MAXIMUM ATTAINABLE ACCURACY OF INEXACT SADDLE POINT SOLVERS

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ITERATIVE SOLUTION OF SADDLE POINT PROBLEMS - DEFINITION

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$A \in \mathcal{R}^{n,n}$ is symmetric positive definite,
$B \in \mathcal{R}^{n,m}$ is of full column rank ($m \leq n$)

Benzi, Golub, Liesen: Numerical solution of saddle point problems, Acta Numerica (2005), 1–137.

OUTLINE

1. **TWO** SOLUTION APPROACHES: SCHUR COMPLEMENT REDUCTION METHOD AND NULL- SPACE PROJECTION METHOD
2. **TWO** MAIN REASONS: INEXACT SOLVES OF INNER ITERATION LOOPS AND ROUNDING ERRORS IN FINITE PRECISION ARITHMETIC
3. **TWO** MAIN EFFECTS: DELAY OF CONVERGENCE AND **LIMIT ON THE ACCURACY OF COMPUTED APPROXIMATE SOLUTIONS**

THE SCHUR COMPLEMENT METHOD

$$x = A^{-1}(f - By)$$

$$\begin{pmatrix} A & B \\ 0 & B^T A^{-1} B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ B^T A^{-1} f \end{pmatrix}$$

$$Ax_k + By_k = f$$

$$\begin{pmatrix} 0 \\ r_k^{(y)} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} - \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x - x_k \\ y - y_k \end{pmatrix}$$

$$\|x - x_k\|_A = \|y - y_k\|_{B^T A^{-1} B}.$$

THE SCHUR COMPLEMENT SCHEME

$$y_0, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} A^{-1}(f - By_0) \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ r_0^{(y)} \end{pmatrix} = \begin{pmatrix} 0 \\ -B^T x_0 \end{pmatrix}$$

$k = 0, 1, \ldots$

$$y_{k+1} = y_k + \alpha_k p_k^{(y)}$$

$$\begin{pmatrix} 0 \\ r_{k+1}^{(y)} \end{pmatrix} = \begin{pmatrix} 0 \\ r_k^{(y)} \end{pmatrix} - \alpha_k \begin{pmatrix} 0 \\ B^T A^{-1} B p_k^{(y)} \end{pmatrix}$$

THE COMPUTATION OF x_{k+1}: THREE DIFFERENT IMPLEMENTATIONS

$$x_{k+1} = A^{-1}(f - By_{k+1})$$

$$x_{k+1} = x_k + \alpha_k(-A^{-1}Bp_k^{(y)})$$

$$x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$$

INEXACT SOLUTION OF SYSTEMS WITH A: INNER-OUTER ITERATION

$$A\tilde{x} = \tilde{y}$$

$$(A + \Delta A)\overline{x} = \tilde{y}$$

$$\|\Delta A\| \leq \tau \|A\|$$

Szyld, Simoncini 2002, Sleijpen, van den Eshof 2002

SCHUR COMPLEMENT APPROACH: BEHAVIOR IN FINITE PRECISION ARITHMETIC

$$\begin{pmatrix} \bar{x}_{k+1} \\ \bar{y}_{k+1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \bar{r}^{(y)}_{k+1} \end{pmatrix}$$

$$\|x - \bar{x}_{k+1}\|_A$$

$$\leq \gamma_1 \|f - A\bar{x}_{k+1} - B\bar{y}_{k+1}\| + \gamma_2 \|B^T A^{-1} f - B^T A^{-1} B\bar{y}_{k+1})\|$$

EXACT ARITHMETIC:

$$\|f - A\bar{x}_{k+1} - B\bar{y}_{k+1}\| = 0$$

$$\|B^T A^{-1} f - B^T A^{-1} B\bar{y}_{k+1}\| \to 0$$

RESIDUAL OF THE SCHUR COMPLEMENT SYSTEM

$$\|B^T A^{-1} f - B^T A^{-1} B \overline{y}_{k+1} - \overline{r}^{(y)}_{k+1}\|$$

$$\leq \mathcal{O}(u)\|A^{-1}\|\|B\|(\|f\| + \|B\|\bar{Y}_{k+1})$$

$$+[\ (k+3)\tau\ + \mathcal{O}(u)]\kappa(A)\|B\|\bar{X}_{k+1}$$

$$\bar{Y}_{k+1} = \max_{i=0,\ldots,k+1}\{\|\overline{y}_i\|\}$$

$$\bar{X}_{k+1} \equiv \max_{i=0,\ldots,k+1}\{\|\overline{x}_0\|,\ \|\overline{\alpha}_i \overline{p}^{(x)}_i\|\}$$

$$\|x - \bar{x}_{k+1}\|_A, \|y - \overline{y}_{k+1}\|_{B^T A^{-1} B} \sim \mathcal{O}(\tau)$$

APPROXIMATE SOLUTION COMPUTED BY DIRECT SUBSTITUTION WITH FORMULA

$$x_{k+1} = A^{-1}(f - By_{k+1})$$

$$\|f - A\overline{x}_{k+1} - B\overline{y}_{k+1}\| \leq \mathcal{O}(u)(\|f\| + \|B\|\|\overline{y}_{k+1}\|) + \tau \ \|A\|\|\overline{x}_{k+1}\|$$

$$\|-B^T\overline{x}_{k+1} - \overline{r}^{(y)}_{k+1}\| \leq \mathcal{O}(u)\|A^{-1}\|\|B\|(\|f\| + \|B\|\bar{Y}_{k+1})$$

$$+[\ (k+4)\tau\ + \mathcal{O}(u)]\kappa(A)\|B\|\bar{X}_{k+1}$$

$$\bar{X}_{k+1} \equiv \max_{i=0,\ldots,k+1}\{\|\overline{x}_0\|,\ \|\overline{x}_{k+1}\|,\ \|\overline{\alpha}_i\overline{p}^{(x)}_i\|\}$$

THE UPDATED APPROXIMATE SOLUTION WITH FORMULA $x_{k+1} = x_k + (-A^{-1}Bp_k^{(y)})$

$$\|f - A\overline{x}_{k+1} - B\overline{y}_{k+1}\| \leq \mathcal{O}(u)(\|f\| + \|B\|\bar{Y}_{k+1})$$

$$+[\ (k+2)\tau\ + \mathcal{O}(u)]\|A\|\bar{X}_{k+1}$$

$$\| - B^T\overline{x}_{k+1} - \overline{r}_{k+1}^{(y)}\| \leq \mathcal{O}(u) \quad \|A^{-1}\|\|B\|(\|f\| + \|B\|\bar{Y}_{k+1})$$

$$+ \mathcal{O}(u) \quad \kappa(A)\|B\|\bar{X}_{k+1}$$

$$\bar{X}_{k+1} \equiv \max_{i=0,\ldots,k+1}\{\|\overline{x}_i\|\}$$

DIRECT SUBSTITUTION WITH CORRECTION VIA FORMULA $x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$

$$\|f - A\bar{x}_{k+1} - B\bar{y}_{k+1}\| \leq \mathcal{O}(u) \quad (\|f\| + \|A\|\bar{X}_{k+1} + \|B\|\bar{Y}_{k+1})$$

$$\|-B\bar{x}_{k+1} - \bar{r}^{(y)}_{k+1}\| \leq \mathcal{O}(u)\|A^{-1}\|\|B\|(\|f\| + \|B\|\bar{Y}_{k+1})$$
$$+[\ (k+4)\tau\ + \mathcal{O}(u)]\kappa(A)\|B\|\bar{X}_{k+1}$$

$$\bar{X}_{k+1} \equiv \max_{i=0,\ldots,k+1}\{\|\bar{x}_0\|,\ \|\bar{x}_{k+1}\|,\ \|\bar{\alpha}_i\bar{p}_i^{(x)}\|\}$$

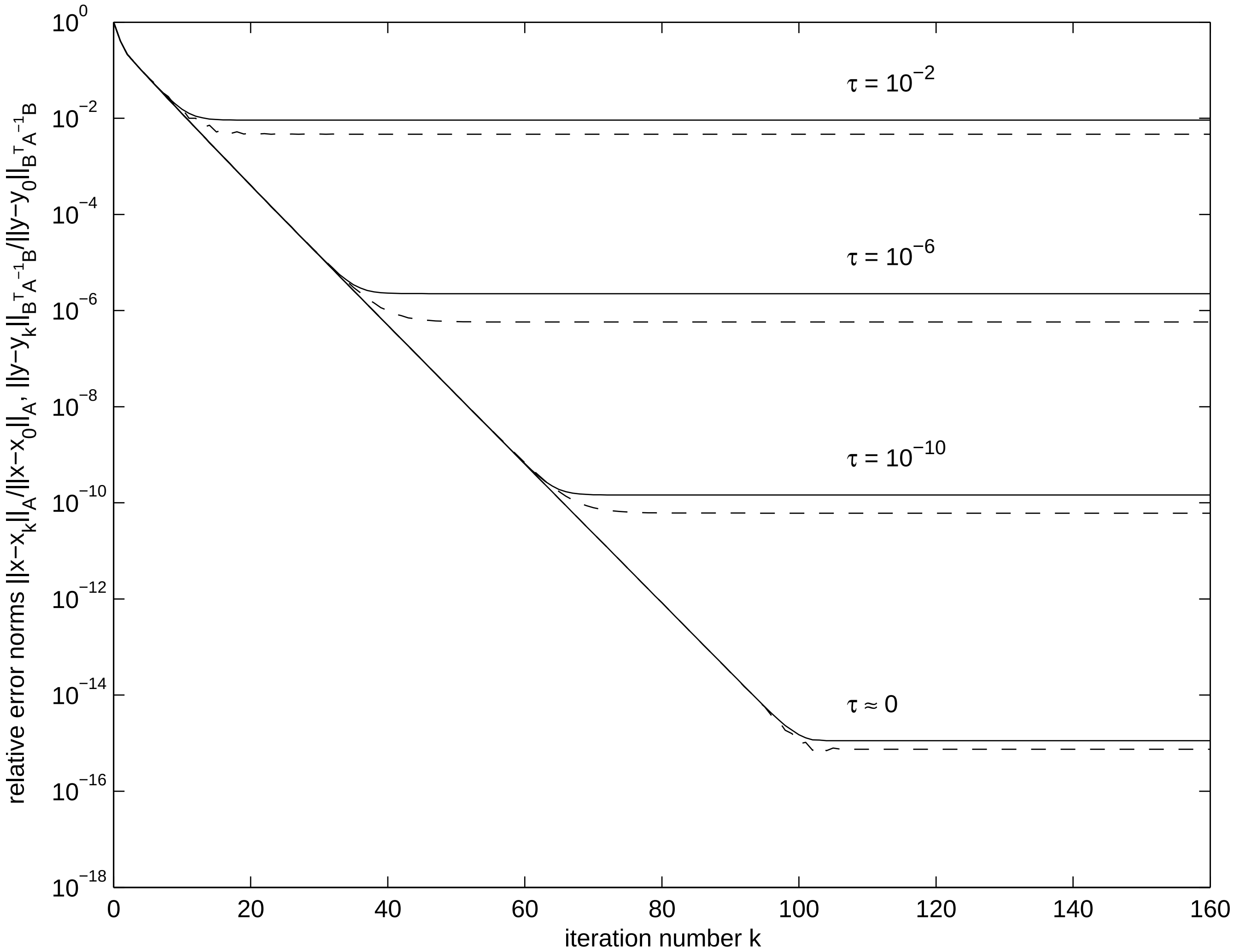
NUMERICAL EXPERIMENTS: SIMPLE MODEL PROBLEM

$$A = tridiag(1, 4, 1) \in \mathsf{R}^{25,25}, \; B = rand(25, 5) \in \mathsf{R}^{25,5}$$
$$f = rand(25, 1) \in \mathsf{R}^{25}$$

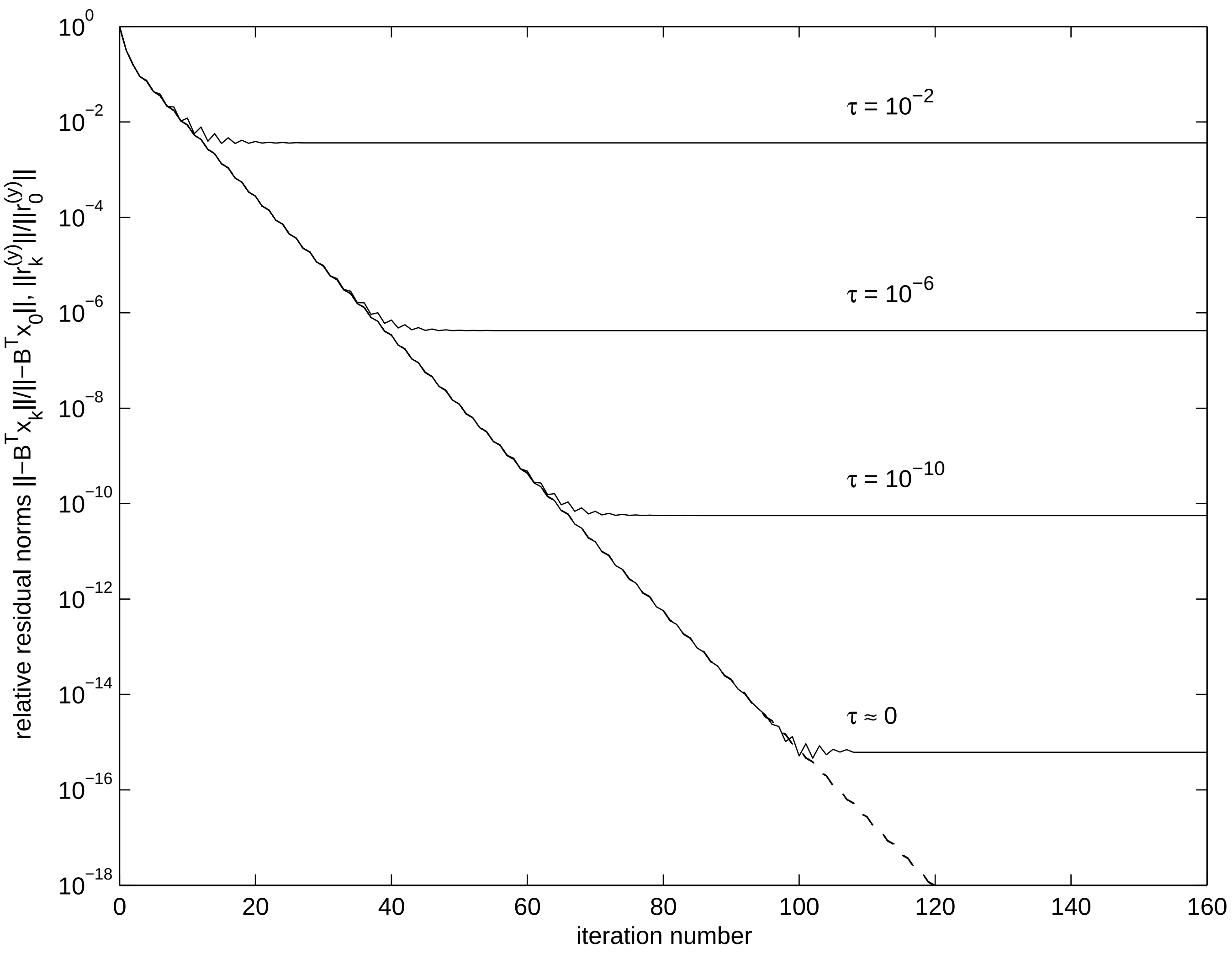
$$\sigma(A) \subset [2.0146, 5.9854]$$
$$sv(B) \subset [0.3016, 2.4580]$$

- The Schur complement system $B^T A^{-1} B y = B^T A^{-1} f$ solved with the conjugate gradient method or the steepest descent method

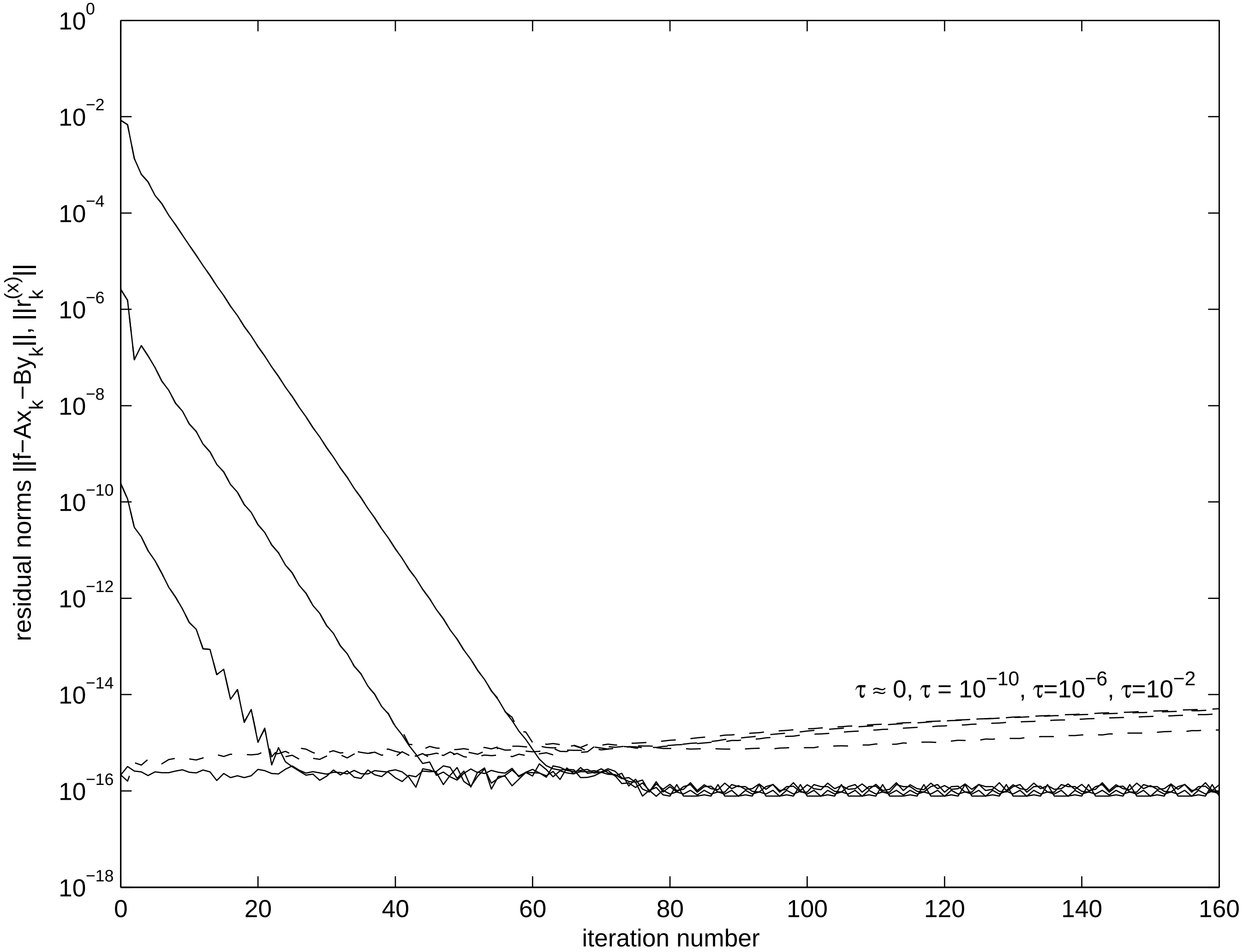
- The systems with the matrix blocks A and $B^T B$ solved inexactly with the conjugate gradient method and stopping criterion based on the backward error equal to τ



$\tau = 10^{-2}$
$\tau = 10^{-6}$
$\tau = 10^{-10}$
$\tau \approx 0$
relative error norms $\|x-x_k\|_A/\|x-x_0\|_A$, $\|y-y_k\|_{B^TA^{-1}B}/\|y-y_0\|_{B^TA^{-1}B}$
iteration number k
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
0
20
40
60
80
100
120
140
160



relative residual norms ||−B^T x_k||/||−B^T x_0||, ||r_k^(y)||/||r_0^(y)||
iteration number
τ = 10^−2
τ = 10^−6
τ = 10^−10
τ ≈ 0



residual norms ||f−Ax_k−By_k||, ||r_k^(x)||
iteration number
τ ≈ 0, τ = 10^−10, τ=10^−6, τ=10^−2
10^0
10^−2
10^−4
10^−6
10^−8
10^−10
10^−12
10^−14
10^−16
10^−18
0
20
40
60
80
100
120
140
160

REFERENCES

P.Jiránek, M.Rozložník, *Maximum attainable accuracy of inexact saddle point solvers*, in preparation.

THANK YOU FOR YOUR ATTENTION!