Numerical Stability of Iterative Methods

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Abstract

Iterative methods for linear systems are often the time-critical component in the solution of large-scale problems in computational science and engineering. In recent years a large amount of work has been devoted to Krylov subspace methods which are among the most widely used iterative schemes. Significantly less attention, however, has been paid to their numerical stability. Rounding errors occurring in finite-precision implementations of iterative methods can have two main effects on their numerical behavior. They can delay the rate of convergence given by the (theoretical) properties of a solved system and there is a limitation to the accuracy of computed approximate solution which does not decrease below a certain level (called usually maximum attainable or limiting accuracy). A comprehensive survey of error analysis for stationary iterative methods was given by Higham in his book, who addressed the problem how small can be the forward or backward error for various classical schemes. Several results for Krylov subspaces methods on both the accuracy of computed solution and the rate of convergence have been presented in last decades. Based on the pioneering papers of Paige on the (symmetric) Lanczos process, Greenbaum, Strakoš and others worked on a rounding error analysis of the CG method. These articles gave rise to the main result on the delay of convergence (due to rounding errors) which is essentially given by the rank-deficiency of the computed basis vectors. The maximal attainable accuracy of various iterative schemes with short recurrences

has been analyzed by Greenbaum, van der Vorst, Strakoš, Gutknecht and others. In this contribution we will review the main results on the numerical behavior of the GMRES method, the most widely known and used representative of nonsymmetric Krylov subspace methods. This method consists of constructing the basis of associated Krylov subspace and then solving the transformed Hessenberg least squares problem at each iteration step. In the talk we analyze different computational variants of the Arnoldi process used in the orthogonalization part of GMRES, including its Householder (HH), classical (CGS) and modified (MGS) Gram-Schmidt implementation. We will examine how important is the orthogonality of computed Arnoldi vectors and to what extent its has an influence on the accuracy of different implementations of GMRES. In particular, we show that there is an important relation to the relative backward error, which gives us the link between the loss of orthogonality in the MGS Arnoldi and the convergence of GMRES. Using this result we will prove that the most usual MGS-GMRES implementation is backward stable. This theoretically justifies the observed fact that the linear independence of Arnoldi vectors in MGS-GMRES is effectively maintained until the convergence to the level of limiting accuracy. Based on a recently obtained bound for the loss of orthogonality in the CGS process, an analogous statement can be formulated also for CGS-GMRES. Presented results lead to important conclusions about the practical use of the GMRES and other iterative methods.

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