Transfer of the boundary conditions for boundary value problems for selfadjoint differential equations of $2n$th order.

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The idea of the transfer of conditions for a very general boundary value problem for a system of differential equations with inner and transition conditions originates in various methods. The present paper is devoted to a special problem and the methods are chosen in order to exploit the additional information, above all the symmetry and some sign properties.

Consider the boundary value problem for a selfadjoint differential equation of $2n$th order

$$\sum_{i=0}^{n}(-1)^i(p_{n-i}(t)y^{(i)}(t))^{(i)} = q(t) \qquad \text{a.e. in} \quad (a,b), \tag{1}$$

where the coefficients of the equation satisfy the following requirements:
$q(t), 1/p_0(t) \in \mathcal{L}\ (a,b)$, and $p_i(t) \in \mathcal{L}\ (a,b)$ for $i = 1, ..., n$.

The quasiderivatives are defined to the following manner:
$y^{[k]}(t) = y^{(k)}(t) \qquad \text{for} \quad k = 0, 1, ..., n-1,$
$y^{[n]}(t) = p_0(t)y^{(n)}(t),$
$y^{[n+j]}(t) = p_j(t)y^{(n-j)}(t) - (y^{[n+j-1]}(t))', \quad \text{for} \quad j = 1, ..., n.$
Put $x_i(t) = y^{[i-1]}(t)$ for $i = 1, ..., 2n$ and the vector $x(t) = (x_1(t), ..., x_{2n}(t))^T$.

Consider the boundary condition for the differential equation (1) in the form

$$W_1x(a) + W_2x(b) = w, \tag{2}$$

where W_1 and W_2 are square matrices of order $2n$ and the vector w has $2n$ components. We divide the matrices W_1 and W_2 into blocks:

$$W_1 = (B_1, B_2), \qquad W_2 = (B_3, B_4),$$

where the matrices B_i $(i = 1, ..., 4)$ have n columns. Let T be the square matrix of order n defined as

$$T = \begin{bmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 1 & 0 \\ ... & & & & \\ 0 & 1 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \end{bmatrix} \text{ for } n \geq 2$$

and $T = 1$ for $n = 1$ (i.e., the matrix has the 1's on the adjoint diagonal and zeros everywhere else).

Now we can formulate two known lemmas (see [6]).

Lemma 1 *The necessary and sufficient condition for the problem (1), (2) to be selfadjoint is*

$$B_1TB_2^T - B_2TB_1^T = B_3TB_4^T - B_4TB_3^T \tag{3}$$

$$\text{and rank } (W_1, W_2) = 2n.$$

Lemma 2 *Let* $p_i(t) \geq 0$ *a.e. in* (a, b) *for* $i = 0, ..., n$. *The necessary and sufficient condition for the selfadjoint problem (1), (2) to be positive semidefinite is that the matrix*

$$B_1TB_2^T - B_3TB_4^T$$

is negative semidefinite.

First we will replace the equation of the $2n$-th order by the system of $2n$ equations of the first order in a standard way. The definition of quasiderivatives implies that the introduced vector $x(t)$ satisfies the differential equation

$$x'(t) + \begin{bmatrix} 0 & -1 & 0 & ... & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & -1 & ... & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ ... & & & & & & & & & & & \\ 0 & 0 & 0 & ... & 0 & -1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & -1/p_0 & 0 & 0 & ... & 0 & 0 \\ ... & & & & & & & & & & & \\ 0 & 0 & 0 & ... & 0 & -p_1 & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & -p_2 & 0 & 0 & 0 & 1 & ... & 0 & 0 \\ ... & & & & & & & & & & & \\ 0 & -p_{n-1} & 0 & ... & 0 & 0 & 0 & 0 & 0 & ... & 0 & 1 \\ -p_n & 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -q(t) \end{bmatrix} \tag{4}$$

a.e. in (a, b).
That means:

$$x'(t) + A(t)x(t) = f(t) \qquad \text{a.e. in} \quad (a, b), \tag{5}$$

The equation (4) is of the form (5). Let the matrix $A(t)$ stand for the corresponding matrix of the equation (4) and the vector $f(t)$ stand for the corresponding right-hand side.

Now, we transform this problem to the problem with separated boundary conditions. Let us introduce the vector $z(t)$:

$$z(t) = \begin{pmatrix} x(t) \\ x(a+b-t) \end{pmatrix} \forall t \in \langle a, b \rangle$$

We obtain for the vector function $z(t)$:

$$z'(t) + \underbrace{\begin{pmatrix} A(t) & 0 \\ 0 & -A(a+b-t) \end{pmatrix}}_{\hat{A}(t)} z(t) = \underbrace{\begin{pmatrix} f(t) \\ -f(a+b-t) \end{pmatrix}}_{\hat{f}(t)} \tag{6}$$

a.e. in $\left(a, \frac{a+b}{2}\right)$. This vector function fulfils separated boundary conditions.

$$(W_1, W_2)z(a) = w \tag{7}$$

$$(I, -I)z\left(\frac{a+b}{2}\right) = o. \tag{8}$$

Theorem 1 *Let $\hat{D}(t)$ be an absolutely continuous $2n \times 4n$ matrix and $\hat{d}(t)$ an absolutely continuous vector with $2n$ components, satisfying the equations*

$$\hat{D}'(t) = \hat{D}(t)\hat{A}(t)$$
$$\hat{d}'(t) = \hat{D}(t)\hat{f}(t)$$

and the initial conditions

$$\hat{D}(a) \quad = \quad (W_1, W_2), \qquad \hat{d}(a) = w,$$

Then

$$\hat{D}(t)z(t) = \hat{d}(t) \quad \text{for every } t \in \langle a, \frac{a+b}{2} \rangle \tag{9}$$

and for every solution of the problem (6),(7).

This theorem brings us to the transfer of the left boundary condition (7) on the whole interval $\left\langle a, \frac{a+b}{2} \right\rangle$. The equation (9) is called the transferred condition. We say that the matrix $\hat{D}(t)$ and the vector $\hat{d}(t)$ realize the transfer of the condition (7). Analogously we can formulate the theorem on the transfer of the right boundary condition.

Theorem 2 *Let $\hat{C}(t)$ be an absolutely continuous $2n \times 4n$ matrix and $\hat{c}(t)$ an absolutely continuous vector with $2n$ components, satisfying the equations*

$$\hat{C}'(t) = \hat{C}(t)\hat{A}(t)$$
$$\hat{c}'(t) = \hat{C}(t)\hat{f}(t)$$

and the initial conditions (this time at the point $\frac{a+b}{2}$)

$$\hat{C}\left(\frac{a+b}{2}\right) = (I, \, -I), \qquad \hat{c}\left(\frac{a+b}{2}\right) = o,$$

Then

$$\hat{C}(t)z(t) = \hat{c}(t) \quad \text{for every } t \in \left\langle a, \frac{a+b}{2} \right\rangle$$

and for every solution of the problem (6),(8).

Matrix $\hat{D}(t)$ could be divided in to four blocks: $\hat{D}(t) = \left(\hat{D}_1(t), \hat{D}_2(t), \hat{D}_3(t), \hat{D}_4(t)\right)$, where $\hat{D}_i(t)$ are $(2n \times n)$. It could be shown, that for all t hold:

1. The matrix $\hat{D}_1(t)T\hat{D}_2^T(t) - \hat{D}_3(t)T\hat{D}_4^T(t)$ is negative semidefinite
2. The rank $\hat{D}(t) = 2n$

The properties 1, 2 imply that matrix $[\hat{D}_1(t) - \hat{D}_2(t)T, \hat{D}_3(t) + \hat{D}_4(t)T]$ is nonsingular for all $t \in \left\langle a, \frac{a+b}{2} \right\rangle$. If we multiply the transferred condition $\hat{D}(t)z(t) = \hat{d}(t)$ by the matrix

$$[\hat{D}_1(t) - \hat{D}_2^T(t)T, \hat{D}_3(t) + \hat{D}_4^T(t)T]^{-1} = K(t)$$

from the left, we obtain transferred condition in canonical form. The matrix $K(t) \cdot [\hat{D}_1(t)\hat{D}_3(t)] = \hat{G}(t)$ is symmetric with all eigenvalues in the interval $\langle 0, 1 \rangle$. Than

$$K(t) \cdot [\hat{D}_2(t), \, \hat{D}_4(t)] = (I - \hat{G}(t)) \begin{pmatrix} -T & 0 \\ 0 & T \end{pmatrix}.$$

The transfer of boundary conditions is realized by finding the symmetric matrix $\hat{G}(t)$. It is the solution of Riccati equation with all eigenvalues in the intervalu $\langle 0,1\rangle$. We denote $\hat{g}(t) = K(t)\cdot\hat{d}(t)$. Than we obtain this function by solving certain linear equation.

Analogously we can divide the matrix $\hat{C}(t)$ in to the blocks: $\hat{C}(t) = \Big(\hat{C}_1(t), \hat{C}_2(t), \hat{C}_3(t), \hat{C}_4(t)\Big)$, where $\hat{C}_i(t)$ are $(2n\times n)$. It could be shown, that for all t hold:

1. The matrix $\hat{C}_1(t)T\hat{C}_2^T(t) - \hat{C}_3(t)T\hat{D}_4^T(t)$ is positive semidefinite
2. The rank $\hat{C}(t) = 2n$

This properties imply that matrix $[\hat{C}_1(t)+\hat{C}_2(t)T, \hat{C}_3(t)-\hat{C}_4(t)T]$ is nonsingular for all $t\in\left\langle a, \frac{a+b}{2}\right\rangle$. If we multiply the transferred condition $\hat{C}(t)z(t) = \hat{c}(t)$ by the matrix

$$[\hat{C}_1(t)+\hat{C}_2^T(t)T, \hat{C}_3(t)-\hat{C}_4^T(t)T]^{-1} = L(t)$$

from the left, we obtain transferred condition in canonical form. The matrix $L(t)\cdot[\hat{C}_1(t)\hat{C}_3(t)] = \hat{H}(t)$ is symmetric with all eigenvalues in the interval $\langle 0,1\rangle$. Than

$$L(t)\cdot[\hat{C}_2(t)\hat{C}_4(t)] = (I-\hat{H}(t))\begin{pmatrix} -T & 0 \\ 0 & T \end{pmatrix}.$$

Then we must find the symmetric matrix $\hat{G}(t)$, which is a solution of the Riccati equation with all eigenvalues in the intervalu $\langle 0,1\rangle$. Let us denote $\hat{h}(t) = L(t)\cdot\hat{c}(t)$. Than we obtain this function by solving certain linear equation.

We have presented algorithms that lead to solving the Riccati differential equations and have shown that for a series of problems these equations possess a unique solution on tho whole interval in question and, moreover, that these equations are represented by symmetric matrices whose eigenvalues lie in $\langle 0,1\rangle$. The methods used so far had to check whether the solutions to the Riccati equations exist on the whole interval in question or whether they do not exceed some a priori given barriers, which is not necessary in our canonical transfer of conditions.

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