Numerical behavior of inexact saddle point solvers

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We consider a saddle point problem with the symmetric $2 \times 2$ block form

$$
\begin{pmatrix}
A & B \\
B^T & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
f \\
0
\end{pmatrix}.
$$

- $A$ is a square $n \times n$ nonsingular (symmetric positive definite) matrix,
- $B$ is a rectangular $n \times m$ matrix of (full column) rank $m$.

Applications: mixed finite element approximations, weighted least squares, constrained optimization etc. [Benzi, Golub, and Liesen, 2005].
inexact solutions of inner systems + rounding errors → inexact saddle point solver

- inexact method
- exact method

maximum attainable accuracy

error / residual

iteration number
• Compute $y$ as a solution of the Schur complement system

$$B^T A^{-1} By = B^T A^{-1} f,$$

• compute $x$ as a solution of

$$Ax = f - By.$$ 

Systems with $A$ are solved inexactly, the computed solution $\bar{u}$ of $Au = b$ is interpreted an exact solution of a perturbed system

$$(A + \Delta A)\bar{u} = b + \Delta b, \quad \|\Delta A\| \leq \tau \|A\|, \quad \|\Delta b\| \leq \tau \|b\|, \quad \tau \kappa(A) \ll 1.$$
choose \( y_0 \), solve \( Ax_0 = f - By_0 \)

compute \( \alpha_k \) and \( p_k(y) \)

\[
y_{k+1} = y_k + \alpha_k p_k(y)
\]

solve \( Ap_k(x) = -Bp_k(y) \)

**back-substitution:**

**A:** \( x_{k+1} = x_k + \alpha_k p_k(x) \),

**B:** solve \( Ax_{k+1} = f - B y_{k+1} \),

**C:** solve \( Au_k = f - Ax_k - By_{k+1} \),

\[
x_{k+1} = x_k + u_k.
\]

\[
r_{k+1}^{(y)} = r_k^{(y)} - \alpha_k B^T p_k(x)
\]
The limiting (maximum attainable) accuracy is measured by the ultimate (asymptotic) values of:

1. the Schur complement residual: \( B^T A^{-1} f - B^T A^{-1} B y_k \);
2. the residuals in the saddle point system: \( f - A x_k - B y_k \) and \( -B^T x_k \);
3. the forward errors: \( x - x_k \) and \( y - y_k \).

**Numerical example:**

\[ A = \text{tridiag}(1, 4, 1) \in \mathbb{R}^{100 \times 100}, \quad B = \text{rand}(100, 20), \quad f = \text{rand}(100, 1), \]

\[ \kappa(A) = \|A\| \cdot \|A^{-1}\| = 7.1695 \cdot 0.4603 \approx 3.3001, \]

\[ \kappa(B) = \|B\| \cdot \|B^\dagger\| = 5.9990 \cdot 0.4998 \approx 2.9983. \]
Accuracy in the outer iteration process

\[ B^T (A + \Delta A)^{-1} B \hat{y} = B^T (A + \Delta A)^{-1} f, \]

\[ \| B^T A^{-1} f - B^T A^{-1} B \hat{y} \| \leq \frac{\tau \kappa(A)}{1 - \tau \kappa(A)} \| A^{-1} \| \| B \| ^2 \| \hat{y} \|. \]

\[ \| - B^T A^{-1} f + B^T A^{-1} By_k - r_k^{(y)} \| \leq \frac{O(\tau) \kappa(A)}{1 - \tau \kappa(A)} \| A^{-1} \| \| B \| (\| f \| + \| B \| Y_k). \]
\[-B^T A^{-1} f + B^T A^{-1} B y_k = -B^T x_k - B^T A^{-1} (f - A x_k - B y_k)\]

\[
\| f - A x_k - B y_k \| \leq \frac{O(\alpha_1) \kappa(A)}{1 - \tau \kappa(A)} (\| f \| + \| B \| Y_k),
\]

\[
\| -B^T x_k - r_k^{(y)} \| \leq \frac{O(\alpha_2) \kappa(A)}{1 - \tau \kappa(A)} \| A^{-1} \| \| B \| (\| f \| + \| B \| Y_k),
\]

\[
Y_k \equiv \max\{\| y_i \| \mid i = 0, 1, \ldots, k\}.
\]

<table>
<thead>
<tr>
<th>Back-substitution scheme</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
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</thead>
<tbody>
<tr>
<td><strong>A:</strong> Generic update</td>
<td>$\tau$</td>
<td>$u$</td>
</tr>
<tr>
<td>$x_{k+1} = x_k + \alpha_k p_k^{(x)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>B:</strong> Direct substitution</td>
<td>$\tau$</td>
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<td>$x_{k+1} = A^{-1} (f - B y_{k+1})$</td>
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<tr>
<td><strong>C:</strong> Corrected dir. subst.</td>
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<td>$\tau$</td>
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additional system with $A$
Generic update: $x_{k+1} = x_k + \alpha_k p_k(x)$
Direct substitution: $x_{k+1} = A^{-1}(f - By_{k+1})$
Corrected direct substitution: \( x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1}) \)
Forward error of computed approximate solution

\[ \| x - x_k \| \leq \gamma_1 \| f - Ax_k - By_k \| + \gamma_2 \| - B^T x_k \|, \]
\[ \| y - y_k \| \leq \gamma_2 \| f - Ax_k - By_k \| + \gamma_3 \| - B^T x_k \|, \]

\[ \gamma_1 = \sigma_{\min}^{-1}(A), \quad \gamma_2 = \sigma_{\min}^{-1}(B), \quad \gamma_3 = \sigma_{\min}^{-1}(B^T A^{-1} B). \]
Conclusions

- All bounds of the limiting accuracy depend on the maximum norm of computed iterates, cf. [Greenbaum, 1997].
- The accuracy measured by the residuals of the saddle point problem depends on the choice of the back-substitution scheme [J, R, 2008].
- Care must be taken when solving nonsymmetric systems [J, R, 2008b].

The residuals in the outer iteration process and the forward errors of computed approximations are proportional to the backward error in solution of inner systems.
Thank you for your attention.

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Null-space projection method

- compute $x \in N(B^T)$ as a solution of the projected system
  \[(I - \Pi)A(I - \Pi)x = (I - \Pi)f,\]
- compute $y$ as a solution of the least squares problem
  \[By \approx f - Ax,\]

$\Pi$ is the orthogonal projector onto $R(B)$.

The least squares with $B$ are solved inexactly, i.e. the computed solution $\bar{v}$ of $Bv \approx c$ is an exact solution of a perturbed least squares problem

\[(B + \Delta B)\bar{v} \approx c + \Delta c, \quad \|\Delta B\| \leq \tau\|B\|, \quad \|\Delta c\| \leq \tau\|c\|, \quad \tau\kappa(B) \ll 1.\]
choose $x_0$, solve $By_0 \approx f - Ax_0$

compute $\alpha_k$ and $p_k(x) \in N(B^T)$

$x_{k+1} = x_k + \alpha_k p_k(x)$

<table>
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<tr>
<th>solve $Bp_k(y) \approx r_k(x) - \alpha_k Ap_k(x)$</th>
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<td>back-substitution:</td>
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<td><strong>A:</strong> $y_{k+1} = y_k + p_k(y)$,</td>
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<tr>
<td><strong>B:</strong> solve $By_{k+1} \approx f - Ax_{k+1}$,</td>
</tr>
<tr>
<td><strong>C:</strong> solve $Bv_k \approx f - Ax_{k+1} - By_k$,</td>
</tr>
<tr>
<td>$y_{k+1} = y_k + v_k$.</td>
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</tbody>
</table>

$r_{k+1}^{(x)} = r_k^{(x)} - \alpha_k Ap_k(x) - Bp_k(y)$
Accuracy in the saddle point system

\[ \| f - Ax_k - By_k - r_k^{(x)} \| \leq \frac{O(\alpha_3)\kappa(B)}{1 - \tau\kappa(B)} (\| f \| + \| A \| X_k), \]

\[ \| - B^T x_k \| \leq \frac{O(\tau)\kappa(B)}{1 - \tau\kappa(B)} \| B \| X_k, \]

\[ X_k \equiv \max\{\| x_i \| | i = 0, 1, \ldots, k\}. \]

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\{ additional least square with B \}
Generic update: $y_{k+1} = y_k + p_k^{(y)}$
Direct substitution: \( y_{k+1} = B^\dagger(f - Ax_k + 1) \)

\[
\begin{align*}
\tau &= O(u) \\
\tau &= 10^{-2} \\
\tau &= 10^{-6} \\
\tau &= 10^{-10}
\end{align*}
\]

Iteration number
Relative residual norms \( \|f - Ax_k - By_k\|/\|f - Ax_0 - By_0\| \), \( \|r(x)_k\|/\|r(x)_0\| \)
Corrected direct substitution: \[ y_{k+1} = y_k + B^\dagger (f - Ax_{k+1} - By_k) \]

