A posteriori error estimates for hierarchical bases

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Abstract

A new a posteriori energy based error estimates were derived. They approximate known estimates but they are more suitable for the two- or multi-level iterative computing algorithms dealing with hierarchical finite element methods. The constants needed for the estimates are computed for the case of p-hierarchical space of piecewise bilinear and biquadratic basis functions. Several examples are introduced.

1. A posteriori estimates for elliptic problems and hierarchical bases

We present a basic comparison of the a posteriori error estimates for elliptic problems when considering two different types of hierarchical refinements. We deal with a h-hierarchical piecewise bilinear basis and with a p-hierarchical basis of piecewise quadretic functions on rectangles. The operator is a generalized laplacian with homogenous boundary conditions. Moreover, we introduce a new type of the estimate which exploits the quantities computed within two-level iterative algorithms.

It is to find $\hat{u}$ in some Hilbert space H, such that

$$a(\hat{u}, v) = f(v) \tag{1}$$

for all $v \in H$.

Bilinear form $a(.,.)$ is elliptic and positive definite in H, $f(.)$ is a linear functional in H. Let the energy norm be

$$|||v||| = \sqrt{a(v,v)}.$$

Let U_h be a finite-dimensional subspace in H and let U_h be generated by a set of finite element basis functions characterized by h. Let us choose some larger space V_h, $U_h \subset V_h \subset H$.

The approximate solution v_h is defined by

$$a(v_h, v) = f(v), \tag{2}$$

$v \in V_h$.

Let the saturation assumption

$$|||\hat{u} - v_h||| \leq \beta |||\hat{u} - u_h|||$$

be valid. We assume a hierarchical decomposition of V_h

$$V_h = U_h \oplus W_h$$

and the strengthened Cauchy - Bunyakowski - Schwarz (CBS) inequality

$$|a(u,w)| \leq \gamma |||u|||\,|||w|||$$

for all $u \in U_h$, $w \in W_h$, where γ is less than one and independent on h.

The energy norm of the error $\hat{u} - u_h$ in W_h can be esimated by the energy norm of e_h such that

$$a(e_h, w) = f(w) - a(u_h, w)$$

for all $w \in W_h$.

Theorem 1. [2] We have

$$|||e_h|||^2 \leq |||\hat{u} - u_h|||^2 \leq \frac{1}{(1-\beta^2)(1-\gamma^2)} |||e_h|||^2.$$

The solution of (2) $v_h \in V_h$ can be decomposed uniquely into

$$v_h = \bar{u}_h + \bar{w}_h,$$

where $\bar{u}_h \in U_h$ and $\bar{w}_h \in W_h$. The preconditioning matrix for the iterative two-level method may be as follows

$$\begin{pmatrix} A_w & 0 \\ A_{uw} & S \end{pmatrix} \begin{pmatrix} I & A_w^{-1} A_{wu} \\ 0 & I \end{pmatrix},$$

where

$$A = \begin{pmatrix} A_w & A_{wu} \\ A_{uw} & A_u \end{pmatrix}$$

is a stiffness matrix of (2) decoupled accordingly to the splitting of V_h into W_h and U_h and S is an approximation of $A_u - A_{uw} A_w^{-1} A_{wu}$.

Our aim is to find the relations among the energy norms of e_h, $\hat{u} - u_h$, and some other expressions which may arise (or at least their approximate values) during the multilevel iterative solution processes. We get the following theorem.

Theorem 2. Under the introduced assumptions, we get

$$(1-\gamma^2)|||w_h||| \leq |||e_h|||,$$

$$|||w_h||| - \gamma |||u_h - \bar{u}_h||| \leq |||e_h|||,$$

and

$$|||e_h||| \leq |||w_h||| + \gamma |||u_h - \bar{u}_h|||.$$

As a probably more interesting and useful topic, the local a posteriori error estimates can be studied, e.g. [5].

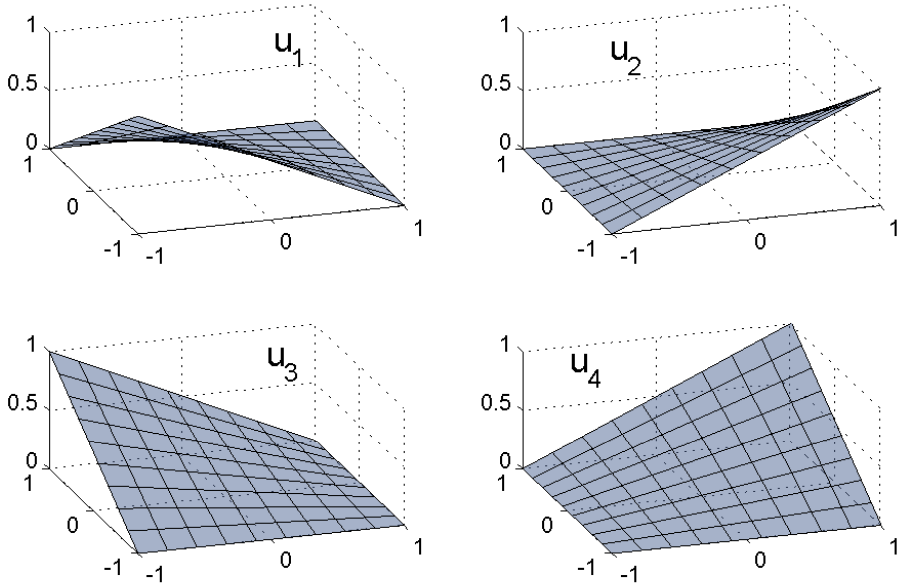

Figure 1: *Four basis functions of U_h on a macroelement.*

2. Comparison of the estimates for $h-$ and $p-$hierarchical bases for linear and bilinear finite element functions

We consider two different hierarchical finite element function spaces

$$V_h^l = U_h \oplus W_h^l \quad \text{and} \quad V_h^q = U_h \oplus W_h^q.$$

The space U_h consists of piecewise bilinear functions with rectangular supports, see Figure 1. The space W_h^l includes complementary bilinear functions with smaller supports (of the size of a quarter of the coarse ones, $W_h^l \subset U_{\frac{h}{2}}$), while the space W_h^q involves piecewise polynomial functions of the second order [1], see Figure 2. The numbers of the degrees of freedom for finite elements corresponding to V_h^l and to V_h^q are equal.

The saturation constants β_l and β_q can be substituted by the quantities $\frac{1}{4}$ and h for spaces V_h^l and to V_h^q, respectively. The CBS constants γ_l and γ_q are

$$\gamma_l = \frac{\sqrt{3}}{2} \quad \text{and} \quad \gamma_q = \frac{5}{6},$$

respectively, when $a(.,.)$ is a generalized laplacian and its coefficients are positive and piecewise constant on the coarse elements. When the operator $a(.,.)$ and the functions correspond to the izotropic problem, the constants are

$$\gamma_l = \frac{1}{2} \quad \text{and} \quad \gamma_q = \frac{5}{11}.$$

Remark. As we can see from [4], we have

$$\gamma_l = \frac{\sqrt{3}}{2}\gamma_q$$

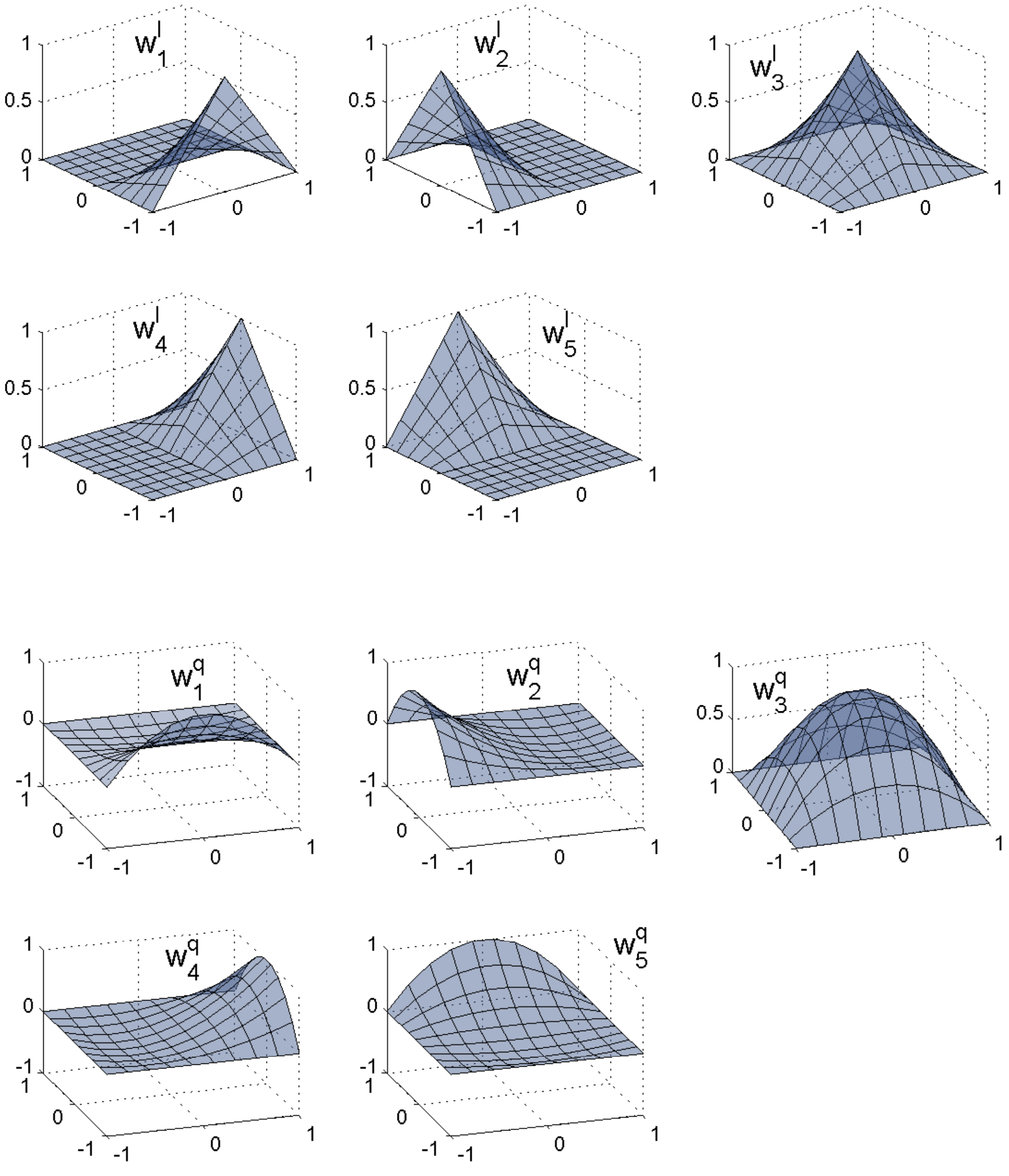

Figure 2: *Five basis functions* $w_1^l, \ldots, w_5^l$ *of* W_h^l *and five basis functions* $w_1^q, \ldots, w_5^q$ *of* W_h^q*, the* h *and* p *refinements of* U_h *on a macroelement.*

for the uniform estimates of the CBS constants for linear finite element functions with triangular supports and their refinements with either h-hierarchical functions (γ_l) or p-hierarchical (quadratic) functions (γ_q). Using the results of [3], this trivially yields $\gamma_q = 1$ for this type of elements.

As introduced in this part, different relations are obtained in the case of rectangular supported finite element functions. The relations between γ_l and γ_q are

$$\gamma_l = \frac{3\sqrt{3}}{9}\gamma_q \quad \text{and} \quad \gamma_l = \frac{5\sqrt{3}}{9}\gamma_q$$

for the anisotropic problems and isotropic problems, respectively.

3. Examples

According to Theorems 1 and 2, we observe the following five quantities

$$q_1 = |||w_h||| - \gamma|||u_h - \bar{u}_h|||, \quad q_2 = |||e_h|||, \quad q_3 = |||\hat{u} - u_h|||,$$

$$q_4 = \frac{|||e_h|||}{\sqrt{(1-\beta^2)(1-\gamma^2)}}, \quad q_5 = \frac{|||w_h||| + \gamma|||u_h - \bar{u}_h|||}{\sqrt{(1-\beta^2)(1-\gamma^2)}},$$

which are ordered according to the introduced formulas,

$$q_1 \leq q_2 \leq q_3 \leq q_4 \leq q_5.$$

We are interested in the accuracy of the introduced estimates. Particularly we observe how q_2 and q_4 are approximated by q_1 and q_5, respectively. In Test 1 the solution is a polynomial of a low order, while in Test 2, the solution has a sharp extremum of the form $1/((a-x)^2(b-y)^2+\delta)$.

Table 1: Error estimates for the h-hierarchical basis in Test 1.

dof	q_1	q_2	q_3	q_4	q_5
15^2	0.0082	0.0084	0.0107	0.0122	0.0125
30^2	0.0040	0.0040	0.0051	0.0059	0.0060
60^2	0.0020	0.0020	0.0025	0.0029	0.0029

Table 2: Error estimates for the p-hierarchical basis in Test 1.

dof	q_1	q_2	q_3	q_4	q_5
15^2	0.0102	0.0106	0.0107	0.0144	0.0150
30^2	0.0050	0.0051	0.0051	0.0070	0.0071
60^2	0.0025	0.0025	0.0025	0.0034	0.0035

Table 3: Error estimates for an h-hierarchical basis in Test 2.

dof	q_1	q_2	q_3	q_4	q_5
15^2	0.8352	0.9432	1.1188	1.9482	2.2482
30^2	0.4351	0.4605	0.5406	0.9512	1.0117
60^2	0.2200	0.2271	0.2660	0.4691	0.4848

Table 4: Error estimates for the p-hierarchical basis in Test 2.

dof	q_1	q_2	q_3	q_4	q_5
15^2	1.0423	1.1004	1.1188	2.6956	2.8569
30^2	0.5238	0.5386	0.5406	1.3192	1.3578
60^2	0.2620	0.2657	0.2660	0.6508	0.6601

The lower a posteriori estimates obtained by using the space V_h^q are more accurate compared to the estimates which use V_h^l. This fact is partly compensated by a greater density of the stiffness matrix and by a worse conditioning of the diagonal block that corresponds to the finer space. Nevertheless, the quantities q_2 and q_4 are well approximated by q_1 and q_5, respectively, when using both of the spaces V_h^l and V_h^q.

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