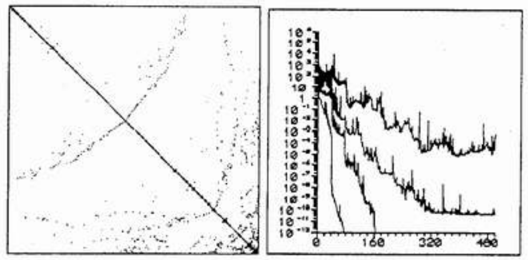
Singular Value Decomposition, Application in Image Deblurring



SNA 2006, Moninec – Martin Plešinger

Outline

1. Singular value decomposition
2. Solving linear algebraic systems using SVD
3. Introduction to the image deblurring
4. Regularization tools
5. Examples in MATLAB

1. Singular value decomposition

SVD theorem:

Consider a (real) matrix

$$A \in \mathcal{R}^{n\times m}, \quad r = \mathrm{rank}\,(A) \leq \min\,\{n, m\}\,.$$

A has

m	columns of length n ,
n	rows of lenght m ,
r	is the maximal number of linearly independent columns and rows of A .

There exists a decomposition of A in the form

$$A = U \Sigma V^T, \quad U \in \mathcal{R}^{n \times n}, \quad V \in \mathcal{R}^{m \times m},$$

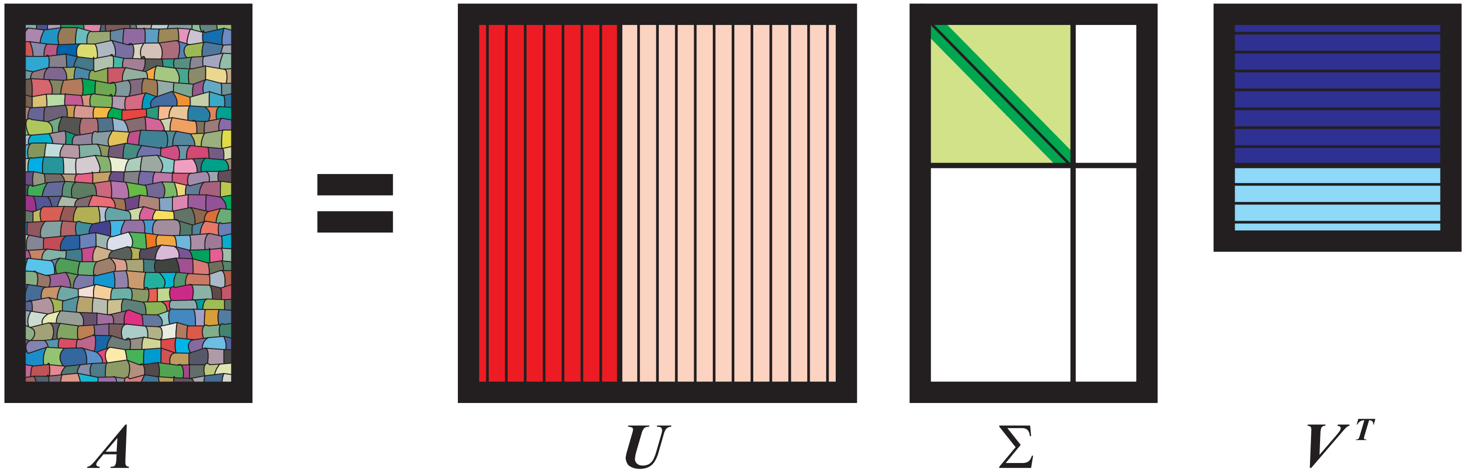
where U, V are orthogonal matrices,

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^{n \times m},$$

$$\Sigma_r = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_r) \in \mathcal{R}^{r \times r},$$

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0\,.$$

Singular value decomposition – the matrices:



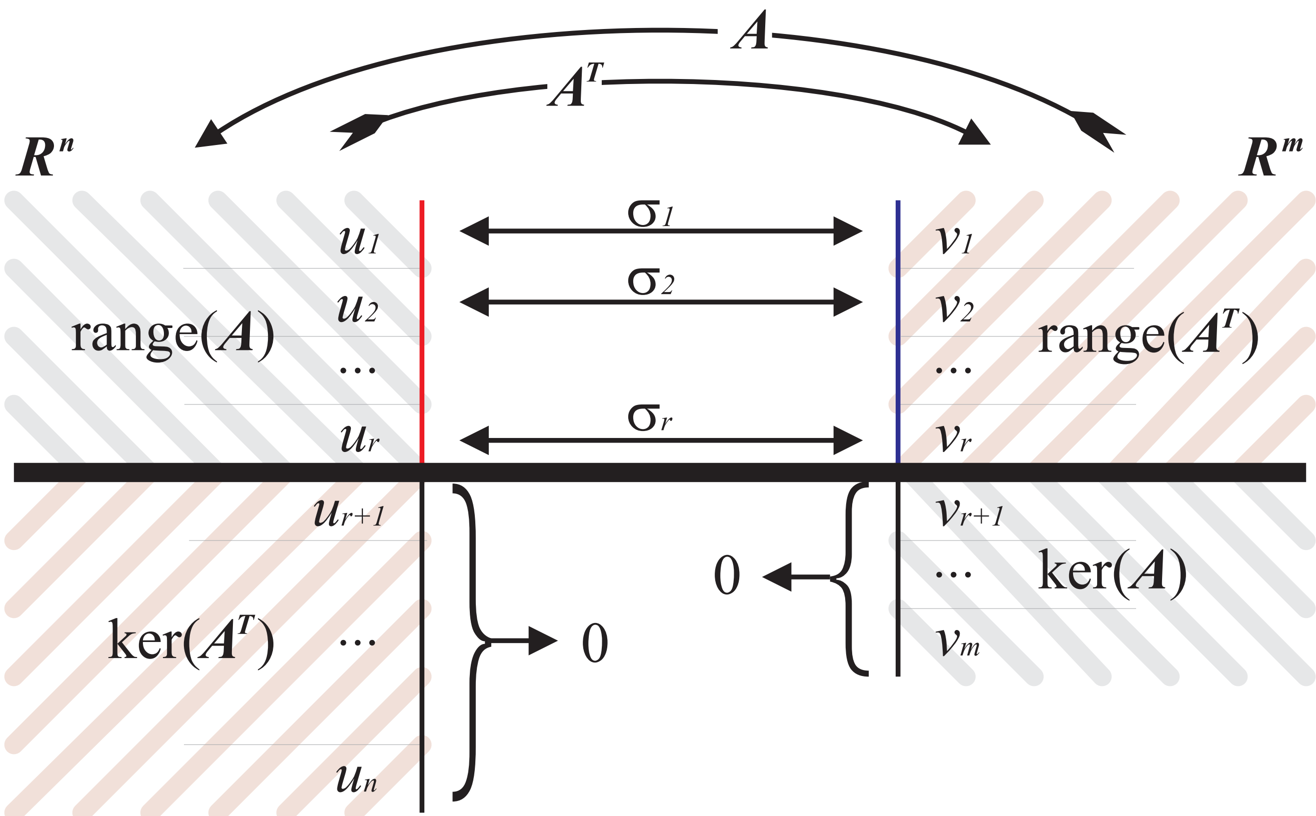
$\{u_i\}_{i=1,\ldots,n}$ are **left singular vectors** (columns of U),
$\{v_i\}_{i=1,\ldots,m}$ are **right singular vectors** (columns of V),
$\{\sigma_i\}_{i=1,\ldots,r}$ are **singular values** of A.

It can be shown that

$$\operatorname{span}(u_1, \ldots, u_r) \equiv \operatorname{range}(A) \subset \mathcal{R}^n,$$
$$\operatorname{span}(v_{r+1}, \ldots, v_m) \equiv \ker(A) \subset \mathcal{R}^m,$$

$$\operatorname{span}(v_1, \ldots, v_r) \equiv \operatorname{range}(A^T) \subset \mathcal{R}^m,$$
$$\operatorname{span}(u_{r+1}, \ldots, u_n) \equiv \ker(A^T) \subset \mathcal{R}^n.$$

Singular value decomposition – the subspaces:



Uniqueness:

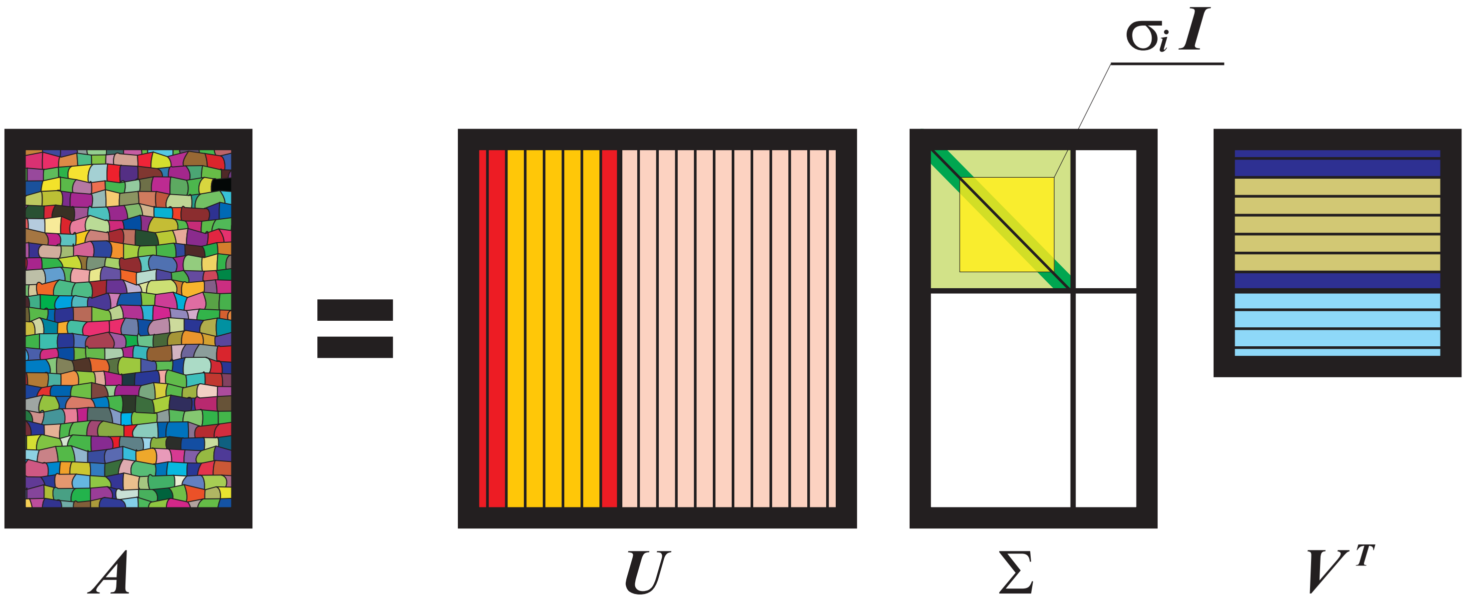
The singular values σ_i and the matrix Σ are for a given A unique.

The orthogonal factors U, V are unique up to multiplication by block-diagonal orthogonal matrices W and T,

$$W = \mathrm{blockdiag}\,(R_1, \ldots, R_l, \tilde{W}) \;\in\; \mathcal{R}^{n\times n},$$
$$T = \mathrm{blockdiag}\,(R_1, \ldots, R_l, \tilde{T}) \;\in\; \mathcal{R}^{m\times m},$$

where l is the number of **distinct singular values**,
size of R_i is given by the multiplicity of σ_i , $i = 1, \ldots, l$,
size of $\tilde{W}$ and $\tilde{T}$ is the dimension of $\ker(A^T)$ and $\ker(A)$ respectively.

Singular value decompostition – uniqueness:



The outer product (dyadic) form:

Using notations

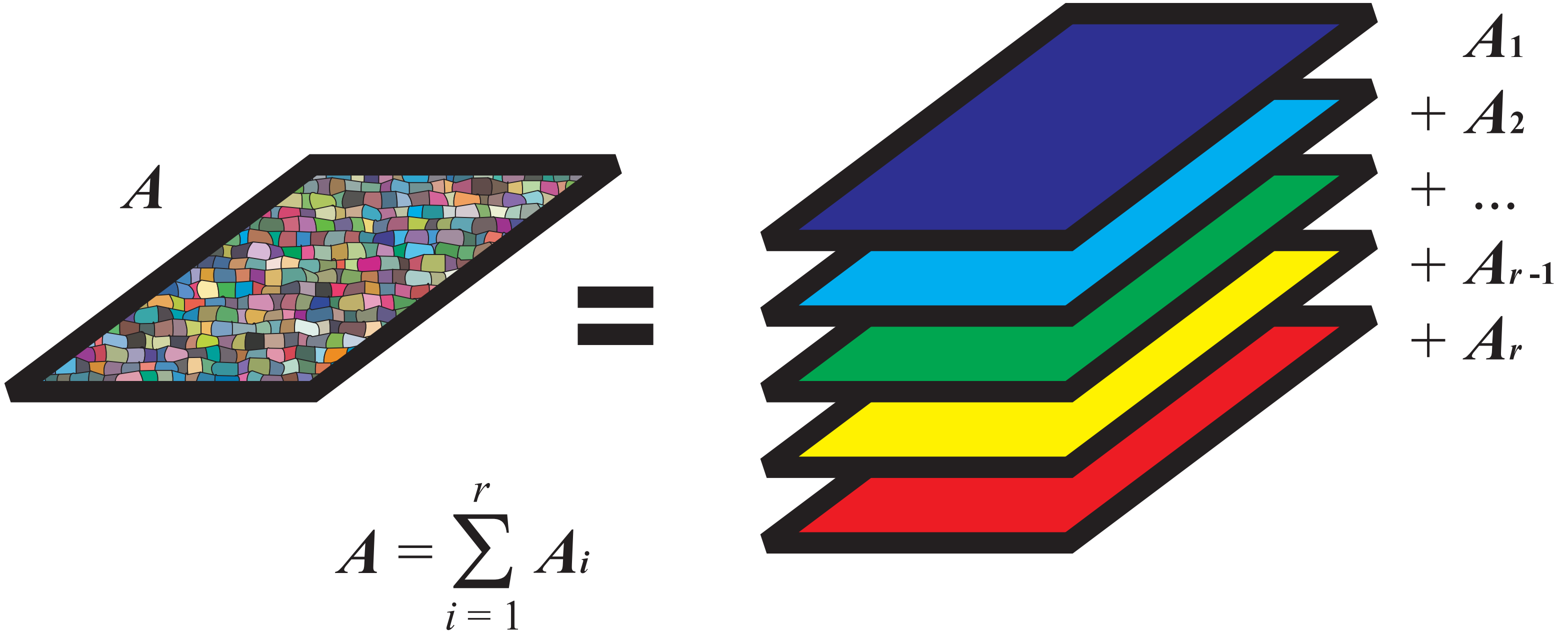
$$U \equiv [U_r | U_{n-r}] , \quad U_r = [u_1, \ldots, u_r] ,$$

$$V \equiv [V_r | V_{m-r}] , \quad V_r = [v_1, \ldots, v_r] ,$$

we can rewrite A as a sum of rank-one matrices in the **dyadic form**

$$A \ = \ U \, \Sigma \, V^T \ = \ U_r \, \Sigma_r \, V_r^T \ = \ \sum_{i=1}^{r} u_i \sigma_i v_i^T \ \equiv \ \sum_{i=1}^{r} A_i \, .$$

Matrix A as a sum of rank-one matrices:



Optimal approximation of A with a rank-k matrix S_k:

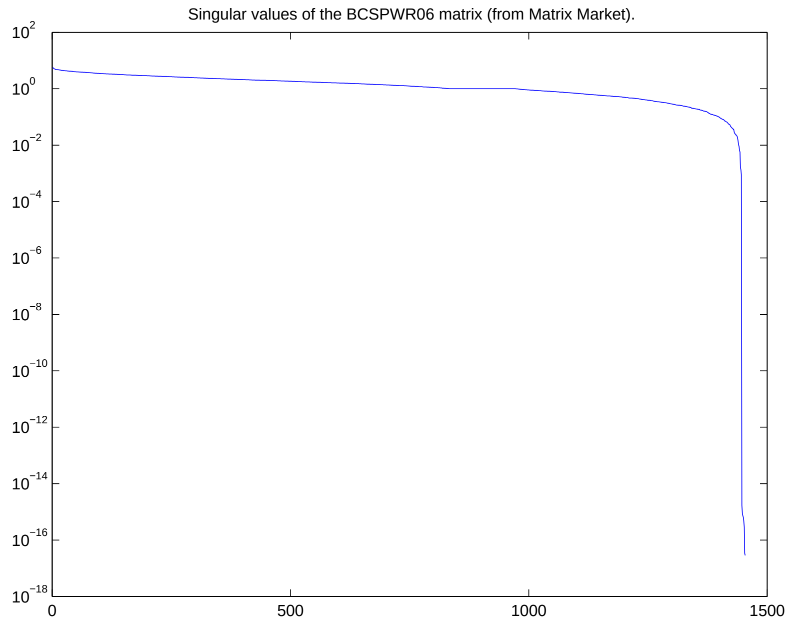
The sum of the first k dyadic terms is the best rank-k approximation of the matrix A ,

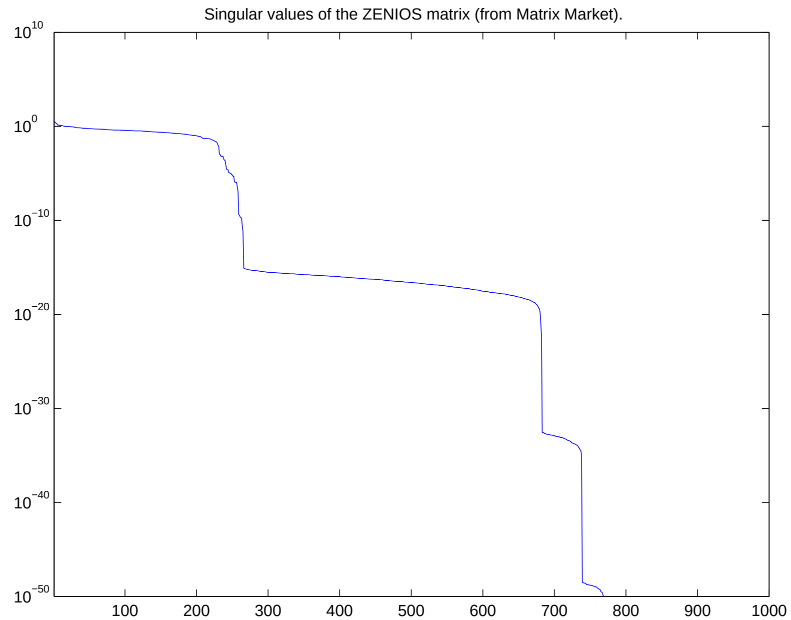
$$S_k = \arg \min_{X \in \mathcal{R}^{n\times m}} \{\|A - X\|_2 \, , \ \ \mathrm{rank}\,(X) \leq k\} = \sum_{i=1}^{k} u_i \sigma_i v_i^T ,$$

in the sense of minimizing the 2-norm of the approximation error.

SVD reveals the dominating information encoded in a matrix. This allows to aproximate A, under some circumstances, with a low-rank matrix.

Different possible distributions of singular values:





The hardly (left) and the easily (right) approximable matrices (BCSPWR06 and ZENIOS from the Harwell-Boeing Collection).

2. Solving linear algebraic systems using SVD

We consider a square nonsingular matrix $A \in \mathcal{R}^{n\times n}$,
with its SVD $A = U\,\Sigma\,V^T$, $\Sigma = \mathrm{diag}\,(\sigma_1, \dots, \sigma_n)$.

The inverse matrix can be expressed as

$$A^{-1} \;=\; V\,\Sigma^{-1}\,U^T \;=\; \sum_{i=1}^{n} v_i \sigma_i^{-1} u_i^T, \qquad \Sigma^{-1} \;=\; \mathrm{diag}\,(\sigma_1^{-1}, \dots, \sigma_n^{-1})\,.$$

With SVD of the given nonsingular matrix A, we can easily compute the solution of a linear algebraic system $Ax = b$,

$$x \;=\; A^{-1}b \;=\; \sum_{i=1}^{n} \left(\frac{u_i^T b}{\sigma_i}\right) v_i\,.$$

For A rectangular

$$\Sigma^\dagger \equiv \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^{m\times n}, \qquad A^\dagger \equiv V\,\Sigma^\dagger U^T \in \mathcal{R}^{m\times n}.$$

The matrix $A^\dagger$ is called the **Moore-Penrose pseudoinverse**.
It is the unique solution to the problem

$$A^\dagger = \arg \min_{X\in\mathcal{R}^{m\times n}} \left\{\|AX - I_n\|_F\right\}.$$

The matrix $A^\dagger$ satisfy

$$A\,A^\dagger = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^{n\times n}, \qquad A^\dagger\,A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^{m\times m}.$$

Using SVD, we can solve a general (over- or underdetermined) linear system $Ax \approx b$ **in the least square sense**

$$x_{\mathrm{LS}} = A^{\dagger} b = \sum_{i=1}^{r} \left(\frac{u_i^T b}{\sigma_i} \right) v_i \, ,$$

where x_{LS} is the **minimum norm least squares solution**

$$x_{\mathrm{LS}} = \arg\min_{x} \left\{ \|b - Ax\|_2 \, , \quad x \perp \ker(A) \right\} ,$$

i. e., x_{LS} has the smallest 2-norm of all vectors x that make $\|b - Ax\|_2$ minimal.

The solution is expressed as the **linear combination of the right singular vectors** v_i, weighted by the projections of the b to the left singular subspaces, divided by the corresponding singular values σ_i .

3. Introduction to the image deblurring

It exists an unknown preimage $\mathbf{X}$ (h pixels high, w pixels wide), we have only its blurred image $\mathbf{B}$ (same size), and we represent these images as matrices X, $B \in \mathcal{R}^{h\times w}$.

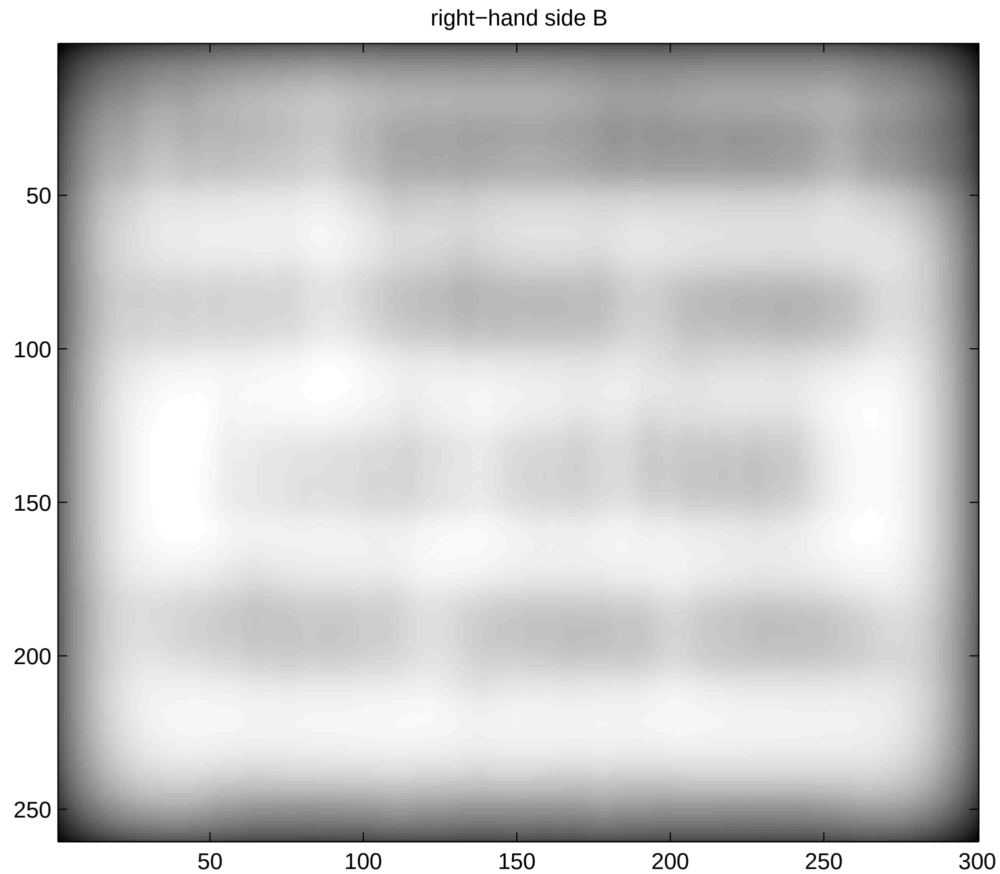
Let the blurring proccess is linear and independent in columns and rows. Then it can be described by the matrix equation

$$A_C \, X \, A_R^T = B,$$

where $A_C \in \mathcal{R}^{h\times h}$ is the column blurring factor,
$A_R \in \mathcal{R}^{w\times w}$ is the row blurring factor.

In practial problems, they arise from the corresponding real imaging systems.

An example of a blurred image (the right-hand side B):



The matrix equation $A_C X A_R^T = B$ can be rewritten in the common (linear system) form

$$A\,x \;=\; b\,,$$

where

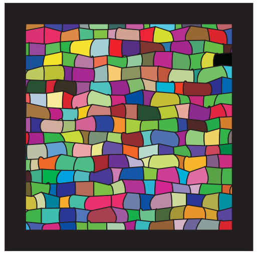
$$A = A_R \otimes A_C\,, \qquad A \in \mathcal{R}^{hw\times hw}\,,$$

and

$$x = \mathrm{vec}\,(X)\,, \qquad b = \mathrm{vec}\,(B)\,, \qquad x\,,b \in \mathcal{R}^{hw}\,,$$

symbol $\otimes$ denotes the **Kronecker product**.
Vectors x and b store the matrices X, respectively B, column by column.

Kronecker product $A = A_R \otimes A_C$:

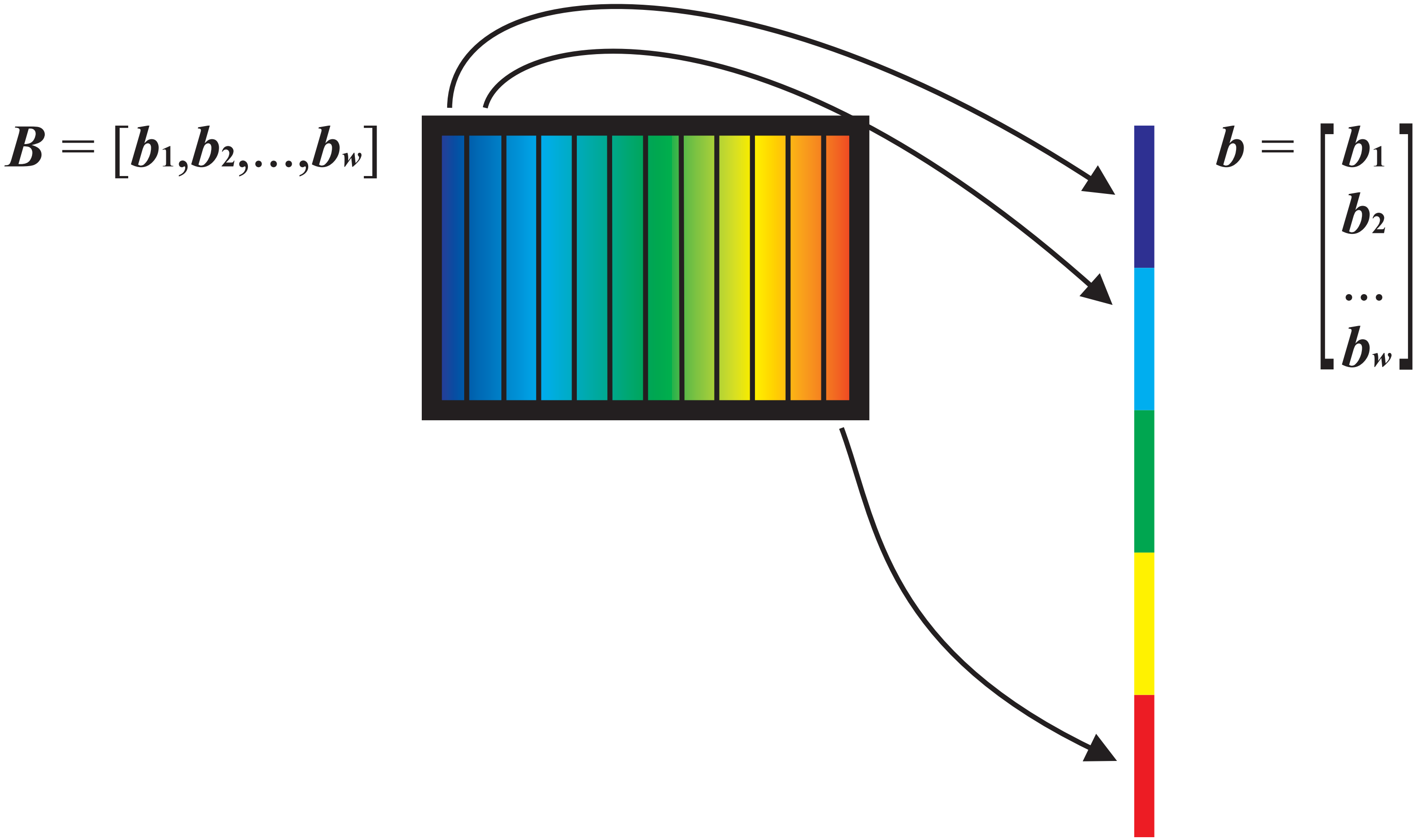
$A_C =$ 

$A_R =$ 

$$A = A_R \otimes A_C =$$

$$= \left[\begin{array}{c|c} a_{R,11}\, A_C & a_{R,12}\, A_C \\ \hline a_{R,21}\, A_C & a_{R,22}\, A_C \end{array} \right] =$$

Right-hand side reshaping $B \rightarrow b = \mathrm{vec}\,(B)$**:**



A **naive solution**

$$x_{\,\mathrm{naive}} \;=\; A^{-1}b \;=\; \sum_{i=1}^{hw} \left(\frac{u_i^T b}{\sigma_i} \right) v_i$$

does not give the original image (preimage). Why?

The right-hand side b (B) contains noise, it can be split as

$$b \;=\; b_{\,\mathrm{exact}} + b_{\,\mathrm{noise}}\,,$$

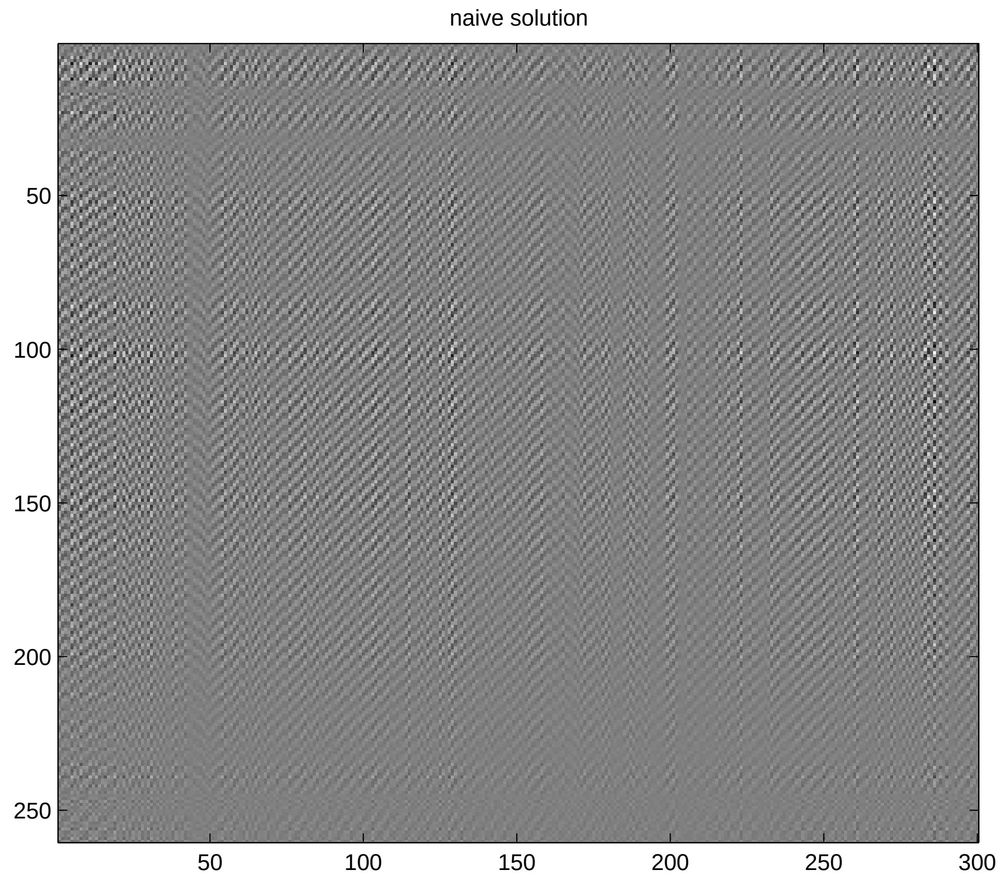
and therefore

$$x_{\,\mathrm{naive}} \;=\; A^{-1}b \;=\; x_{\,\mathrm{exact}} \;+\; A^{-1}b_{\,\mathrm{noise}}\,,$$

where often

$$\|x_{\,\mathrm{exact}}\| \;\ll\; \|A^{-1}b_{\,\mathrm{noise}}\|\,.$$

The naive solution $x_{\mathrm{naive}} = A^{-1}b$ (reshaped to the original size):



Discrete Picard condition:

If the image resolution is refined, the smallest singular values of A approach zero.

Continous Picard condition says: with increasing i, the projections $u_i^T b$ decay faster than the singular values σ_i . Expression

$$\lim_{h,w \to \infty} \left\{ \sum_{i=1}^{hw} \left(\frac{u_i^T b_{\text{exact}}}{\sigma_i} \right) v_i \right\}$$

represents in the limit a real continous image, and therefore

$$\lim_{i \to \infty} \left(\frac{u_i^T b_{\text{exact}}}{\sigma_i} \right) = 0 .$$

But the noise is random,
noise need not (and it does not!) satisfy the Picard condition.

It means, that the sum

$$x_{\text{naive}} = A^{-1}b = \sum_{i=1}^{hw} \left(\frac{u_i^T b}{\sigma_i} \right) v_i$$

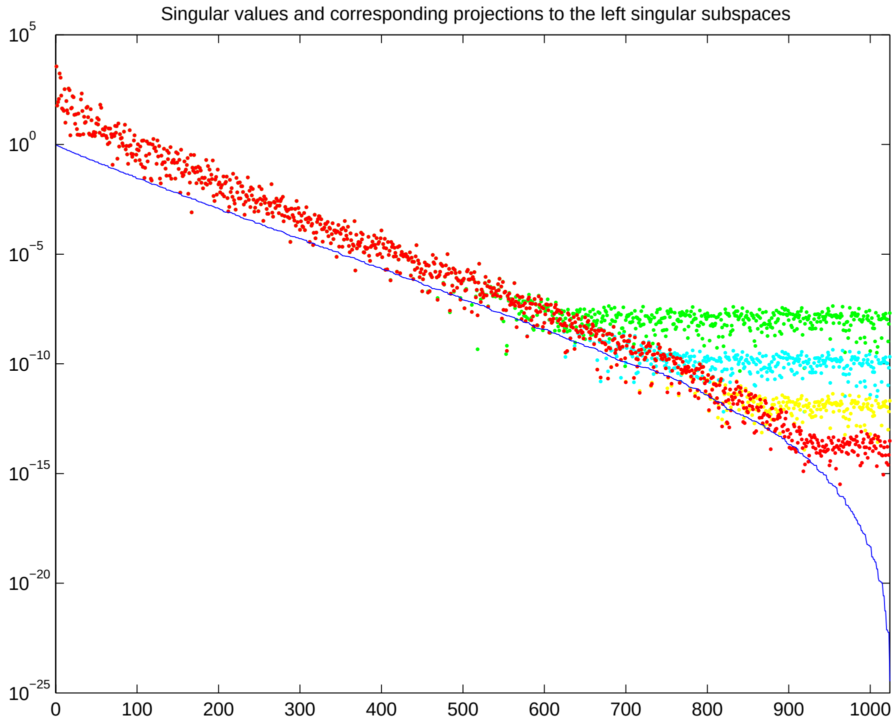
has the wanted, well bounded part x_{exact}, and the unwanted, randomly influenced, in the limit possibly unbounded part $A^{-1}b_{\text{noise}}$.

The domination of the second part

$$\|x_{\text{exact}}\| \ll \|A^{-1}b_{\text{noise}}\|$$

is the consequence of violating the Picard condition.

Singular values σ_i of A and projections $u_i^T b$:



Right-hand sides with different noise levels.

4. Regularization tools

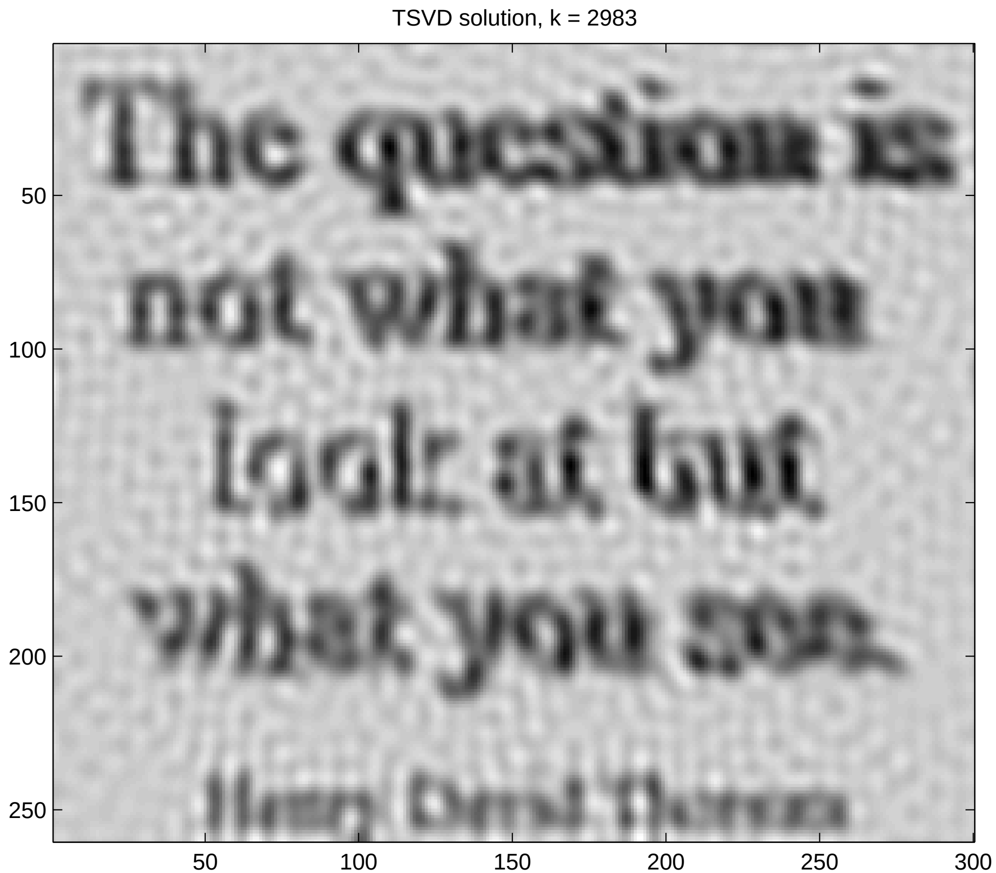
Truncated SVD regularized solution is given by

$$x_k = S_k^\dagger b = \sum_{i=1}^{k} \left(\frac{u_i^T b}{\sigma_i} \right) v_i \,,$$

where S_k is for the given k the best rank-k approximation of A.
An optimal k depends on the matrix properties and on the noise level.

The dyadic SVD form of A is truncated after the first k terms, which leads to the minimum norm least squares solution for the modified problem.

The TSVD solution (reshaped to the original size):



The most widely used method is **Tikhonov regularization**.
The Tikhonov solution is given by the minimization problem

$$x_\lambda = \arg\min_x \left\{ \|b - Ax\|_2^2 + \lambda^2 \|x\|_2^2 \right\},$$

or, equivalently, by solving the least squares problem

$$\begin{bmatrix} A \\ \lambda I \end{bmatrix} x \approx \begin{bmatrix} b \\ 0 \end{bmatrix},$$

for a given parameter $\lambda > 0$ (depending on the noise).

We will see the Tikhonov regularized solution of the given example at the end of this presentation.

Regularization by the spectral filtering:

Truncated SVD and Tikhonov regularization belong to a wide class of regularization methods, called spectral filtering.

These methods generate solution in the form

$$x_{\text{filt}} = \sum_{i=1}^{hw} \phi(i) \left(\frac{u_i^T b}{\sigma_i} \right) v_i \,,$$

where $\phi(i)$ is some **filter function**.

TSVD filter function:

$$\phi_{\mathrm{TSVD}}(i) = \left\{ \begin{array}{ll} 1 & \text{if } i \leq k \\ 0 & \text{if } i > k \end{array} \right\} \quad \text{for the given } k\,.$$

Tikhonov filter function:

$$\phi_{\mathrm{Tikhonov}}(i) = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \quad \text{for the given } \lambda\,.$$

Of course, the naive solution can be expressed as the filtered solution with the filter function $\phi_{\mathrm{naive}}(i) = 1$.

Comment: solution reshaping

All of the above-mentioned solutions x (naive or regularized) represent in fact images. Please recall that x is a linear combinations of the right singular vectors v_i .

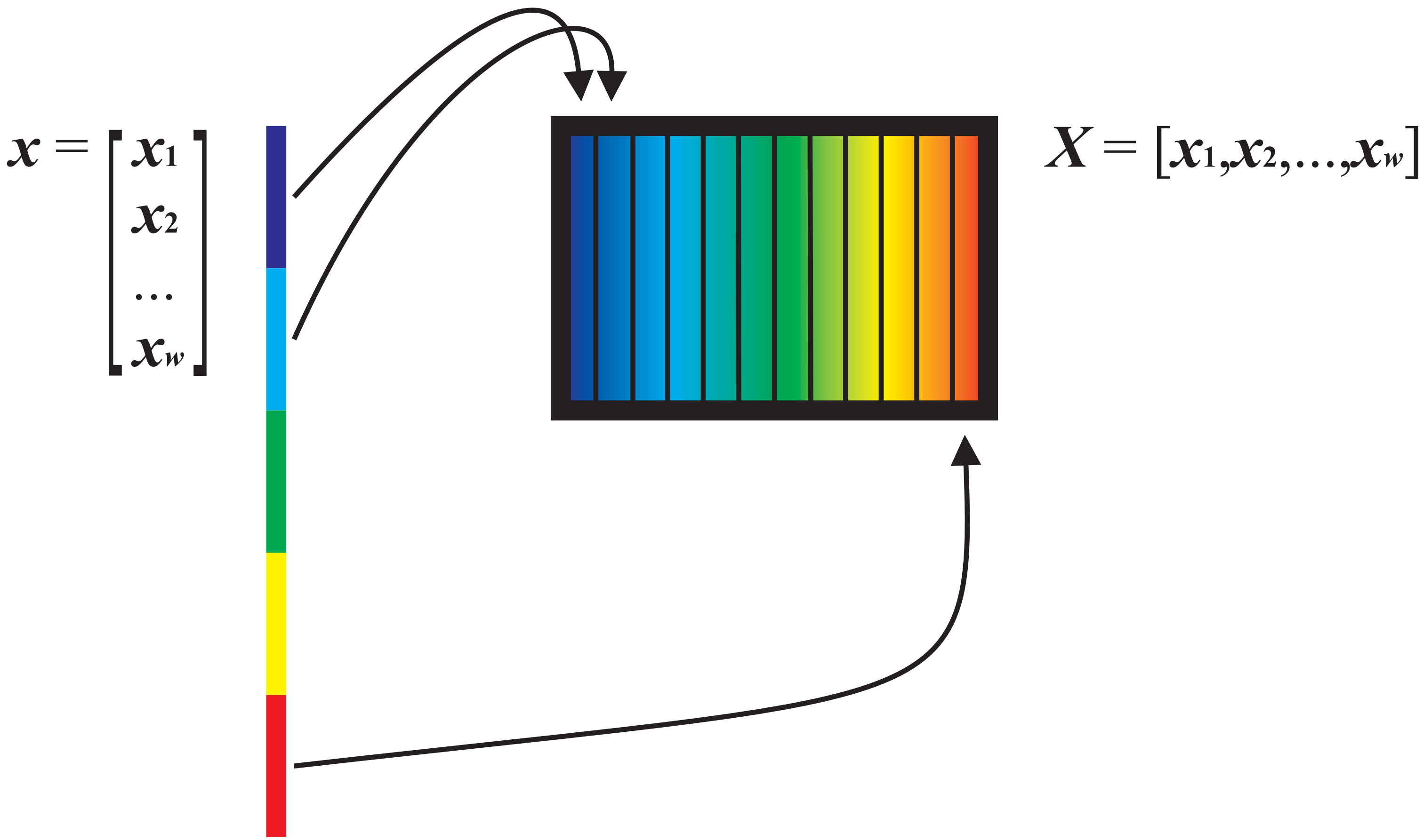
If we reshape $x \in \mathcal{R}^{hw}$ into the original form, $X \in \mathcal{R}^{h \times w}$,
we can rewrite the expression for filtered solution

$$X_{\text{filt}} = \sum_{i=1}^{hw} \phi(i) \left(\frac{u_i^T b}{\sigma_i} \right) V_i \,,$$

where

$V_i \in \mathcal{R}^{h \times w}$ are something like "**base images**" .

Solution reshaping $x = \mathrm{vec}\,(X) \;\rightarrow\; X$:



6. Examples in MATLAB

1st show:

- right hand side B,
- naive solution x_{naive},
- singular values of A and projections $u_i^T b$,
- TSVD filter and solution X_k.
- Tikhonov filter and solution X_λ

2nd show:

- base images V_i,
- progressive TSVD solution X_i, $i = 1, \ldots, hw$.

References

(01) Gene H. Golub, Charles F. van Loan,
Matrix Computations,
Baltimore, The Johns Hopkins University Press, 1983, 1989, 1996.

(02) Sabine Van Huffel, Joos Vandewalle,
The Total Least Squares Problem,
SIAM, 1991.

(03) Per Christian Hansen,
Rank-Deficient and Discrete Ill-Posed Problems,
SIAM, 1998.

(04) Per Christian Hansen, James Nagy, Diana O'Leary,
Deblurring Images – Matrices, Spectra and Filtering,
SIAM, 2006.

(05) C. C. Paige, Zdeněk Strakoš,
Core problem in linear algebraic systems,
SIAM Journal on Matrix Analysis and Appllications, 2005.

THANK YOU
FOR YOUR ATTENTION