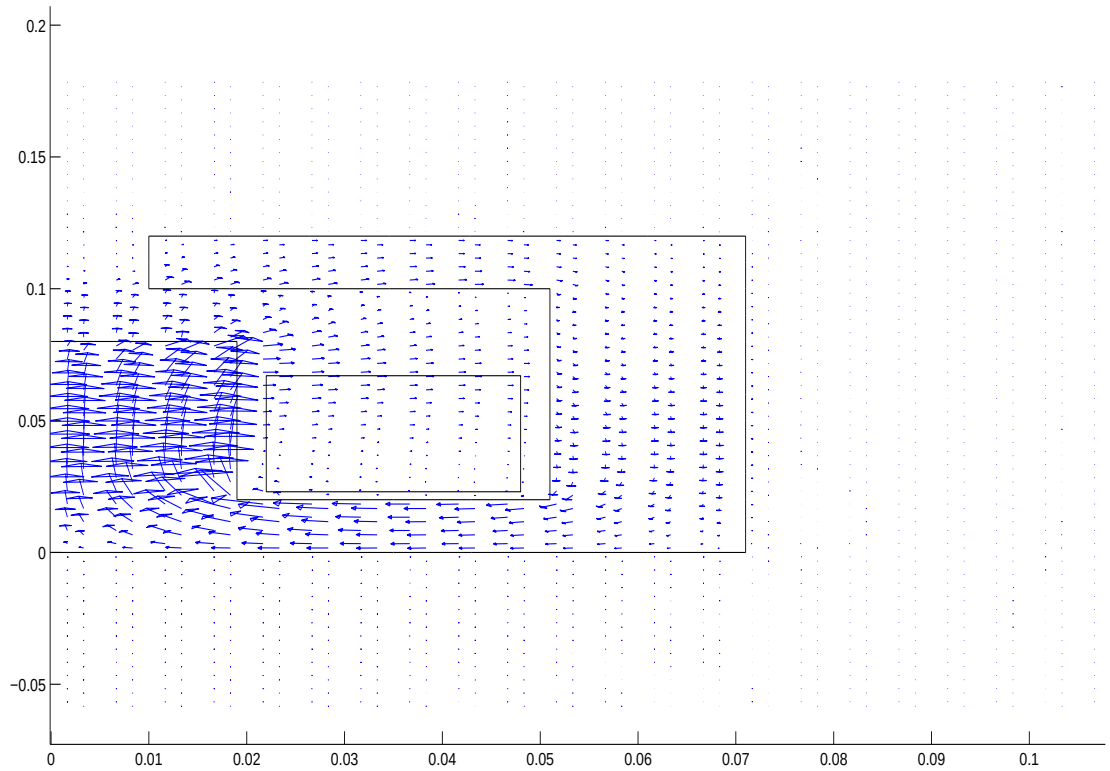
$$V_{ij} = -\int_0^1 \int_0^\pi \frac{\cos(t)(n_{i1}x_{i1} + n_{i3}(x_{i3} - z_j(p))r_j(p)\sqrt{r_j'(p)^2 + z_j'(p)^2}}{2\pi\sqrt{[(x_{i1} - r_j(p)\cos(t))^2 + (r_j(p)\sin(t))^2 + (x_{i3} - z_j(p))^2]^3}}\, dt\, dp$$

$$N_i = -\frac{J}{\nu_0}\int_0^\pi \int_{\underline{r}}^{\overline{r}} \int_{\underline{z}}^{\overline{z}} \frac{n_{i1}(x_{i1}\cos(t) - r) + n_{i3}\cos(t)(x_{i3} - z)}{2\pi\sqrt{[(x_{i1} - r\cos(t))^2 + (r\sin(t))^2 + (x_{i3} - z)^2]^3}} r\, dz\, dr\, dt$$

such that $\mathbf{V}[\boldsymbol{\lambda}(\mathbf{y})|_{\sigma_j}](\mathbf{x}_i) = (0, V_{ij}\lambda_j, 0)$ and $\mathbf{N}\left[\frac{1}{\nu_0}\mathbf{J}(\mathbf{y})\right](\mathbf{x}_i) = (0, N_i, 0)$ at $\mathbf{x}_i \in \sigma_i$.

Boundary Integral Equation Approach

BIE solution



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A Coupling Scheme for Linear Axisymmetrix Magnetostatics

FEM equation

Denote $\omega := \{\mathbf{x} := (r,z) \in \mathbb{R}^2 \ : \ (r,0,z) \in \Omega\}$, $\mathbf{n} := (n_r, n_z)$ outer normal to $\partial\omega$.
Find $u(r,z) \in H^1(\omega)$ such that $\forall v(r,z) \in H^1(\omega)$:

$$\int_{\omega} \nu \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \frac{1}{r^2}\frac{\partial (ru)}{\partial r}\frac{\partial (rv)}{\partial r} \right) r\, dr dz - \int_{\partial\omega} \left(n_z \frac{\partial u}{\partial z} + n_r \frac{1}{r}\frac{\partial (ru)}{\partial r} \right) v\, ds(p) = 0.$$

FEM–BIE Neumann and Dirichlet coupling equations

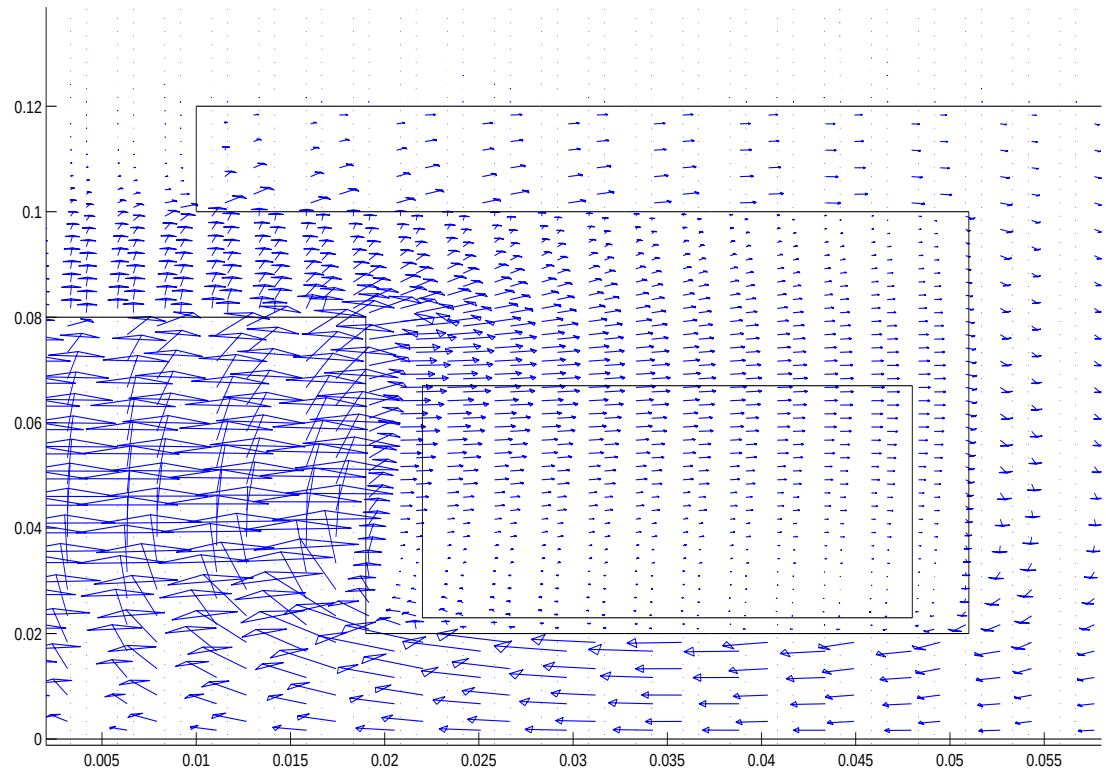
$$\nu_{\mathrm{r}} \left(n_z(\mathbf{x})\frac{\partial u(\mathbf{x})}{\partial z} + n_r(\mathbf{x})\frac{1}{r(\mathbf{x})}\frac{\partial (r(\mathbf{x})u(\mathbf{x}))}{\partial r} \right) + \left[-\frac{1}{2}I + V \right](\lambda)(\mathbf{x}) = N(J/\nu_0)(\mathbf{x}) \text{ on } \partial\omega$$

$$u(\mathbf{x}) + V_{\mathrm{D}}(\lambda)(\mathbf{x}) = N_{\mathrm{D}}(J/\nu_0)(\mathbf{x}) \text{ on } \partial$$

where $V_{\mathrm{D}}(\lambda)(\mathbf{x}) = \int_{\partial\omega} \frac{\lambda(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|}\, ds(\mathbf{y})$ and $N_{\mathrm{D}}(J/\nu_0)(\mathbf{x}) = \frac{1}{\nu_0}\int_{\omega_{\mathbf{J}}} \frac{J(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|}\, dS(\mathbf{y})$.

A Coupling Scheme for Linear Axisymmetrix Magnetostatics

FEM–BIE least–square solution



5673 FEM DOFs, 504 BIE DOFs

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References

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