DUAL VARIABLE APPROACH FOR MIXED-HYBRID FINITE ELEMENT APPROXIMATION OF THE POTENTIAL FLUID FLOW PROBLEM

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OUTLINE

1. POTENTIAL FLUID FLOW PROBLEM: SYMMETRIC INDEFINITE (AUGMENTED) SYSTEMS, PARTICULAR STRUCTURE OF THE PROBLEM: STRUCTURAL AND SPECTRAL PROPERTIES
2. NULL-SPACE (DUAL) APPROACH: NULL-SPACE BASIS, PROJECTION + PRECONDITIONING THE (INDEFINITE) PROJECTED SYSTEM
3. NUMERICAL RESULTS, CONCLUSIONS

Potential fluid flow problem

Darcy's law, continuity equation in Ω:

$$\mathbf{u} \;=\; -\mathbf{A}\;\nabla p,$$

$$\nabla \cdot \mathbf{u} \;=\; q.$$

Dirichlet and Neumann boundary conditions on $\partial\Omega$:

$$p = p_D \quad \text{on } \partial\Omega_D,$$

$$-\mathbf{n}.(\mathbf{A}\nabla p) = \mathbf{n}.\mathbf{u} = u_N \quad \text{on } \partial\Omega_N.$$

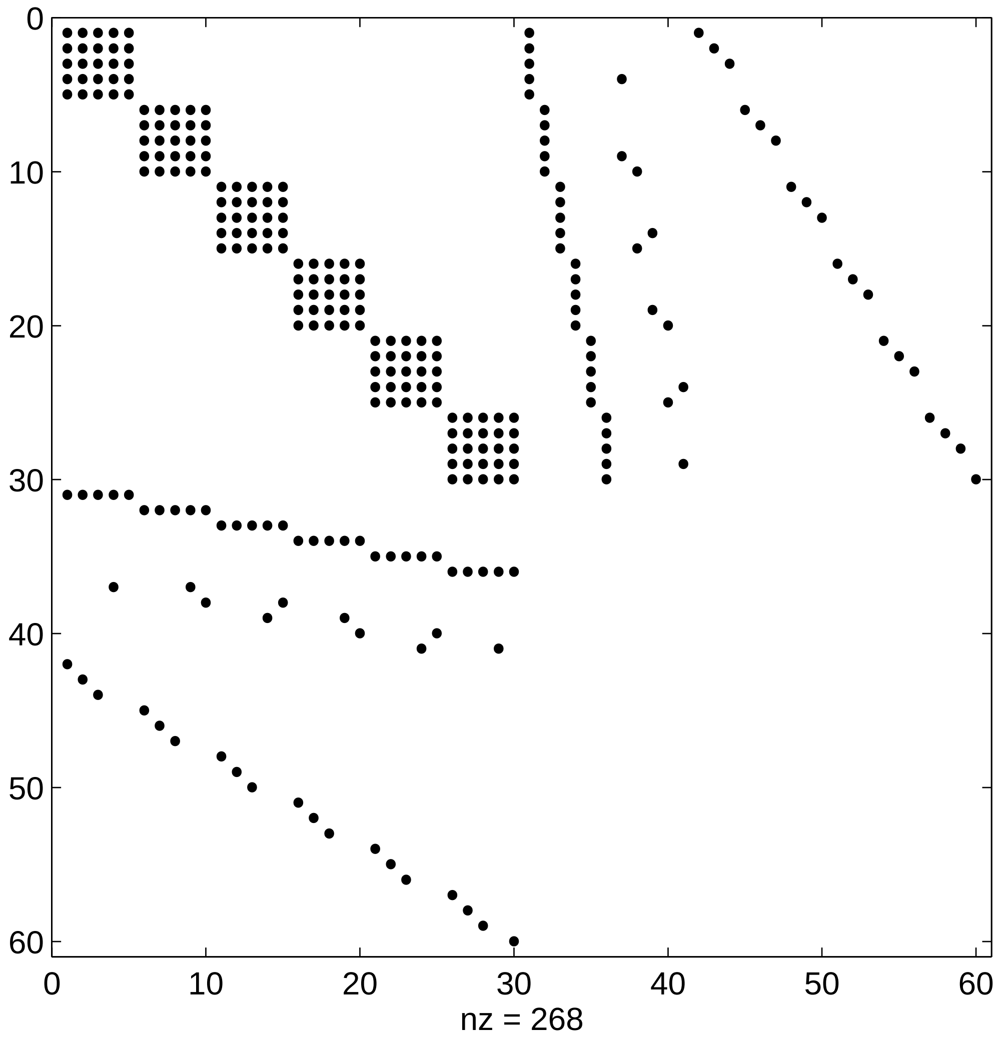
Discretization, hybrid formulation and matrix properties

- Low(est)-order Raviart-Thomas elements: $\mathbf{p}$ is element-wise constant, $\mathbf{u}$ is element-wise linear
- hybrid formulation: Lagrange multipliers λ, natural condensation of unknowns corresponding to non-Dirichlet faces
- larger, but more transparent system matrix, enables aposteriori updates in the matrix

Mixed-hybrid finite element discretization: large structured symmetric indefinite systems

$$\begin{pmatrix} A & B & (C_1\ C_2) \\ B^T & & \\ (C_1\ C_2)^T & & \end{pmatrix} \begin{pmatrix} u \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

- A element-wise block diagonal symmetric positive definite
- B face-element incidence matrix, $B^T B = 5 * I$
- C_1 has orthogonal columns, $C_1^T C_1 = 2 * I$
- C_2 is orthogonal matrix, $C_2^T C_2 = I$


0
10
20
30
40
50
60
0
10
20
30
40
50
60
nz = 268

Structured symmetric indefinite systems: spectral properties of matrix blocks

$$\sigma(A) \subset [\frac{c_1}{h}, \frac{c_2}{h}]$$

$$sv(B\ C) \subset [c_3 h, c_4]$$

Conditioning of the whole indefinite matrix after appropriate diagonal scaling of the matrix: $\mathrm{O}(h^{-2})$

Maryška, R, Tůma 1995, 1996

Null-space (dual) approach: motivation

- $(B\ C)$ is an incomplete incidence matrix of certain graph
- fixed geometry of the domain, iterative change of material (physical) properties (solving inverse problems or sequences of time-dependent or nonlinear problems)
- use divergence-free finite elements (null-space approach embedded in formulation) vs. fully algebraic mixed or mixed-hybrid approach

Approach based on a null-space basis of the block $(B\ C)^T$

- Find a null-space basis Z of the matrix block $(B\ C)^T$ such that $(B\ C)^T Z = 0$
- Find some particular solution of the $(B\ C)^T u_1 = (q_2; q_3)$
- Solve (iteratively) the (symmetric positive definite) projected system

$$Z^T A Z u_2 = Z^T (q_1 - A u_1)$$

- Find unknown vectors $(p; \lambda)$ such that $(B\ C)(p; \lambda) = q_1 - Au$; $u = u_1 + Z u_2$

Choices for the null-space basis

- (fundamental) cycle null-space basis
  based on incidence vectors of cycles in the mesh: find a (shortest path) spanning tree; form cycles using non-tree edges

- orthogonal null space basis
  based on QR decomposition of $(B\ C)$: based on sparse QR (MA49 from HSL); projected system with $Z^T AZ$ independent of the mesh size h

Fundamental cycle null-space basis: spectral properties

$$sv(Z) \subset [1, \frac{c_5}{h^2}]$$

$$\sigma(Z^T \mathbf{A} Z) \subset [c_1, \frac{c_2 c_5^2}{h^4}]$$

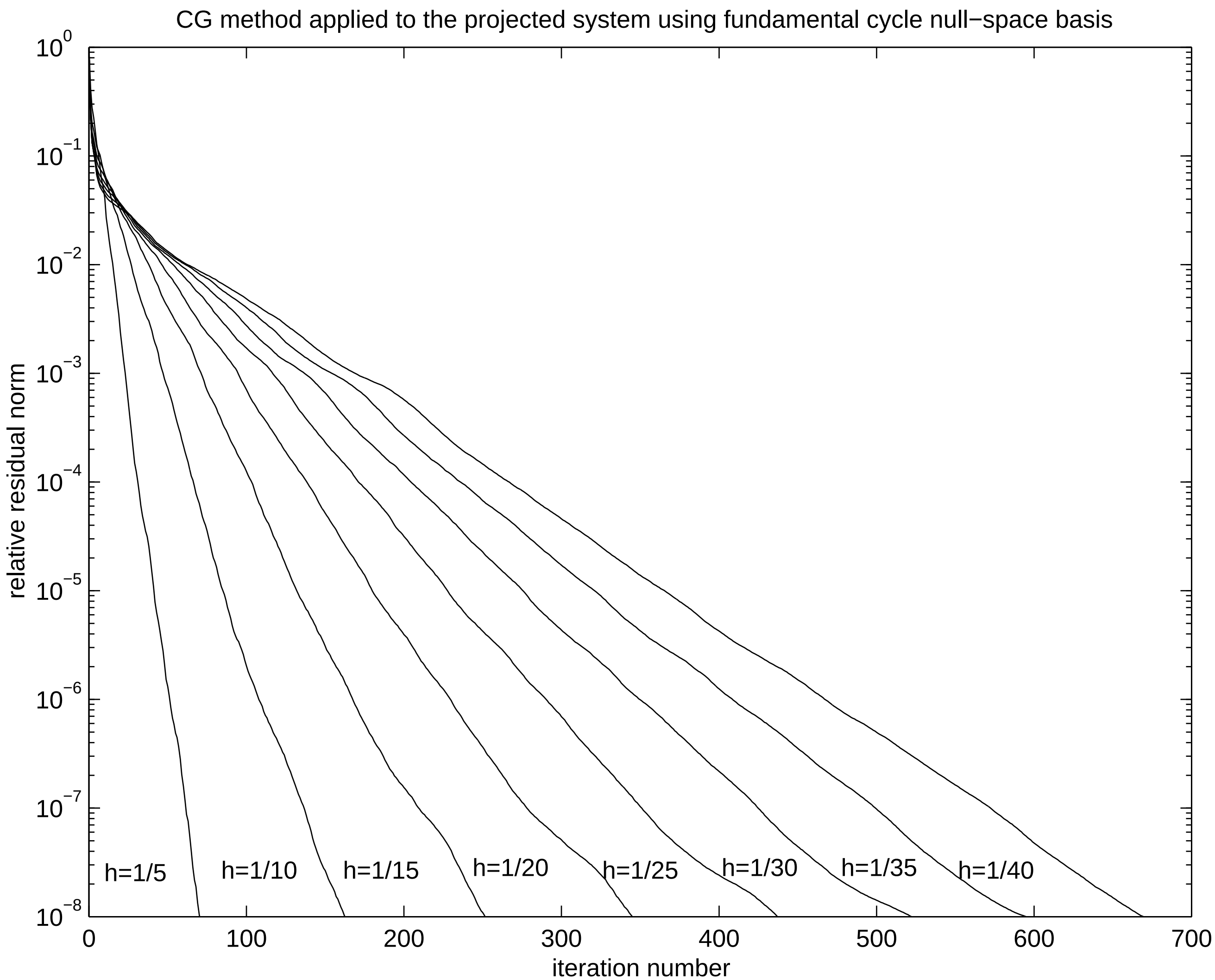
$$\Downarrow$$

$$\lim_{n \to +\infty} \left(\frac{\|r_n\|}{\|r_0\|} \right)^{\frac{1}{n}} \leq 1 - c_6 h^2$$

FUNDAMENTAL CYCLE NULL-SPACE APPROACH: UNPRECONDITIONED AND SMOOTHED CG APPLIED TO THE PROJECTED SYSTEM

in practice better than in theory

$$\lim_{n \to \infty} \left(\frac{\|r_n^{CG}\|}{\|r_0\|} \right)^{\frac{1}{n}} \leq 1 - c_6 h^2$$


CG method applied to the projected system using fundamental cycle null−space basis
relative residual norm
iteration number
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0
100
200
300
400
500
600
700
h=1/5
h=1/10
h=1/15
h=1/20
h=1/25
h=1/30
h=1/35
h=1/40

Approach based on a null-space basis of the block C^T

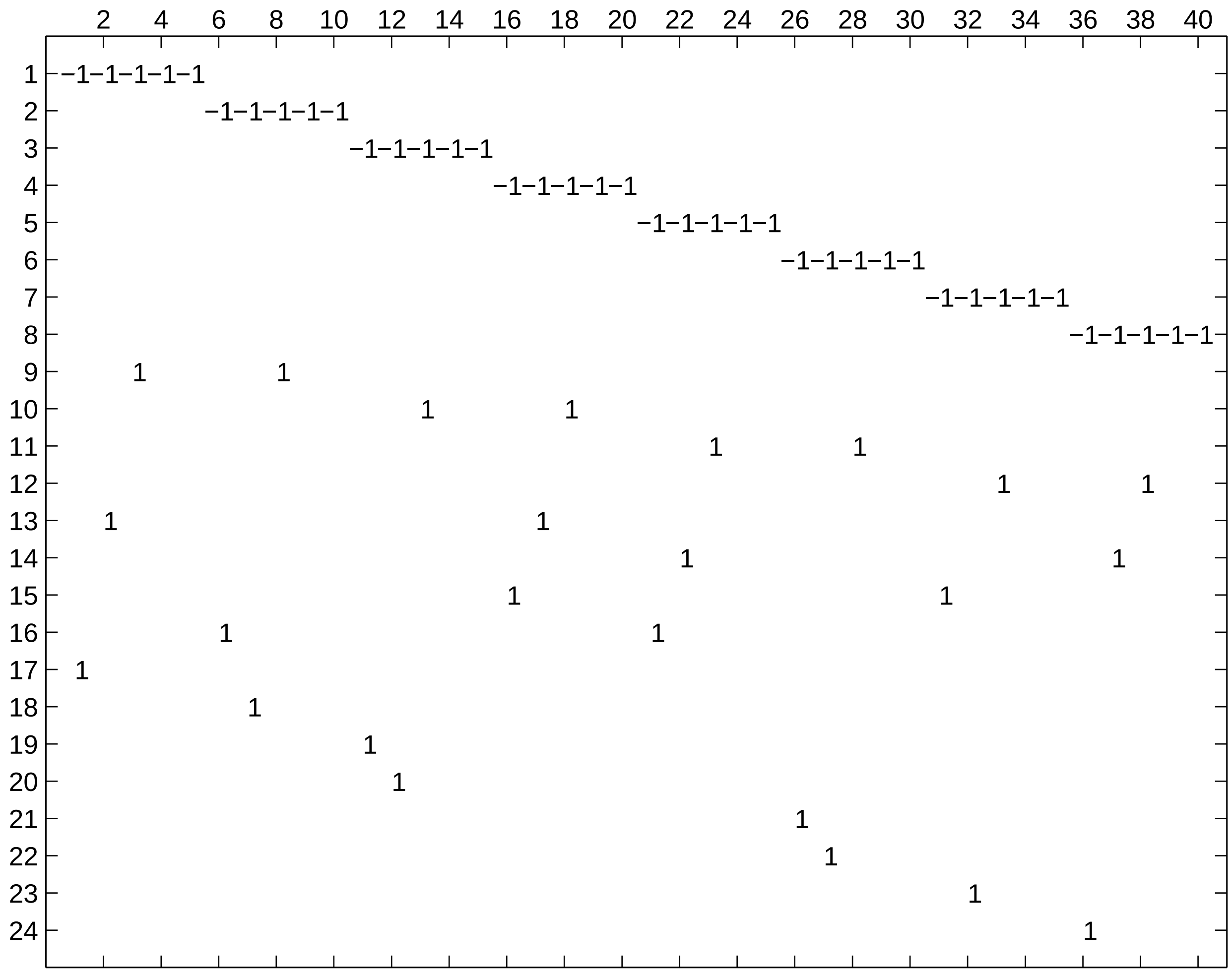
- Find an orthogonal null-space basis Z of the matrix block $(C_1\ C_2)^T$ such that $(C_1\ C_2)^T Z = 0$

- Find some particular solution of $C^T u_1 = q_3$

- Solve (iteratively) the (symmetric indefinite) projected system $\begin{pmatrix} Z^T \mathbf{A} Z & Z^T B \\ B^T Z & \end{pmatrix} \begin{pmatrix} u_2 \\ p \end{pmatrix} = \begin{pmatrix} Z^T(q_1 - \mathbf{A} u_1) \\ q_2 - B^T u_1 \end{pmatrix}$

- Set $u = u_1 + Z u_2$, find unknown vector λ such that $C\lambda = q_1 - Au - Bp$;

Arioli, Maryška, R, Tůma 2004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2 −1−1−1−1−1
3 −1−1−1−1−1
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5 −1−1−1−1−1
6 −1−1−1−1−1
7 −1−1−1−1−1
8 −1−1−1−1−1
9 1 1
10 1 1
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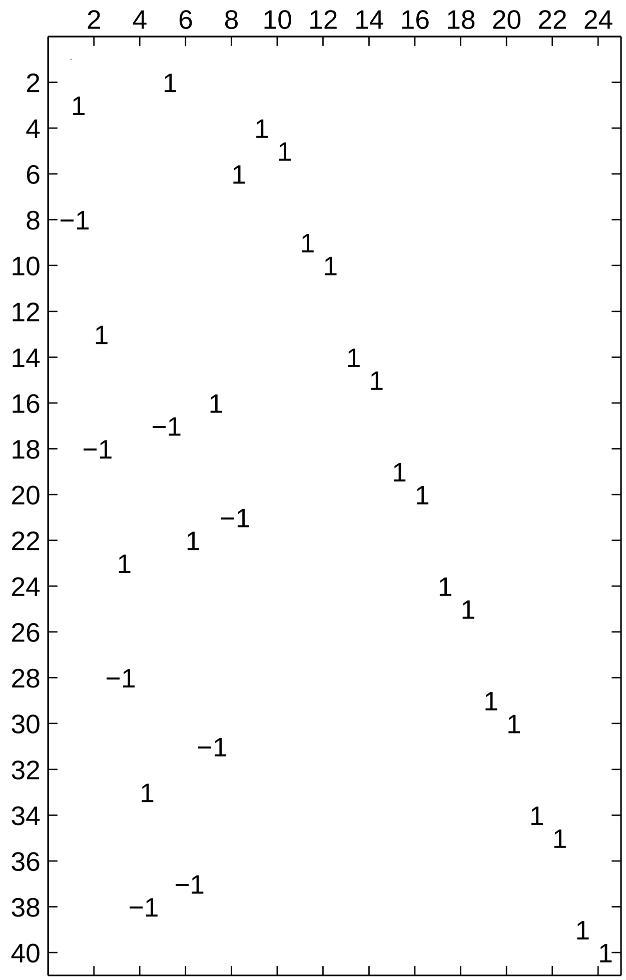
21 1
22 1
23 1
24 1



2 4 6 8 10 12 14 16 18 20 22 24
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
1
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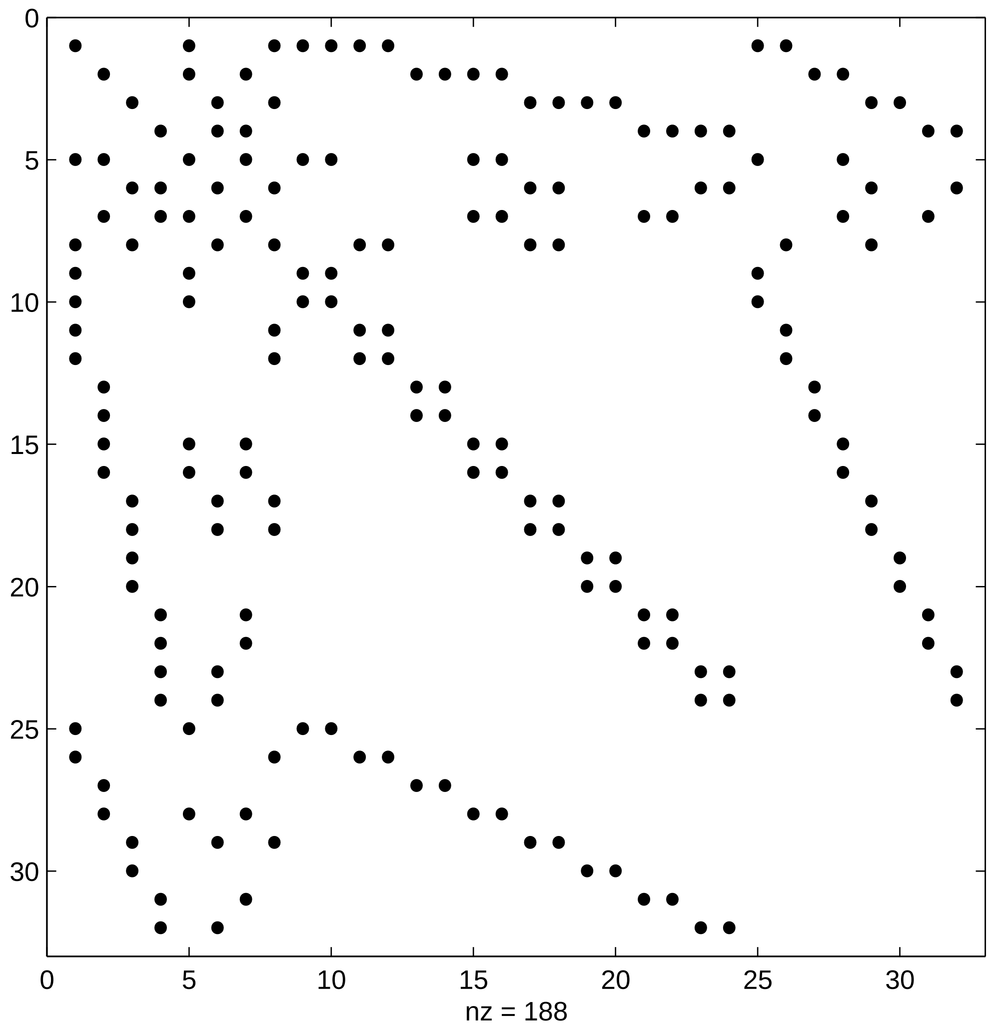
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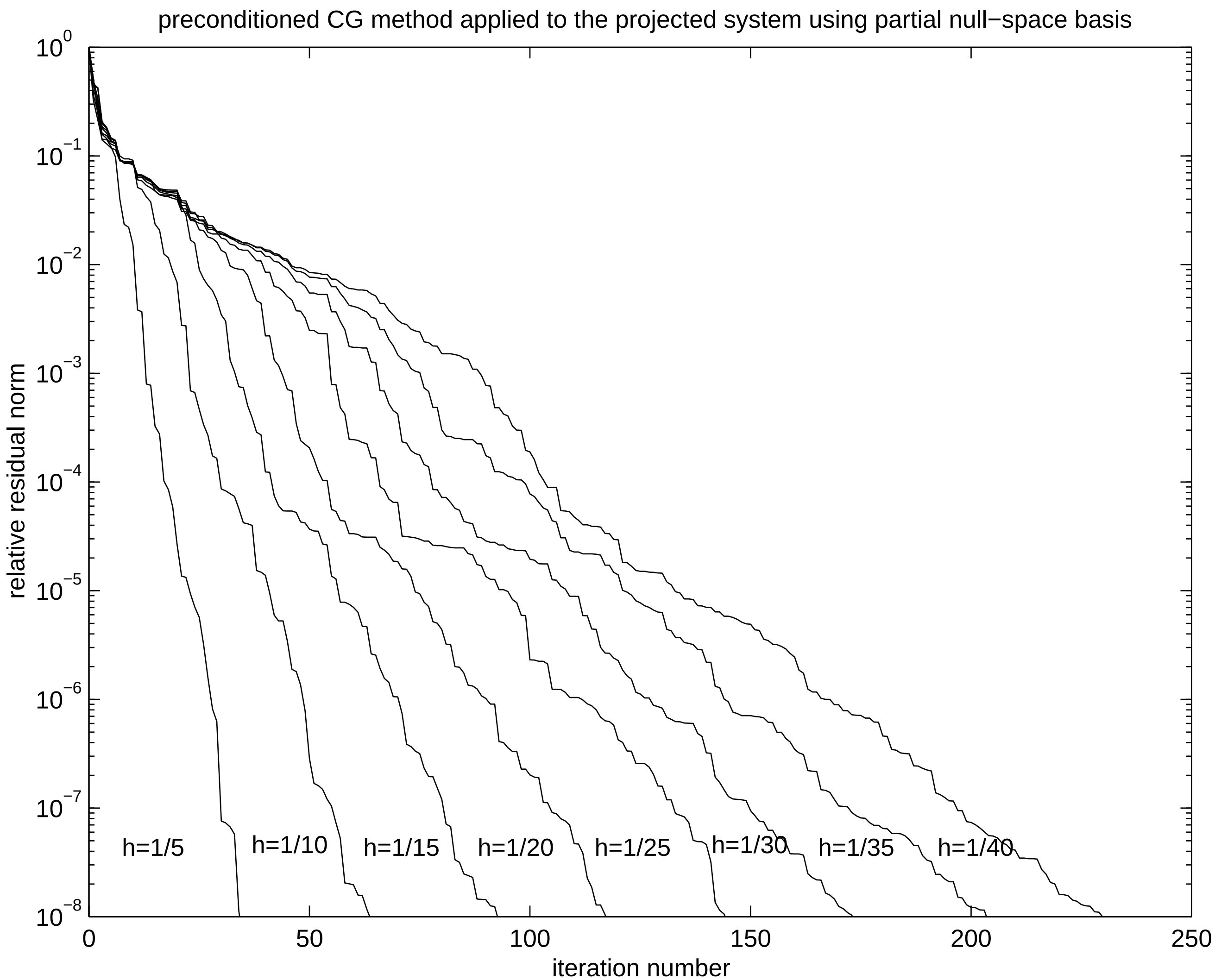
30
nz = 188

PARTIAL NULL-SPACE APPROACH: INDEFINITELY PRECONDITIONED CG APPLIED TO THE PROJECTED INDEFINITE SYSTEM

$$sv(Z^T B) \subset [c_7 h, c_8]$$

$$\Downarrow$$

$$\lim_{n \to \infty} \left(\frac{\|r_n^{CG}\|}{\|r_0\|} \right)^{\frac{1}{n}} \leq 1 - c_9 h$$


preconditioned CG method applied to the projected system using partial null−space basis
relative residual norm
iteration number
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0
50
100
150
200
250
h=1/5
h=1/10
h=1/15
h=1/20
h=1/25
h=1/30
h=1/35
h=1/40

h	memory requirements		iteration counts	
	QR $NNZ(QR)$	FC $NNZ(Z1)$	QR QR/SN	FC UN
1/5	28360 (3e-2)	3360 (7e-3)	22/20 (0.17/0.44)	71 (0.08)
1/10	410466 (0.97)	47120 (0.07)	22/21 (1.87/4.23)	163 (1.57)
1/15	1979203 (9.73)	226780 (0.30)	22/21 (8.48/17.1)	252 (19.9)
1/20	7120947 (59.6)	697840 (0.93)	22/21 (25.0/48.6)	346 (75.9)
1/25	18105131 (237)	1675800 (2.21)	22/21 (57.2/107)	438 (222)
1/30	40837823 (980)	3436160 (4.60)	21/21 (110/214)	523 (510)
1/35	—	6314420 (8.64)	—	596 (1009)
1/40	—	10706080 (14.8)	—	670 (1900)

h	NNZ	implicit IP/IQ	sparse QR $NNZ(QR)$	sparse QR QR/SN
1/5	14375	62/35 (0.05/0.03)	20834 (0.02)	18/14 (0.09/0.09)
1/10	123000	103/64 (0.68/0.48)	356267 (0.35)	19/16 (1.11/0.89)
1/15	424125	144/93 (5.17/3.79)	1840670 (3.14)	21/15 (6.09/4.63)
1/20	1016000	186/118 (20.2/14.2)	6322468 (17.97)	21/15 (18.3/14.94)
1/25	1996875	225/145 (50.8/37.4)	16661544 (86.6)	23/15 (47.0/27.8)
1/30	3465000	260/174 (111/84.2)	40669978 (584)	22/15 (96.7/85.5)
1/35	5518625	295/204 (224/173)	—	—
1/40	8256000	331/230 (383/295)	—	—