Computing plates and some related topics

Ivo Marek
Faculty of Civil Engineering, Czech University of Technology

PANM 13
In honor of Ivo Babuška's 80^{th} birthday
Prague, May 28-31, 2006

Acknowledgement

Petr Mayer and Milan Pultar

The work on which this talk is based was supported by the Program Information Society under Project 1ET400300415, Grant No. 201/02/0595 of the Grant Agency of the Czech Republic and Grant No. MSM 210000010 of the Ministry of Education, Youth and Sports of the Czech Republic.

Outline of the talk

- Motivation
- Some formulations of the problem
- Analytic properties of a mixed formulation
- Reproducing kernels and the biharmonic operator
- Positivity issues in the plate bending problems
- Some properties of the systems of linear algebraic equations obtained via discretization

[1] Aronszajn N. *Theory of reproducing kernels*. Trans. Amer. Math. Soc. **68** (1951), 337-404.

[2] Aronszajn N., Smith K.T. *Characterization of positive reproducing kernels. Applications to Green's functions*. (1957), 611-622.

[3] Bittnar Z., Šejnoha J. *Numerical Methods of Mechanics 1*. Vydavatelství ČVUT, Praha 1992.

[4] Bittnar Z., Šejnoha J. *Numerical Methods of Mechanics 2*. Vydavatelství ČVUT, Praha 1992.

[5] Ciarlet P.G. *The Finite Element Method for Elliptic Problems*. SIAM, Philadelphia 2002.

[6] Ciarlet P.G., Glowinski R. *Dual iterative techniques for solving a finite element approximation of the biharmonic equation*. Computer methods in Applied Mechanics and Engineering **5** (1975) 277 - 295.

[7] Duffin R.J. *On a question of Hadamard concerning superharmonic functions*. Journal of Mathematics and Physics **1** (1949), 253-258.

[8] Hlaváček I. *A mixed finite element method for plate bending with a unilateral obstacle*. Appl. Math. **39**, 25-44 (1994).

[9] Hlaváček I. *Plate bending problems with uncertain input data*. Unpublished manuscript, Srní, September 2005.

[10] Mayer P. *Computational experiments with plates*. Research papers ČVUT K101.05.01. Praha 2005.

1 Definitions and notation

Ω	-	region in $\mathcal{R}^2$
$\partial\Omega$	-	boundary of region Ω
$H^m(\Omega),\ 1 \leq m \leq 2$	-	solution space
$\|v\|_{m,\Omega} = \left(\sum_{\|\alpha\| \leq m} \int_\Omega \|\partial^\alpha v\|^2\right)^{1/2}$	-	norm in the space $H^m(\Omega)$
$H_0^m(\Omega)$	-	zero-trace solution space
$L^{1/2}(\partial\Omega)$	-	trace space

$\mathcal{M}$ - complementary subspace of the Hilbert space $H^1(\Omega)$ with respect to $H_0^1(\Omega)$ thus satisfies relation $H^1(\Omega) = H_0^1(\Omega) \oplus \mathcal{M}$

$(.,.)_{\mathcal{M}}$ - inner product on $\mathcal{M}$ inducing on $\mathcal{M}$ norm equivalent with the norm heredited from $H^1(\Omega)$

2 Biharmonic equation

$$\Delta^2 u = f \qquad \text{in } \Omega \qquad \text{and} \qquad u = 0 = \frac{\partial u}{\partial \nu} \qquad \text{on } \partial\Omega$$

2.1 Formulation of the Problem

Find $u \in H_0^2(\Omega)$ such that relations

$$J(u) = \min\left\{J(v) : v \in H_0^2(\Omega)\right\}, \tag{2.1}$$

hold where

$$J(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 - \int_\Omega f v \ , \quad v \in H_0^2(\Omega). \tag{2.2}$$

In place of (2.2) one can minimize functional

$$\mathcal{J}(v,\psi) = \frac{1}{2}\int_\Omega |\psi|^2 - \int_\Omega f v \tag{2.3}$$

under the assumption that $v \in H_0^2$ and $\psi \in L^2(\Omega)$ and simultaneously

$$\Delta v = \psi.$$

The subspaces corresponding to the variational principle introduced are characterized as follows:

$$\mathcal{V} = \left\{ (v, \psi) \in H_0^1(\Omega) \times L^2(\Omega) : \forall \mu \in H_0^1(\Omega) \text{ satisfying } \beta\left((v, \mu), \mu\right) = 0 \right\},$$

where

$$\beta\left((v, \psi), \mu\right) = \int_\Omega grad v \; grad \mu - \int_\Omega \psi \mu \tag{2.4}$$

A relationship between the natural variational principle (2.1)-(2.2) and that one given by (2.3) reads:

2.2 *Assume u is a solution of Problem 1. Then also*

$$\mathcal{J}(u, -\Delta u) = \min\left\{ \mathcal{J}(v, \phi) : (v, \phi) \in \mathcal{V} \right\}.$$

This fact guarantees a possibility to factorize Problem 1 to a subsequent solving Poisson problems, hence problems of order 2. It is easy to see that function ϕ ca be interpreted as an aproximation or reprezentative of $-\Delta u$.

3 An Algorithm

Input data: f $\in L^2(\Omega)$, $\lambda^0 \in \mathcal{M}$

$1^{(0)}$ Set subsequently $k = 0, 1, ...$

2^0 Find $\phi^k \in H_0^1(\Omega)$ such that in the classical formulation

$$\Delta\phi^k = f \text{ in } \Omega$$

$$\phi^k = \lambda^k \text{ on}$$

in the variational formulation

$$\int_\Omega grad\phi^k grad\mu d\Omega = \int_\Omega f\mu d\Omega \quad \forall\mu \in H_0^1(\Omega)$$

$$\phi^k - \lambda^k \in H_0^1(\Omega)$$

3^0 Find $u^k \in H_0^1(\Omega)$ such that

in the classical formulation

$$\Delta u^k = \phi^k \text{ in } \Omega$$

$$u^k = 0 \text{ on}$$

in the variational formulation

$$\int_\Omega grad u^k grad \mu d\Omega = \int_\Omega \phi^k \mu d\Omega \quad \forall \mu \in H_0^1(\Omega)$$

$$u^k \in H_0^1(\Omega)$$

4^0Find $\lambda^{k+1} \in \mathcal{M}$ such that

in the classical formulation

$$\lambda^{k+1} = \lambda^k + \rho \left[\Delta u^k - \phi^k \right] \text{ on } \partial\Omega$$

in the variational formulation

$$\left(\lambda^{k+1} - \lambda^k, \mu\right)_{\mathcal{M}} = \rho \left[\int_\Omega grad u^k grad \mu d\Omega - \int_\Omega \phi^k \mu d\Omega \right]$$

$$\forall \mu \in \mathcal{M}$$

Convergence of the Algorithm is characterized using the following relations.

3.1 *Assuming parameter ρ is chosen in the interval $(0, 2c^2\sigma^2)$, where*

$$\sigma = \inf \left\{ \frac{\|\Delta v\|_{L^2(\Omega)}}{\|\frac{\partial v}{\partial \nu}\|_{L^2(\Omega)}} : v \in H^2(\Omega) \cap H_0^1(\Omega) \right\}$$

and constant $c > 0$ satisfies relations

$$c\|\mu\|_{L^2(\partial\Omega)} \leq (\mu, \mu)_{\mathcal{M}}^{1/2}, \; \forall \mu \in H^1(\Omega),$$

the quantities appearing in Algorithm 1 fulfil relations

$$\lim_{k\to\infty} \|u^k - u\|_{1,\Omega} = 0,$$

a

$$\lim_{k\to\infty} \|\phi^k + \Delta u\|_{L^2(\Omega)} = 0.$$

4 Discrete formulation

Space of approximate solutions:

$$V_h \subset H^1(\Omega)$$

Space of approximate solutions with zero traces:

$$V_{0h} = \{v_h \in V_h : v_h = 0 \quad \text{na} \quad \partial\Omega\}$$

Space of approximate solutions with constraints:

$$\mathcal{V}_h = \{(v_h, \psi_h) \in V_{0h} \times V_h ; \forall \mu_h \in V_h, \;\; \beta((v_h, \psi_h), \mu_h) = 0\}$$

Space of approximate solutions with constraints and zero traces:

$$\mathcal{W}_h = \{(v_h, \psi_h) \in V_{0h} \times V_h; \forall \mu_h \in V_{0h}, \;\; \beta((v_h, \psi_h), \mu_h) = 0\}$$

Complementary space of V_{0h} in V_h, thus,

$$V_h = V_{0h} \oplus \mathcal{M}_h$$

4.1 Algorithm 1_h

Input data: $f_h \in V_h$, $\lambda_h^0 \in \mathcal{M}$

Set subsequently $k = 0, 1, \ldots$

1^0 *Find* ϕ_h^k *satisfying relations* $\phi_h^k - \lambda_h^k \in V_{0h}$ *and*

$$\forall v_h \in V_{0h}, \quad \int_\Omega grad\ \phi_h^k \ \ grad\ v_h = \int_\Omega f v_h.$$

2^0 *Find* $u_h^k \in V_{0h}$ *such that by*

$$\forall v_h \in V_{0h}, \quad \int_\Omega grad\ u_h^k\ grad\ v_h = \int_\Omega \phi_h^k v_h.$$

3^0 *Find* $\lambda_h^{k+1} \in \mathcal{M}_h$ *such that*

$$\forall \mu_h \in \mathcal{M}_h, \quad \left(\lambda_h^{k+1} - \lambda_h^k, \mu_h\right)_{\mathcal{M}_h} = \rho\ \beta\left((u_h^k, \phi_h^k), \mu_h\right) = 0.$$

Let $A_h : V_h \longrightarrow V_{0h}$ be given by

$$A_h\psi_h = v_h \Leftrightarrow \forall \mu_h \in V_{0h},\ \int_\Omega grad\ v_h\ grad\mu_h = \int_\Omega \psi_h\mu_h$$

Similarly, $B_h : V_h \longrightarrow \mathcal{M}_h$ utilizing the system of equations: B_h

$$\forall \mu_h \in \mathcal{M}_h : (B_h\psi_h, \psi_h)_{\mathcal{M}_h} = \beta\left((A_h\psi_h, \psi_h), \mu_h\right) = 0.$$

Hence, $B_h\psi_h$ is a restriction of $A_h\psi_h$ onto $\partial\Omega$. (We have that $A_h \sim \Delta^{-1}$ with boundary conditions implied from $A_h\psi_h$).

A result similar to that describing convergence for the continuous case is contained in the following

4.2 *Assuming paprameter* ρ *is chosen from the interval* $(0, 2{\sigma_h}^2)$, *where*

$$\sigma_h = \frac{1}{\|B\|},$$

and $\|B\|$ *denotes the operator* L^2*-norm, relations*

$$\lim_{k\to\infty} \left\| u_h^k - u_h \right\|_{V_{0h}} = 0, \quad \lim_{k\to\infty} \left\| \phi_h^k - \phi_h \right\|_{V_h} = 0$$

hold.

5 Finite element spaces

Triangulation $\mathcal{T}_h$ of the region Ω consists of elements K such that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h}$ and each element K satisfies for all triangulations $0 < h \leq h_0$ standard geometric conditions in the sense of [5]. It is assumed also that each element K is an affine picture of a reference element $\hat{K} : K = F_K(\hat{K})$.

Space

$$V_h = \left\{ v_h \in \mathcal{C}(\overline{\Omega}) : \forall K \in \mathcal{T}_h, v_h|_K \in P_K \right\}$$

is generated utilizing the triangulation $\mathcal{T}_h$ where

$$P_K = \left\{ v : K \to \mathcal{R}; v = \hat{v} \circ F_K^{-1}, \forall \hat{v} \in \hat{P} \right\}$$

assuming $\hat{P}$ a finite dimensional space of functions satisfying

$$P_1 \subset \hat{P}$$

where P_1 denotes the set of all polynomials of order ≤ 1 in two variables.

5.1 *As* regular class of triangulations *one understands a system $\{\mathcal{T}_h\}$ satisfying with suitable constants independent of h, α and τ such that*

$$\text{max } \left\{\frac{h(K)}{\delta(K)}\right\} \leq \alpha,$$

$$\tau \text{ max } \{h(K) : K \in \mathcal{T}_h\} \leq \text{min } \{h(K) : K \in \mathcal{T}_h\}, \qquad h = \text{max}\{h(K) : K \in \mathcal{T}_h\}$$

where $h(K) = \text{diameter } K, \delta(K) = \text{sup}\{\text{diameter of the discs inscribed into } K\}$*.*

The above theory has been developed without any requirements concerning the spaces V_h, V_{0h}, $\mathcal{M}_h$. To determining the value of parameter ρ we need some specifications concerning these spaces however. As spaces $\mathcal{M}_h$ we choose in agreement with the choice of the spaces V_h a V_{0h} the following specific ones: Spaces $\mathcal{M}_h$ are subspaces of V_h consisting of functions whose values vanish at all inner nodes of Ω. We then have the following result.

5.2 Theorem *Assume that the inner product $(.,.)_{\mathcal{M}_h}$ is the L^2-inner product on $\partial\Omega$. Further, let V_h a V_{0h} a $\mathcal{M}_h$ be chosen such described above.*

Then

$$\lim_{k\to\infty} \sigma_h = \sigma,$$

where σ is the quantity introduced in the part of this paper devoted to the continuous problem.

6 Reproducing kernels and the biharmonic operator

Consider in a plane domain $\mathcal{D}$ the class $\mathcal{B}$ of all regular harmonic functions (in general complex valued) with a finite norm given by

$$\|h\|^2 = \int\int_{\mathcal{D}} |h(x,y)|^2 dxdy.$$

Class $\mathcal{B}$ possesses a reproducing kernel which will be denoted by $H(z,z_1)$, i.e.

$$f(z) = \int\int_{\mathcal{D}} H(z,z_1)f(z_1)dz_1, \;\forall f \in \mathcal{B}, z = x+iy \in \mathcal{D}.$$

Assume u is a complex-valued function. Then $\hat{u}$ denotes its complex conjugate.

6.1 Theorem [1] *In order that the reproducing kernel $K(x, y)$ of the proper functional Hilbert space $\mathcal{E}$ be nonnegative it is necessary and sufficient that $\mathcal{E}$ have two properties*

(a) *If $u \in \mathcal{E}$, then $\hat{u} \in \mathcal{E}$ and $\|\hat{u}\| = \|u\|$.*

(b) *For each real-valued $u \in \mathcal{E}$ there exists $\tilde{u} \in \mathcal{E}$ such that*

$$\tilde{u}(x) \geq |u(x)| \text{ for all } x \text{ and } \|\tilde{u}\| = \|u\|.$$

7 Positivity issues in plate bending problems

Assume $\mathcal{D} \subset \mathcal{R}^2$ is a region.

Denote by symbol A an operator defined by

$$\begin{cases} Au \equiv \Delta^2 u & \text{for} \quad (x,y) \in \mathcal{D} \\ u(x,y) = 0 = \frac{\partial}{\partial \nu} u(x,y) & \text{for } (x,y) \in \partial\mathcal{D} \end{cases}$$

where

$$(\Delta u)(x,y) \equiv \left[\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \right] u(x,y) \qquad \text{for } (x,y) \in \mathcal{D}.$$

Question (Hadamard) Does the pointwise relation

$$Au(x,y) \geq 0 \quad \text{for all } (x,y) \in \mathcal{D}$$

imply that

$$u(x,y) \geq 0 \quad \text{for all } (x,y) \in \mathcal{D}?$$

Answer Not always!

According to Duffin [7] the answer to the above question is no if $\mathcal{D}$ is suitably chosen rectangle.

The answer to the Hadamard question is affirmative if

- $\mathcal{D}$ is a disc

Moreover, Duffin conjectures [7] that the answer to the Hadamard question is affirmative if $\mathcal{D}$ is a square.

8 Concluding remarks

- The original problem of the fourth order is transferred to iterating on two Poisson equations in the domain Ω and finding solution to one functional equation on the boundary $\partial\Omega$.

- The structure of the Poisson problems to solve is suitable for application of the most efficient (multigrid) methods: Actually, the corresponding matrices are symmetric M-matrices

- The operator governing the functional equation on the boundary $\partial\Omega$ reads

$$\Delta^{-1} P_{\mathcal{M}, H_0^1(\Omega)} \Delta^{-1} \Delta$$

  where $P_{\mathcal{M}, H_0^1(\Omega)}$ denotes the projection of $H^1(\Omega)$ onto $\mathcal{M}$ along $H_0^1(\Omega)$ and is thus symmetrizable and positive semidefinite. A simple Richardson iteration recommended in [5] to compute approximate boundary data for the auxiliary Poisson problem can, if needed, be replaced by some efficient Krylov like method, e.g. *conjugate gradient method*.

- An extra additional gain of the approach described is continuity of the quantity Δu, where u is the true solution of our biharmonic problem. This property of Δu may not be guaranteed even if one applies some conformal FEM.

- In order to use the methods described in the lecture to computing plate problems standard software products utilizing $\mathcal{C}^0$ elements apply .