Generalized Eigenvalue Problem in FEM Modelling of the Resonance Frequencies Piezoelectric Resonators

P. Rálek

Department of Modelling of Processes,

Faculty of Mechatronics and Interdisciplinary Engineering Studies,

Technical University in Liberec, Czech Republic

SNA Ostrava 2005

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- resonance frequencies are subsequently found by solving this algebraic problem
- typically, we are not interested in all eigenvalues (resonance frequencies)
  $\Rightarrow$ for determining of several of them we consider iterative methods

Physical formulation

linear piezoelectric constitutive equations:

- generalized Hook's law

$$\mathrm{T_{ij}} = c_{ijkl}\,\mathrm{S_{kl}} - d_{kij}\,\mathrm{E_k}, \qquad i,j = 1,2,3, \quad (1)$$

equation of the direct piezoelectric effect

$$\mathrm{D}_k = d_{kij}\,\mathrm{S_{ij}} + \varepsilon_{kj}\,\mathrm{E_j}, \qquad k = 1,2,3. \quad (2)$$

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symmetric stress tensor $\mathbf{T}$, symmetric strain tensor $\mathbf{S}$,
vector of intensity of electric field $\mathbf{E}$, vector of electric displacement $\mathbf{D}$

c_{ijkl}, d_{kij}, ε_{ij} ... material tensors ($\mathbf{c}$ is symmetric in all four indices and PD, $\mathbf{d}$ is symmetric in last two indices and ε is symmetric and PD)

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$$\mathrm{S}_{ij} = \frac{1}{2}\left[\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right], \qquad \mathrm{E_k} = -\frac{\partial \tilde{\varphi}}{\partial x_k}, \qquad i, j, k = 1, 2, 3,$$

$\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^{\mathrm{T}}$ is the displacement vector and $\tilde{\varphi}$ is the electric potential

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- governing equation for piezoelectric continuum

$$\varrho \frac{\partial^2 \tilde{u}_i}{\partial t^2} = \frac{\partial \mathrm{T_{ij}}}{\partial x_j} \qquad i = 1, 2, 3, \qquad x \in \Omega, \quad t \in (0, \mathrm{T}), \quad (3)$$

$$\nabla . \, \mathbf{D} = \frac{\partial \mathrm{D_j}}{\partial x_j} = 0, \qquad (4)$$

with density ϱ, volume of the resonator Ω and its boundary Γ.

Physical formulation

- $(1) + (2) + (3) + (4) \Rightarrow$

$$\varrho \frac{\partial^2 \tilde{u}_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(c_{ijkl}\ \frac{1}{2}\left[\frac{\partial \tilde{u}_k}{\partial x_l} + \frac{\partial \tilde{u}_l}{\partial x_k}\right] + d_{kij}\ \frac{\partial \tilde{\varphi}}{\partial x_k}\right) \quad i = 1, 2, 3, \qquad (5)$$

$$0 = \frac{\partial}{\partial x_k}\left(d_{kij}\ \frac{1}{2}\left[\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right] - \varepsilon_{kj}\ \frac{\partial \tilde{\varphi}}{\partial x_j}\right). \qquad (6)$$

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- initial conditions, Dirichlet boundary conditions and Neumann boundary conditions are added:

$$\tilde{u}_i(.,0) = u_i,\ x \in \Omega,\ \ \tilde{u}_i = 0,\ x \in \Gamma_u,\ \ \mathrm{T_{ij}} n_j = f_i,\ x \in \Gamma_f,\ i = 1, 2, 3, \qquad (7)$$
$$\tilde{\varphi}(.,0) = \varphi,\ \tilde{\varphi} = \varphi_D,\ x \in \Gamma_\varphi,\ \ \mathrm{D_k} n_k = q,\ x \in \Gamma_q,$$

where

$$\Gamma_u \cup \Gamma_f = \Gamma,\ \Gamma_u \cap \Gamma_f = \emptyset,\ \Gamma_\varphi \cup \Gamma_q = \Gamma,\ \Gamma_\varphi \cap \Gamma_q = \emptyset.$$

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* standard weak formulation, H_0, using boundary conditions
* discretization using the linear Lagrange finite elements in 3D (with GMSH code)
* system of ODEs for unknown values of $\mathbf{u}$, φ in nodes of discretization results (our c++ code)

Discretization of the problem, BC

$$\begin{aligned} \mathrm{M}\ddot{\mathrm{U}} + \mathrm{K}\mathrm{U} + \mathrm{P}^{\mathrm{T}}\Phi &= \mathrm{F}, \\ \mathrm{P}\mathrm{U} - \mathrm{E}\Phi &= \mathrm{Q}. \end{aligned}$$

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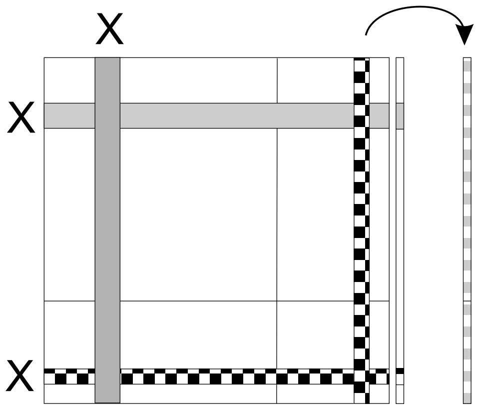

after introduction of Dirichlet boundary conditions, sub-matrices $\mathrm{M},\ \mathrm{K}$ and E are symmetric and positive definite

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$$\begin{pmatrix} \mathrm{K} - \omega^2 \mathrm{M} & \mathrm{P}^{\mathrm{T}} \\ \mathrm{P} & -\mathrm{E} \end{pmatrix} \begin{pmatrix} \mathrm{U} \\ \Phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

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  where ω is the frequency of oscillation
- eigenfrequencies can be computed by solving the generalized eigenvalue problem

$$AX = \lambda BX \quad (8)$$

  with

$$A = \begin{pmatrix} K & P^T \\ P & -E \end{pmatrix}, \; B = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}, \; \lambda = \omega^2,$$

  A being symmetric and B being symmetric and positive semi-definite matrix

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$$\mathrm{K}^{\star}\mathrm{U} = \lambda \mathrm{M}\mathrm{U}, \ \mathrm{K}^{\star} = \mathrm{K} - \mathrm{P}^{\mathrm{T}}\mathrm{E}^{-1}\mathrm{P}.$$

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- it losts the sparseness of the matrices; generalized Schur decomposition

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- what is the shift:

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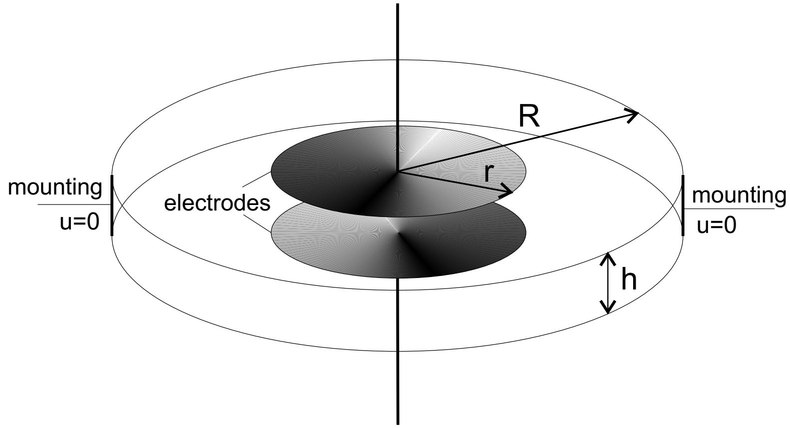
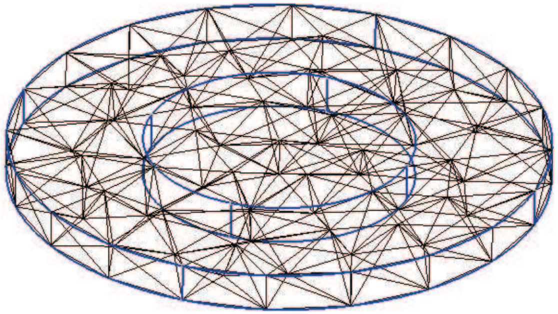
- what is the shift:
- we can focust to wanted part of the spectra - time and memory saving, higher precision

Oscillation of planparallel quartz resonator

- shear vibration mode in x direction





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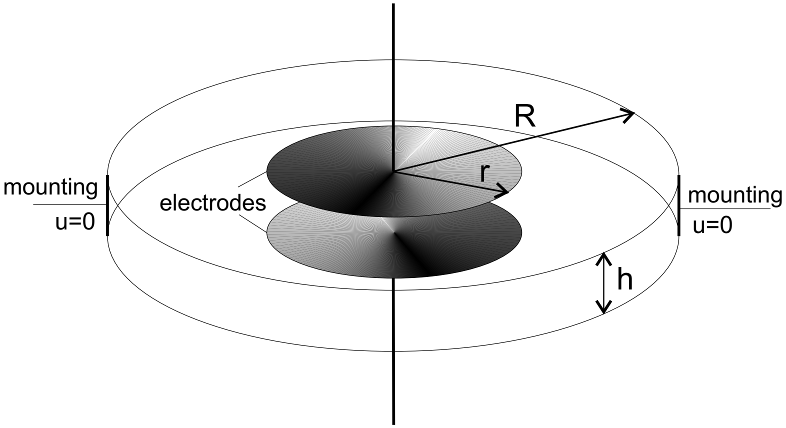

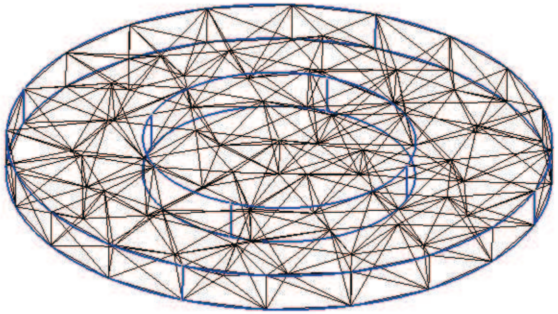

- three different samples

sample	R (mm)	r (mm)	h (mm)	res. freq.
1	7	3.5	0.333	5 MHz
2	3.975	2.5	0.168	10 MHz
3	3.475	1.5	0.0833	20 MHz

Results

- comparison of measured and computed resonance frequencies:

sample	measured res. frequency (kHz)	computed res. frequency (kHz)
1	5000.200	5080
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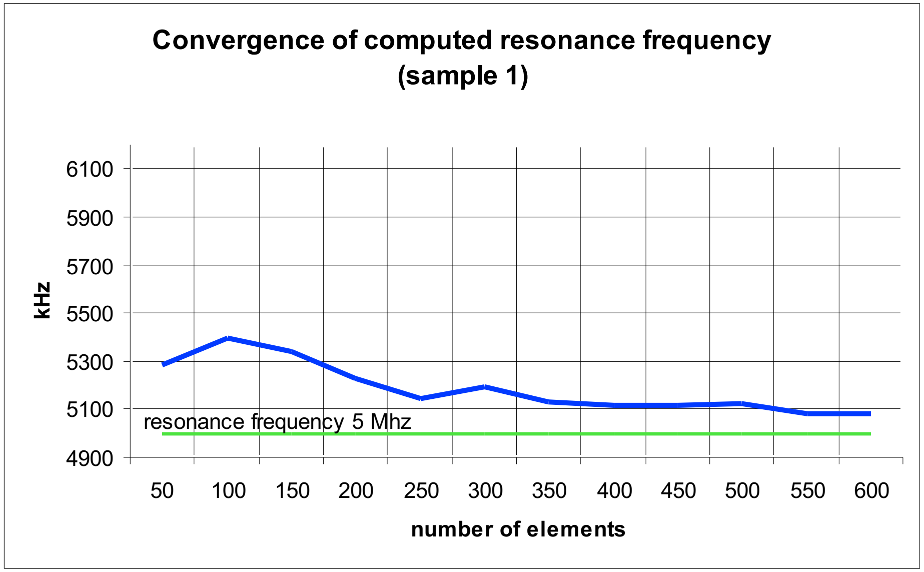

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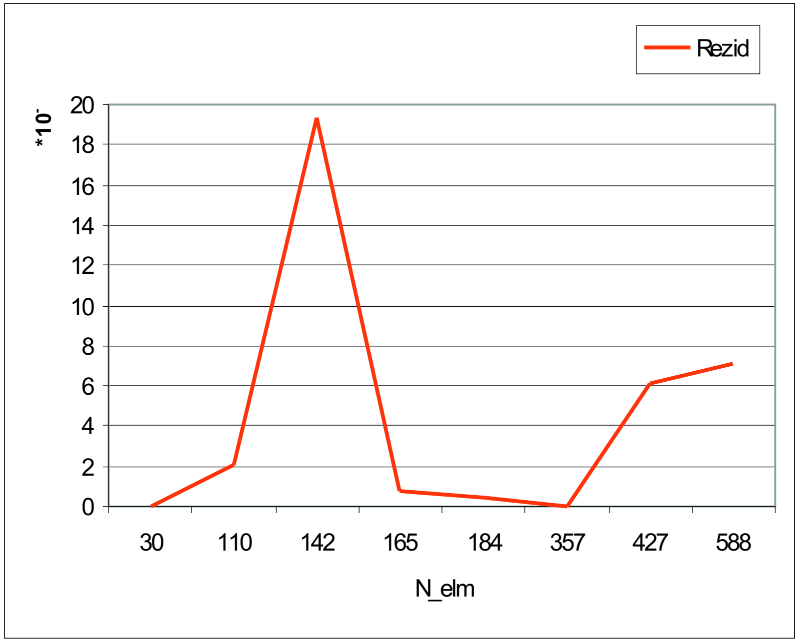

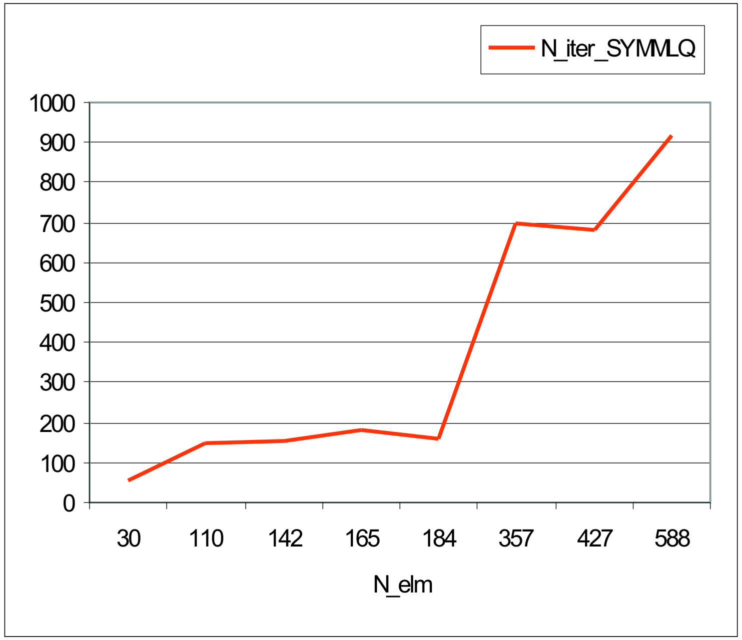

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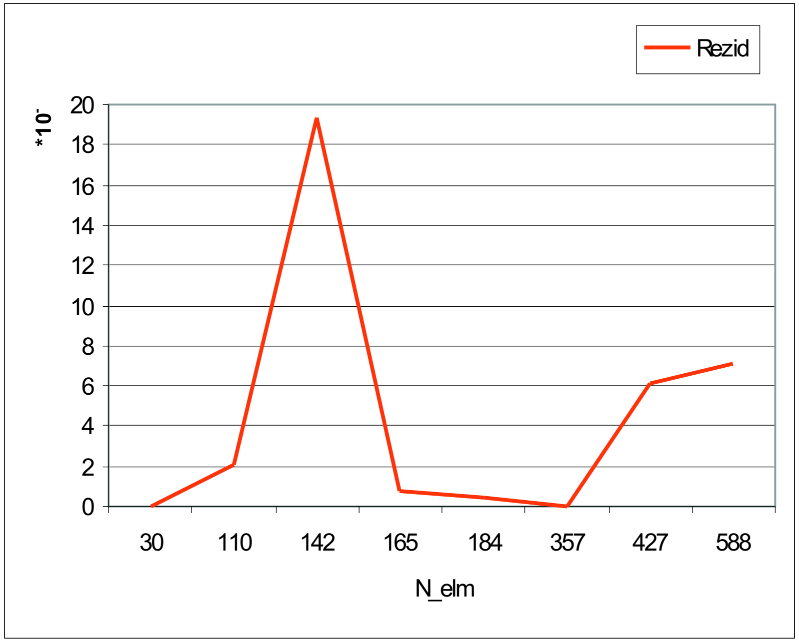

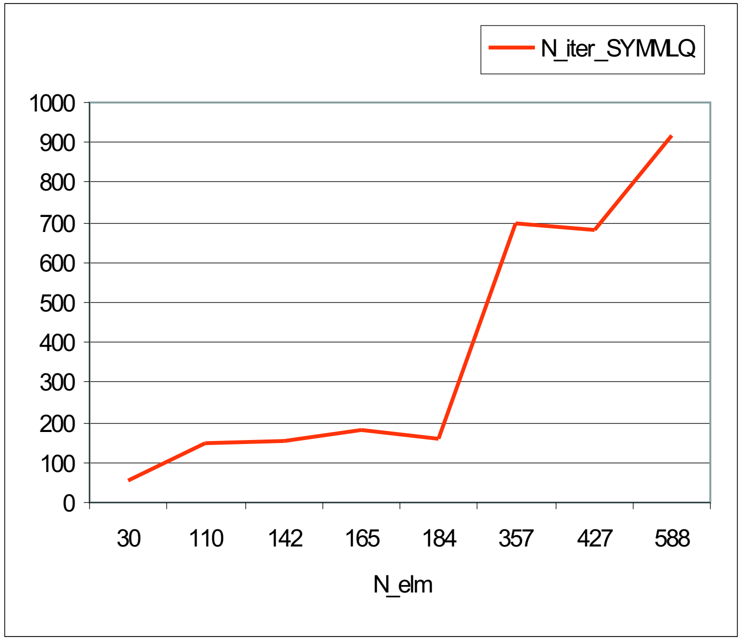

- rezidual about 10^{-13} in worst case

Postprocessing

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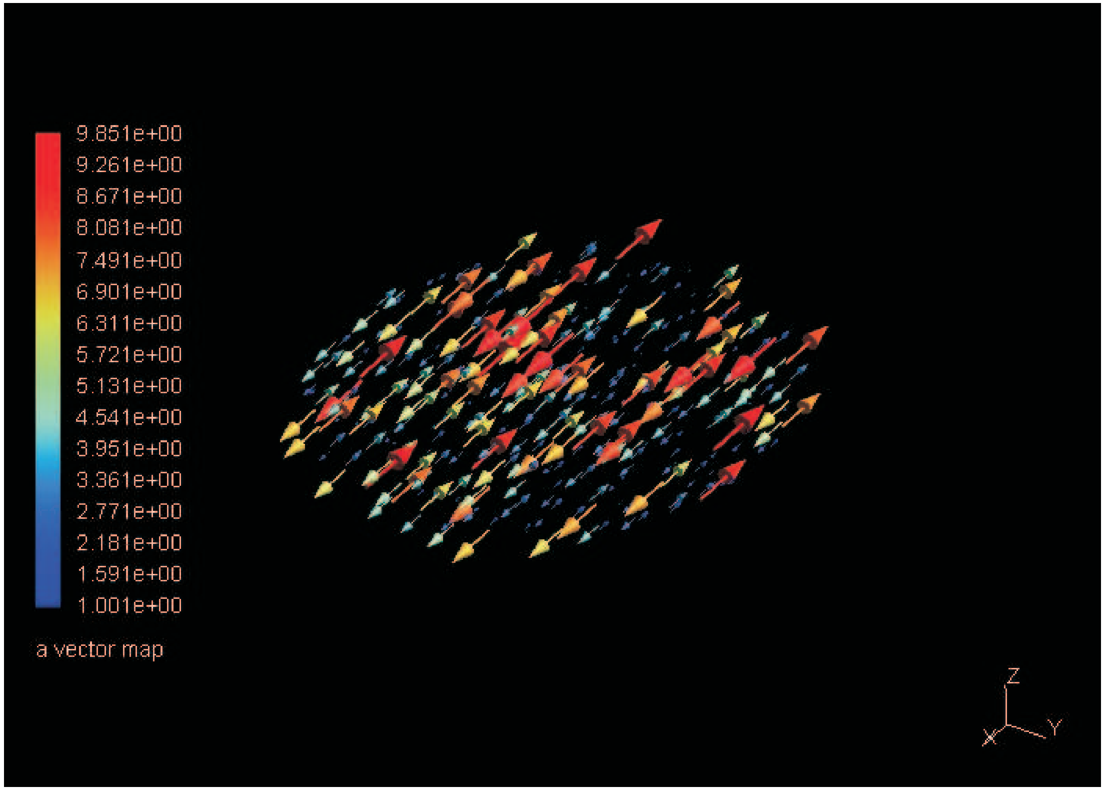

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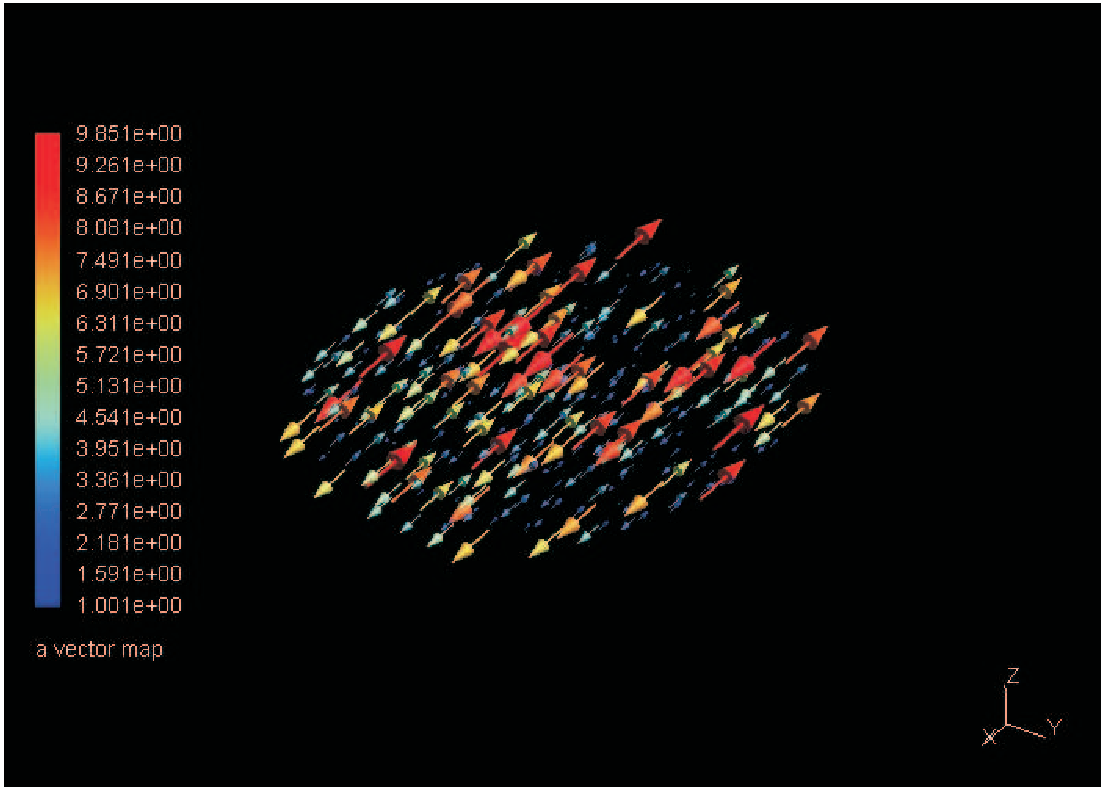

- problems with recognition of vibrational modes - which are the right?

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physicians usually know, what should result and why; focust on practical problems