Numerical Stability of Iterative Methods

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Abstract

Iterative methods for linear systems are often the time-critical component in the solution of large-scale problems in computational science and engineering. In recent years a large amount of work has been devoted to Krylov subspace methods which are among the most widely used iterative schemes. Significantly less attention, however, has been paid to their numerical stability. Rounding errors occurring in finite-precision implementations of iterative methods can have two main effects on their numerical behavior. They can delay the rate of convergence given by the (theoretical) properties of a solved system and there is a limitation to the accuracy of computed approximate solution which does not decrease below a certain level (called usually maximum attainable or limiting accuracy). A comprehensive survey of error analysis for stationary iterative methods was given by Higham in his book, who addressed the problem how small can be the forward or backward error for various classical schemes. Several results for Krylov subspaces methods on both the accuracy of computed solution and the rate of convergence have been presented in last decades. Based on the pioneering papers of Paige on the (symmetric) Lanczos process, Greenbaum, Strakoš and others worked on a rounding error analysis of the CG method. These articles gave rise to the main result on the delay of convergence (due to rounding errors) which is essentially given by the rank-deficiency of the computed basis vectors. The maximal attainable accuracy of various iterative schemes with short recurrences
has been analyzed by Greenbaum, van der Vorst, Strakoš, Gutknecht
and others. In this contribution we will review the main results on the
numerical behavior of the GMRES method, the most widely known
and used representative of nonsymmetric Krylov subspace methods.
This method consists of constructing the basis of associated Krylov
subspace and then solving the transformed Hessenberg least squares
problem at each iteration step. In the talk we analyze different com-
putational variants of the Arnoldi process used in the orthogonaliza-
tion part of GMRES, including its Householder (HH), classical (CGS)
and modified (MGS) Gram-Schmidt implementation. We will exam-
ine how important is the orthogonality of computed Arnoldi vectors
and to what extent its has an influence on the accuracy of different
implementations of GMRES. In particular, we show that there is an
important relation to the relative backward error, which gives us the
link between the loss of orthogonality in the MGS Arnoldi and the
convergence of GMRES. Using this result we will prove that the most
usual MGS-GMRES implementation is backward stable. This the-
oretically justifies the observed fact that the linear independence of
Arnoldi vectors in MGS-GMRES is effectively maintained until the
convergence to the level of limiting accuracy. Based on a recently
obtained bound for the loss of orthogonality in the CGS process, an
analogous statement can be formulated also for CGS-GMRES. Pre-
sented results lead to important conclusions about the practical use
of the GMRES and other iterative methods.

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