Approximation and numerical realization of 2D quasistatic contact problems with local Coulomb friction

J. Haslinger[a], O. Vlach[b], Z. Dostál[b]

$a)$ Charles University, Prague, Czech Republic

$b)$ VŠB-TU Ostrava, Czech Republic

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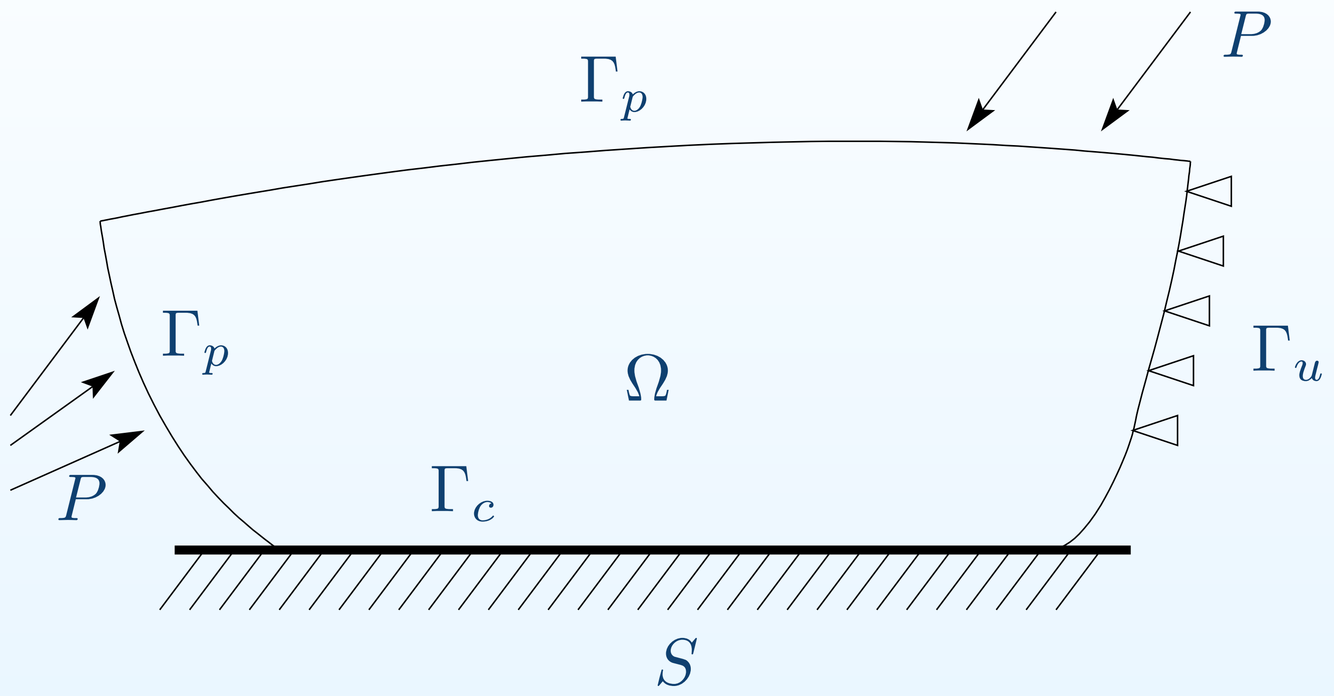
Geometry (in 2D)



$$\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_p \cup \overline{\Gamma}_c$$

Classical formulation

equilibrium equations:

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad in\ \Omega \times (0, T_0), \quad i = 1, 2;$$

linear Hooke's law:

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl}(u), \quad i, j, k, l = 1, 2;$$
$$\varepsilon_{kl}(u) = \tfrac{1}{2}\left(\tfrac{\partial u_k}{\partial x_l} + \tfrac{\partial u_l}{\partial x_k}\right) ;$$

classical boundary conditions:

$$u_i = 0 \quad on\ \Gamma_u \times (0, T_0),\ i = 1, 2;$$
$$T_i := \tau_{ij}\nu_j = P_i \quad on\ \Gamma_p \times (0, T_0),\ i = 1, 2 ;$$

Classical formulation

Conditions on Γ_c

unilateral conditions:

$$u_\nu \leq 0, \quad \tau_\nu \leq 0, \quad u_\nu \tau_\nu = 0 \quad on\ \Gamma_c \times (0, T_0);$$

Coulomb's law of friction:

$$\dot{u}_T(x) = 0 \Rightarrow \ |\tau_T(x)| \ \leq -\mathcal{F}(x)\tau_\nu(x); \qquad x \in \Gamma_c \times (0, T_0)$$

$$\dot{u}_T(x) \neq 0 \Rightarrow \quad \tau_T(x) \ = \mathcal{F}(x)\tau_\nu(x) \,\text{sign}\, \dot{u}_T(x) \ .$$

$$\mathcal{F} \in C^1(\Gamma_c) \, , \quad \mathcal{F} \geq 0$$

initial condition:

$$u(0) = u_0 \quad in\ \Omega \, .$$

Weak formulation

Notation

$$V = \{ v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma_u \} \,, \quad \mathbb{V} = V \times V$$

$$K = \{ v \in \mathbb{V} \,|\, v_\nu \le 0 \text{ a.e. on } \Gamma_c \}$$

$H^{1/2}(\Gamma_c) = V_{|\Gamma_c}$ (*space of traces on* Γ_c *of functions from* V)

$H^{-1/2}(\Gamma_c) = (H^{1/2}(\Gamma_c))'$ (*the dual space to* $H^{1/2}(\Gamma_c)$)

$H^{-1/2}_{-}(\Gamma_c) \ldots$ (*cone of all non–positive elements of* $H^{-1/2}(\Gamma_c)$)

$\langle \,,\, \rangle \ldots$ *duality pairing between* $H^{-1/2}(\Gamma_c)$ *and* $H^{1/2}(\Gamma_c)$

Assumptions:

$$F \in W^{1,2}(0, T_0, (L^2(\Omega))^2), \quad P \in W^{1,2}(0, T_0, (L^2(\Gamma_p))^2)$$

$$a(u, v) := \int_\Omega \tau_{ij}(u) \varepsilon_{ij}(v) \, dx \,, \quad j(\lambda, v) := -\langle \mathcal{F}\lambda, |v_T| \rangle$$

$$L(t)(v) := \int_\Omega F_i(t) v_i \, dx + \int_{\Gamma_p} P_i(t) v_i \, ds \,, \quad u, v \in \mathbb{V} \,, \quad \lambda \in H^{-1/2}(\Gamma_c)$$

Weak formulation

$$\left.\begin{array}{l} \textit{Find } u \in W^{1,2}(0,T_0,\mathbb{V}),\ \lambda \in W^{1,2}(0,T_0,H^{-1/2}(\Gamma_c)) : \\ u(t) \in K \quad \textit{for a.a. } t \in (0,T_0),\ u(0) = u_0 \textit{ in } \Omega \\ a(u(t), v - \dot{u}(t)) + j(\lambda(t), v) - j(\lambda(t), \dot{u}(t)) \geq \\ \quad L(t)(v - \dot{u}(t)) + \langle \lambda(t), v_\nu - \dot{u}_\nu(t) \rangle \\ \qquad \forall v \in \mathbb{V} \textit{ and for a.a } t \in (0,T_0) \\ \langle \lambda(t), z_\nu - u_\nu(t) \rangle \geq 0 \quad \forall z \in K \textit{ and for a.a } t \in (0,T_0) \end{array}\right\} (\mathcal{P})$$

where

$$a(u_0, v - u_0) + j(\lambda_0, v - u_0) \geq L(0)(v - u_0) \ \ \forall v \in K \ ,$$

$$\lambda_0 = \tau_\nu(u_0)_{|\Gamma_c} \ .$$

Classical and weak formulations are formally equivalent and

$$\lambda = \tau_\nu(u)_{|\Gamma_c}$$

[Rocca R. and Coccu M. 01]: If $supp\,\mathcal{F} \subset \Gamma_c$ and $\mathcal{F}$ is small enough, then $(\mathcal{P})$ has at least one solution.

Time discretization

$\Delta t = T_0/n \ldots \ \ldots$ time step, $t_i = i\Delta t$, $u^i := u(t_i)$.

- approximation $\dot{u}^{i+1} \approx \frac{u^{i+1} - u^i}{\Delta t}$
- denoting $w := u^i + \Delta t v \in \mathbb{V}$
- writing u instead u^{i+1} and v instead u^i
- skipping the index i

Leads to a implicit mixed formulation:

$$\left.\begin{array}{l} \textit{Find } u, \lambda \in \mathbb{V} \times H_-^{-1/2}(\Gamma_c) \textit{ such that} \\ a(u, w-u) + j(\underline{\lambda}, w-v) - j(\underline{\lambda}, u-v) \geq \\ \qquad L(w-u) + \langle \lambda, w_\nu - u_\nu \rangle \quad \forall w \in \mathbb{V} \\ \langle \mu - \lambda, u_\nu \rangle \geq 0 \qquad \forall \mu \in H_-^{-1/2}(\Gamma_c) \ . \end{array}\right\} \qquad (\mathcal{Q})$$

Fixed point approach

With any $g \in H_-^{-1/2}(\Gamma_c)$ we associate the auxiliary problem $(\mathcal{Q}(g))$: For $v \in K$ given

$$\left.\begin{array}{l} \textit{Find } u := u(g) \in \mathbb{V},\ \lambda := \lambda(g) \in H_-^{-1/2}(\Gamma_c) \textit{ s.t.} \\ a(u, w-u) + j(g, w-v) - j(g, u-v) \geq \\ \qquad L(w-u) + \langle \lambda, w_\nu - u_\nu \rangle \quad \forall w \in \mathbb{V} \\ \langle \mu - \lambda, u_\nu \rangle \geq 0 \quad \forall \mu \in H_-^{-1/2}(\Gamma_c)\ , \end{array}\right\} \qquad (\mathcal{Q}(g))$$

There exists a unique solution of $(\mathcal{Q}(g))$ for every $g \in H_-^{-1/2}(\Gamma_c)$.

Let $\Phi : H_-^{-1/2}(\Gamma_c) \mapsto H_-^{-1/2}(\Gamma_c)$ be a mapping defined by

$$\Phi(g) = \lambda\,, \qquad g \in H_-^{-1/2}(\Gamma_c)\ .$$

Compairing $(\mathcal{Q})$ and $(\mathcal{Q}(g))$ we see that (u, λ) solves $(\mathcal{Q})$ iff λ is a fixed point of Φ:

$$\Phi(\lambda) = \lambda\ .$$

Method of successive approximations

Let $\lambda^{(0)} \in H_{-}^{-1/2}(\Gamma_c)$ be given, $k := 1$;

if $\lambda^{(k)} \in H_{-}^{-1/2}(\Gamma_c)$, $k \geq 1$ is known,

solve $(\mathcal{Q}(\lambda^{(k)}))$ and set $\lambda^{(k+1)} := \lambda$,

where (u, λ) is a solution of $(\mathcal{Q}(\lambda^{(k)}))$;

$k := k + 1$;

repeat until stopping criterion

Mixed formulation of $(\mathcal{Q}(g))$

Let

$$\Lambda_\nu = H_-^{-1/2}(\Gamma_c)$$

$$\Lambda_T(g) = \{\mu_T \in L^2(\Gamma_c) | \, |\mu_T| \le \mathcal{F} g \; a.e. \; on \; \Gamma_c\} \; , \quad g \in L^2_+(\Gamma_c) \; .$$

Mixed formulation of $(\mathcal{Q}(g))$ reads as follows:

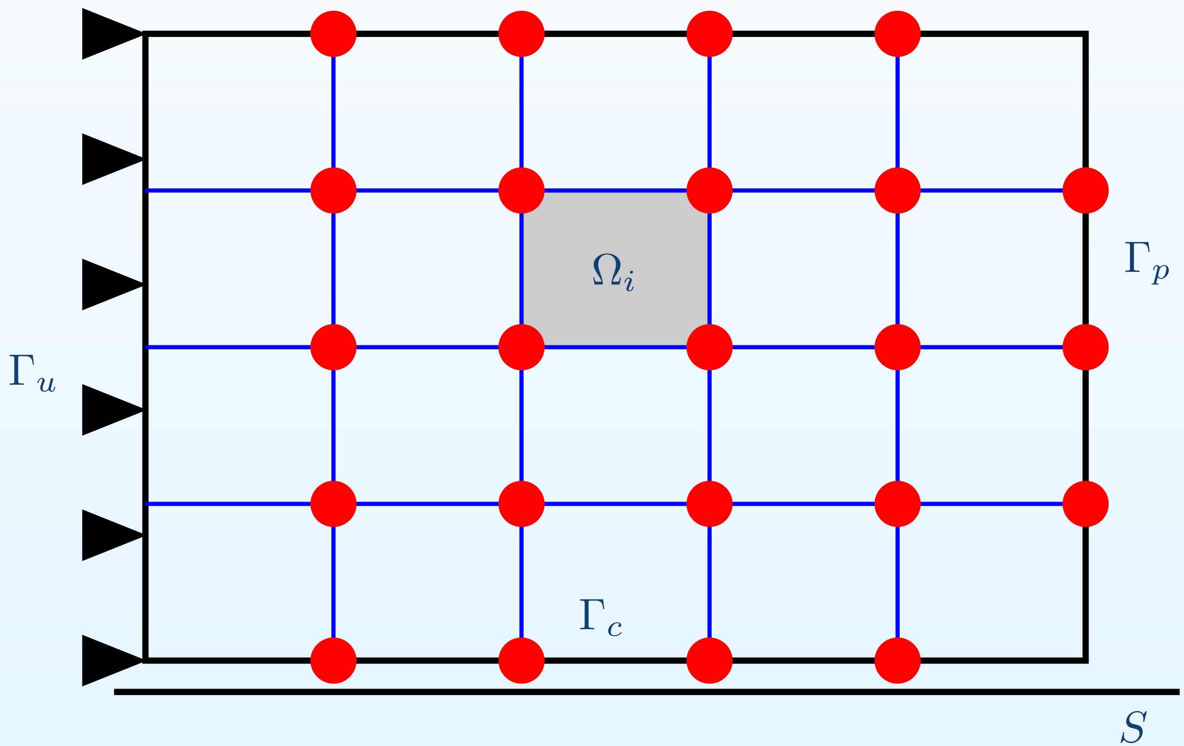
$$\left.\begin{array}{l} \textit{Find } (u, \lambda_\nu, \lambda_T) \in \mathbb{V} \times \Lambda_\nu \times \Lambda_T(g) \textit{ such that} \\ a(u,w) = L(w) + \langle \lambda_\nu, w_\nu \rangle + \langle \lambda_T, w_T \rangle \quad \forall w \in \mathbb{V} \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle \ge 0 \quad \forall \mu_\nu \in \Lambda_\nu \\ \langle \mu_T - \lambda_T, u_T \rangle \ge \langle \mu_T - \lambda_T, v_T \rangle \; \forall \mu_T \in \Lambda_T(g) \; . \end{array}\right\} \qquad (\mathcal{M}(g))$$

It holds:

- $u \in K$ solves $(\mathcal{Q}(g))$;
- $\lambda_\nu = \tau_\nu(u)_{|\Gamma_c}$, $\lambda_T = \tau_T(u)_{|\Gamma_c}$ on Γ_c .

Substructuring by the FETI-DP method

Let $\Omega = \bigcup_{i=1}^{q} \Omega_i$ be a polygonal domain.



$$\Gamma = \bigcup_{k=1}^{p} \Gamma_k\,, \quad \Gamma_k = \partial\Omega_i \cap \partial\Omega_j \;\; \ldots \qquad \textit{skeleton } \Gamma$$

Discretization and FETI-DP

Discretization of primal variables

Let $\mathcal{T}_i$ be a triangulation of $\overline{\Omega}_i$, $i = 1, \ldots, q$, such that

$$\mathcal{T}_{i|\Gamma_k} = \mathcal{T}_{j|\Gamma_k}, \qquad \Gamma_k = \partial\Omega_i \cap \partial\Omega_j \ .$$

System $\mathcal{T} = \{\mathcal{T}_i\}_{i=1}^q$ creates a triangulation of $\overline{\Omega}$.

$$\mathbb{X}_i^r = \{v \in (C(\overline{\Omega}_i))^2 |\ v_{|T} \in (P_1(T))^2\ \forall T \in \mathcal{T}_i,$$

$$v = 0 \ on\ \partial\Omega_i \cap \overline{\Gamma}_u,\ v = 0 \text{ at "corners" } \mathsf{C}\}$$

$$\mathbb{X}^r = \mathbb{X}_1^r \times \mathbb{X}_2^r \times \cdots \times \mathbb{X}_q^r$$

$$\mathbb{X}^c = \{\varphi_a\,,\ a \in \mathsf{C}\}\ ,$$

where $\varphi_a \in \mathbb{V}_h$ is the Courant basis function at a. We set

$$\mathbb{X} = \mathbb{X}^r \oplus \mathbb{X}^c\ .$$

It holds:

$$(v \in \mathbb{X})\ \wedge\ ([v] = 0 \text{ on } \Gamma) \Rightarrow v \in \mathbb{V}\ .$$

Discretization and FETI-DP

Discretization of dual variables

We construct the partition $\Delta\Gamma_c$ of $\overline{\Gamma}_c$ into segments $I \in \Delta\Gamma_c$ so, that there is one–to–one correspondence between contact nodes a_i and segments $I_i \in \Delta\Gamma_c$.

$$L = \{\mu \in L^2(\Gamma_c)| \quad \mu_{|_I} \in P_0(I) \quad \forall I \in \Delta\Gamma_c\}$$

$$\Lambda_\nu = \{\mu \in L| \quad \mu \leq 0 \; a.e. \; on \; \Gamma_c\}$$

$$\Lambda_T = \{\mu \in L| \quad |\mu^i| \leq \mathcal{F} g^i \; \forall I \in \Gamma_c\}$$

$$\langle \mu, v \rangle_{\Gamma_c} := \sum_{i \in \mathcal{I}} \mu^i v(a_i)|I_i|, \quad |I_i| \ldots length \; I_i$$

Discretization and FETI-DP

Discretization of dual variables

To remove discontinuity of functions on $int\,\Gamma$ we introduce a new set Λ_Γ of Lagrange multipliers on Γ:

$$\Lambda_\Gamma = \Lambda_{\Gamma_1} \times \cdots \times \Lambda_{\Gamma_p} \ ,$$

where Λ_{Γ_k}, $k = 1, \ldots, p$ are chosen in such a way that the following conditions are satisfied:

$$\begin{array}{lllll} \mu_k \in \Lambda_{\Gamma_k} & \wedge & \langle \mu_k, w_k \rangle_{\Gamma_k} = 0 & \forall w_k \in W_k & \Rightarrow & \mu_k = 0 \\ w_k \in W_k & \wedge & \langle \mu_k, w_k \rangle_{\Gamma_k} = 0 & \forall \mu_k \in \Lambda_{\Gamma_k} & \Rightarrow & w_k = 0 \end{array}$$

and

$$W_k = \mathbb{X}^r_{i\,|_{\Gamma_k}} = \mathbb{X}^r_{j\,|_{\Gamma_k}} \ , \quad \Gamma_k = \partial\Omega_i \cap \partial\Omega_j$$

Discretization and FETI-DP

The FETI-DP method reads as follows (the index h is omitted):

$$\left.\begin{array}{l} \textit{Find } (u, \lambda_\Gamma, \lambda_\nu, \lambda_T) \in \mathbb{X} \times \Lambda_\Gamma \times \Lambda_\nu \times \Lambda_T(g) \textit{ such that} \\ \sum_{i=1}^{q} a_i(u,w) = \sum_{i=1}^{q} L_i(w) + \langle \lambda_\Gamma, [w] \rangle_\Gamma + \langle \lambda_\nu, w_\nu \rangle_{\Gamma_c} \\ \qquad + \langle \lambda_T, w_T \rangle_{\Gamma_c} \quad \forall w \in \mathbb{X} \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle_{\Gamma_c} + \langle \mu_T - \lambda_T, u_T \rangle_{\Gamma_c} \ge \langle \mu_T - \lambda_T, v_T \rangle_{\Gamma_c} \\ \qquad \forall (\mu_\nu, \mu_T) \in \Lambda_\nu \times \Lambda_T(g) \\ \langle \mu_\Gamma, [u] \rangle_\Gamma = 0 \quad \forall \mu_\Gamma \in \Lambda_\Gamma \ , \end{array}\right\}$$

where

$$\langle \ , \ \rangle_\Gamma := \sum_{k=1}^{p} \langle \ , \ \rangle_{\Gamma_k}$$

and $[v]$ is the jump of v across $int\,\Gamma$.

Discretization and FETI-DP

The previous problem is equivalent to the following system of equations:

$$\left.\begin{array}{l} a_i(u_i, w_i^r) = L_i(w_i^r) + \langle \lambda_\Gamma, [w_i^r] \rangle_{\Gamma \cap \partial\Omega_i} + \langle \lambda_\nu, w_{i\nu}^r \rangle_{\Gamma_c \cap \partial\Omega_i} + \\ \qquad \langle \lambda_T, w_{iT}^r \rangle_{\Gamma_c \cap \partial\Omega_i} \quad \forall w_i^r \in \mathbb{X}_i^r, \; i = 1, \ldots, q \\ \sum_{i=1}^{q} a_i(u_i, w^c) = L(w^c) + \langle \lambda_\nu, w_\nu^c \rangle_{\Gamma_c} + \langle \lambda_T, w_T^c \rangle_{\Gamma_c} \quad \forall w^c \in \mathbb{X}^c \\ + \textit{ conditions for Lagrange multipliers} \end{array}\right\}$$

$$u_i := u_{|\Omega_i} \;, \quad a_i := a_{|\Omega_i} \;, \quad L_i := L_{|\Omega_i}$$

Elimination of $u_i = u_i^r + u_i^c$ leads to a quadratic programming problem for λ_Γ, λ_ν and λ_T with box constraints of the following type:

QP problem with box constraints

$$\left.\begin{array}{l} \textit{Find } \boldsymbol{\lambda} := (\boldsymbol{\lambda}_\Gamma, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_T) \in \mathbb{R}^c \times \mathbb{R}^d_- \times \boldsymbol{\Lambda}_T(\mathbf{g}) \textit{ such that} \\ \mathcal{S}(\boldsymbol{\lambda}) = \min\limits_{\substack{\boldsymbol{\mu}_\Gamma \in \mathbb{R}^c \\ \boldsymbol{\mu}_\nu \in \mathbb{R}^d_- \\ \boldsymbol{\mu}_T \in \boldsymbol{\Lambda}_T(\mathbf{g})}} \mathcal{S}(\boldsymbol{\mu}) \end{array}\right\}$$

where

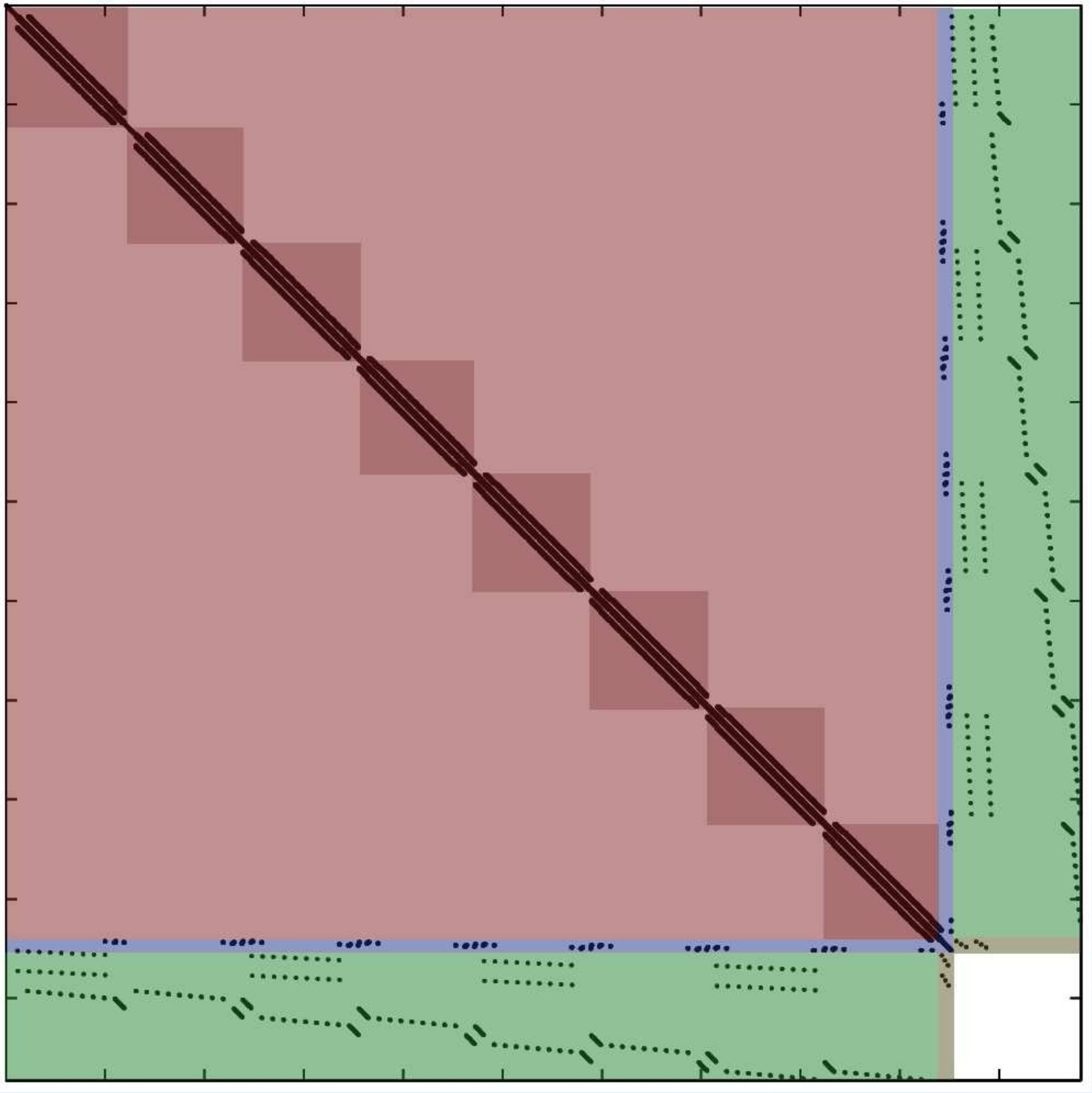
$$\mathcal{S}(\boldsymbol{\mu}) = \frac{1}{2}\boldsymbol{\mu}^\top \mathbf{Q}\boldsymbol{\mu} - \boldsymbol{\mu}^\top \mathbf{h}$$

and

$$\boldsymbol{\Lambda}_T(\mathbf{g}) = \{\boldsymbol{\mu} \in \mathbb{R}^d |\ |\boldsymbol{\mu}_i| \leq \mathbf{g}_i\}, \quad \mathbf{g} \in \mathbb{R}^d_+ .$$

For solving this quadratic programming with simple (box) constraints we use the algorithm MPGRP [Dostál, Schöeberl].

Matrix in Feti-DP saddle point problem

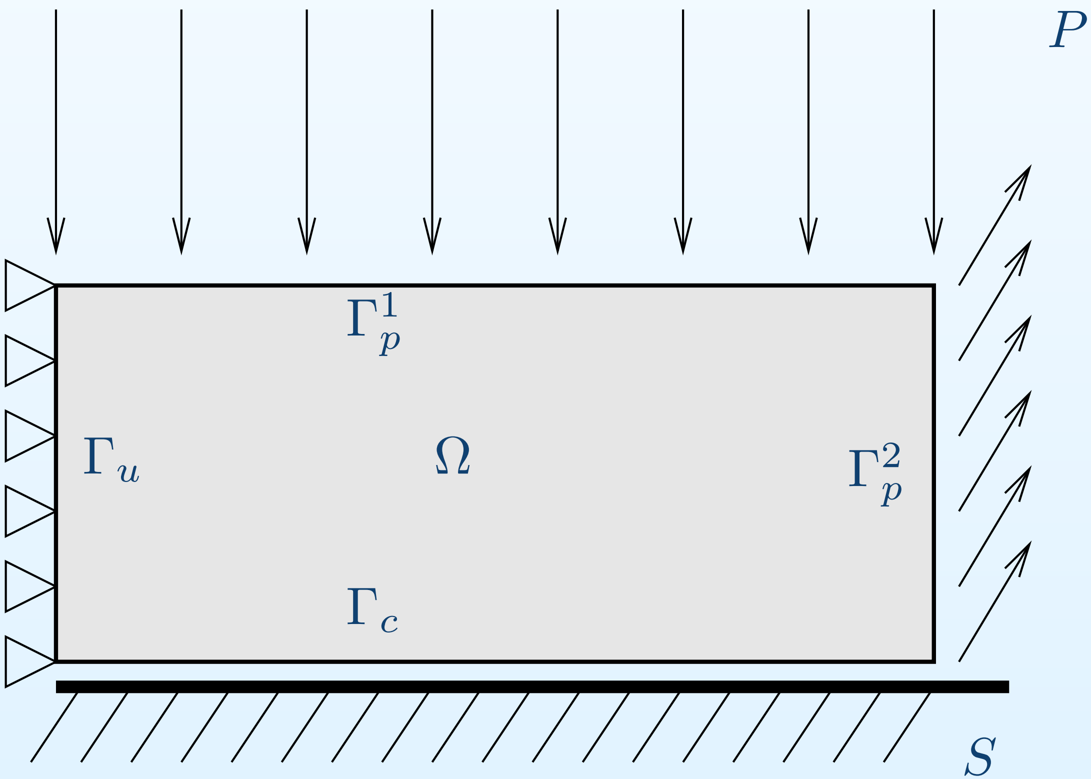


Model example

$\Omega = (0,2) \times (0,1)$ (in meters).
Young's modulus $E = 21.19e10[Pa]$,
Poisson's ratio $\sigma = 0.277$, coefficient of friction $\mathcal{F} = 0.3$

$P_1(t) = 0.$, $P_2(t) = -10.e7\phi(t)$, on Γ_p^1

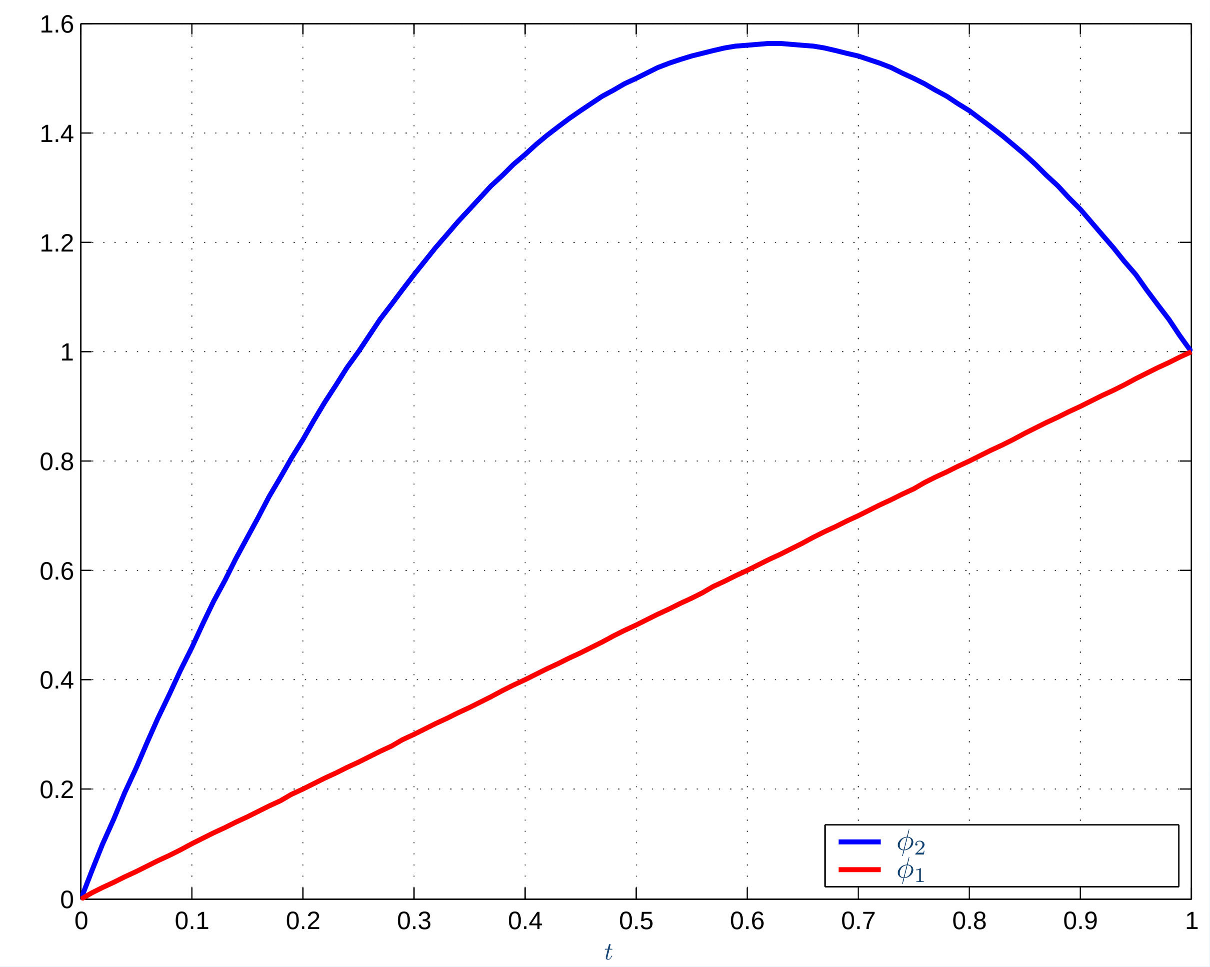
$P_1(t) = 3.e7\phi(t)$, $P_2(t) = 5.e7\phi(t)$, on Γ_p^2



History of loading (characterized by $\phi : [0,1] \to \mathbb{R}^1$)

$\phi_1(t) = t$ *(monotne loading)*

$\phi_2(t) = -4t^2 + 5t$ *(nonmonotne loading)*



Dependence on h and the number of subdomains

n_s . . . number of subdomains

n_p, n_d . . . number of primal, dual variables, respectively

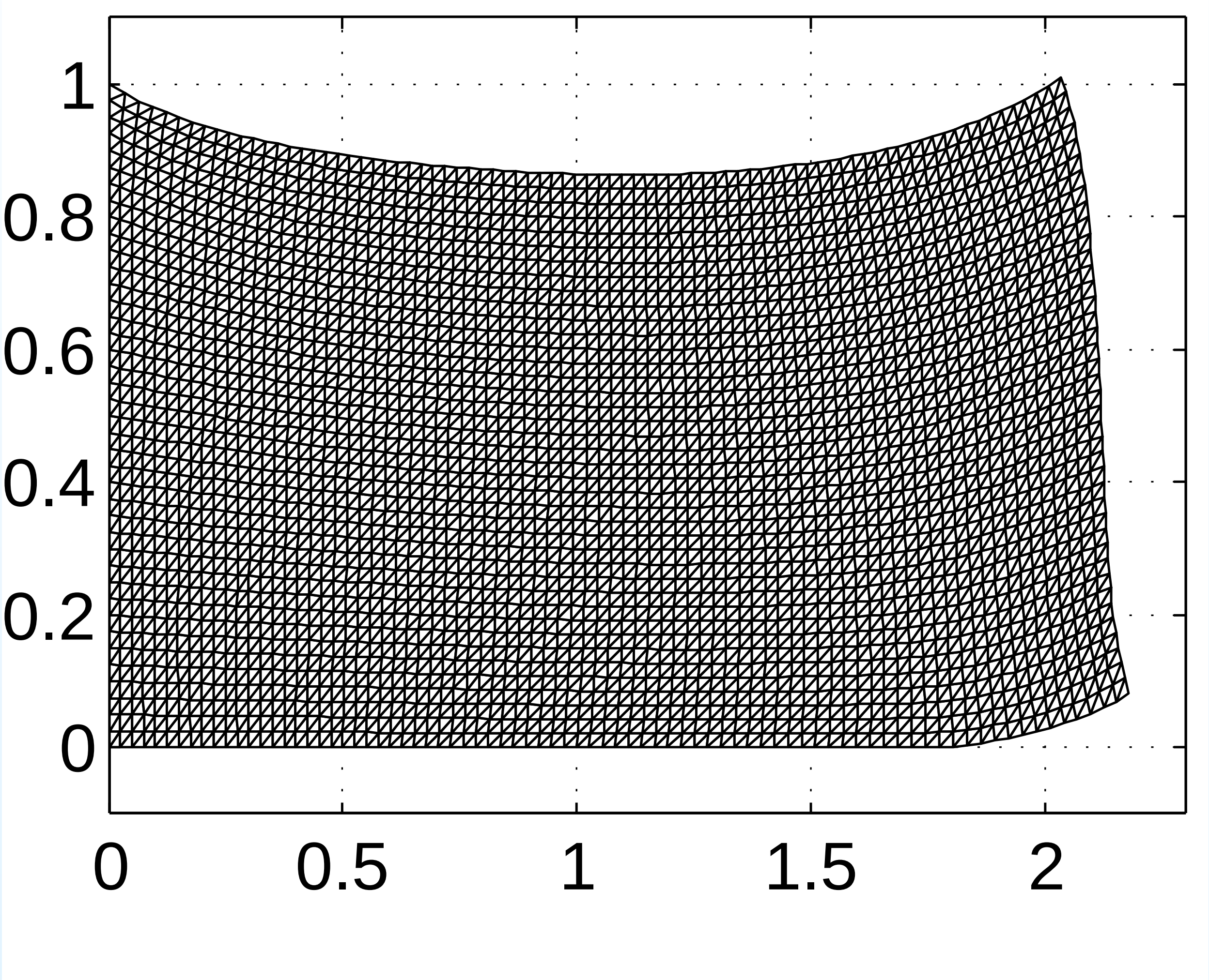
it . . . fixed–point iterations

n_m . . . dual matrix multiplications

n_s	n_p	n_d	it	n_m
8	1936	260	135/120	6515/ 5565
32	7744	1096	136/121	8326/ 7309
128	30974	4496	136/122	11423/ 9844
512	123904	18208	135/122	15206/13237
1024	495616	73280	146/129	22887/19659

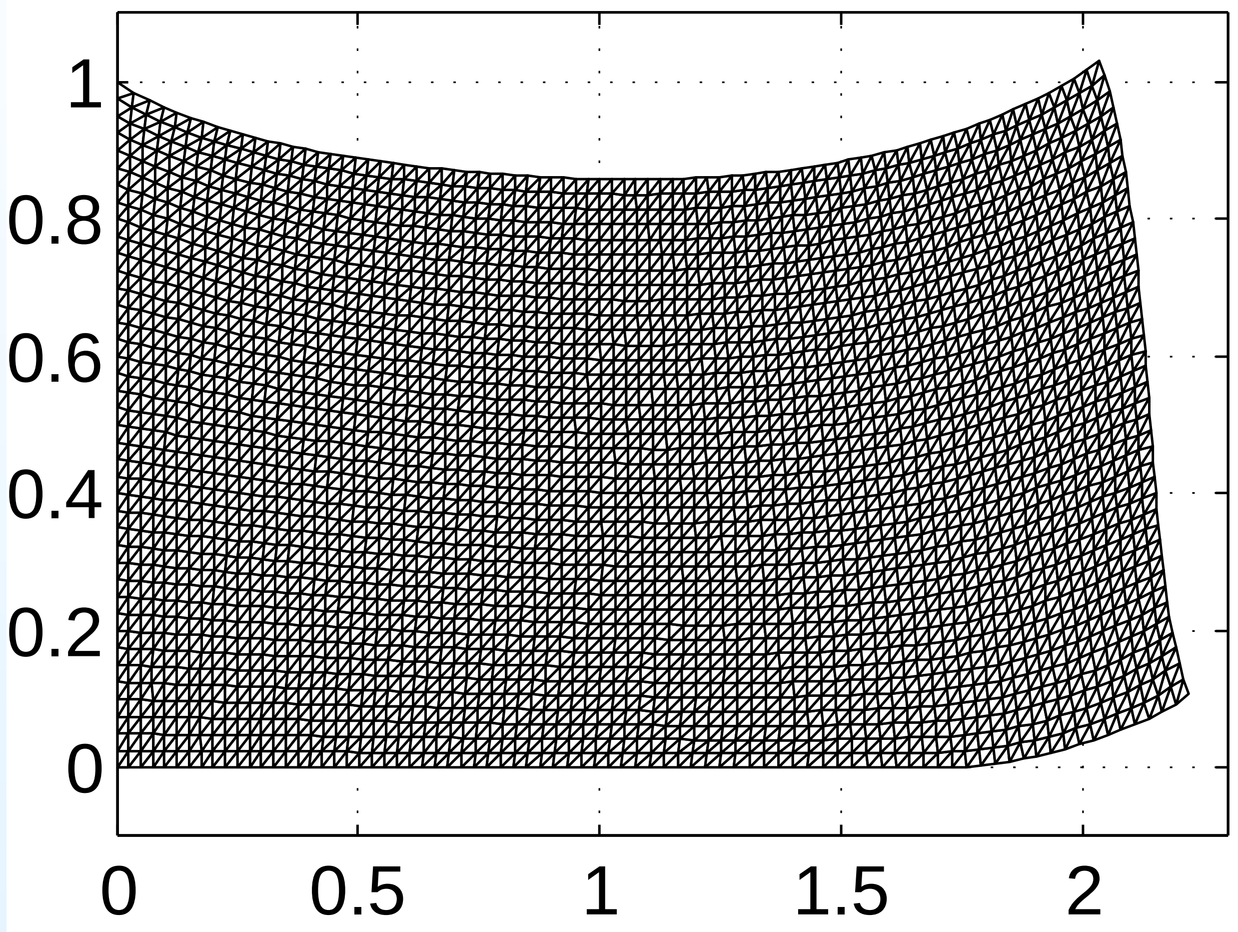
Deformation of Ω at $t = 1$ enlarged $300\times$

ϕ_1

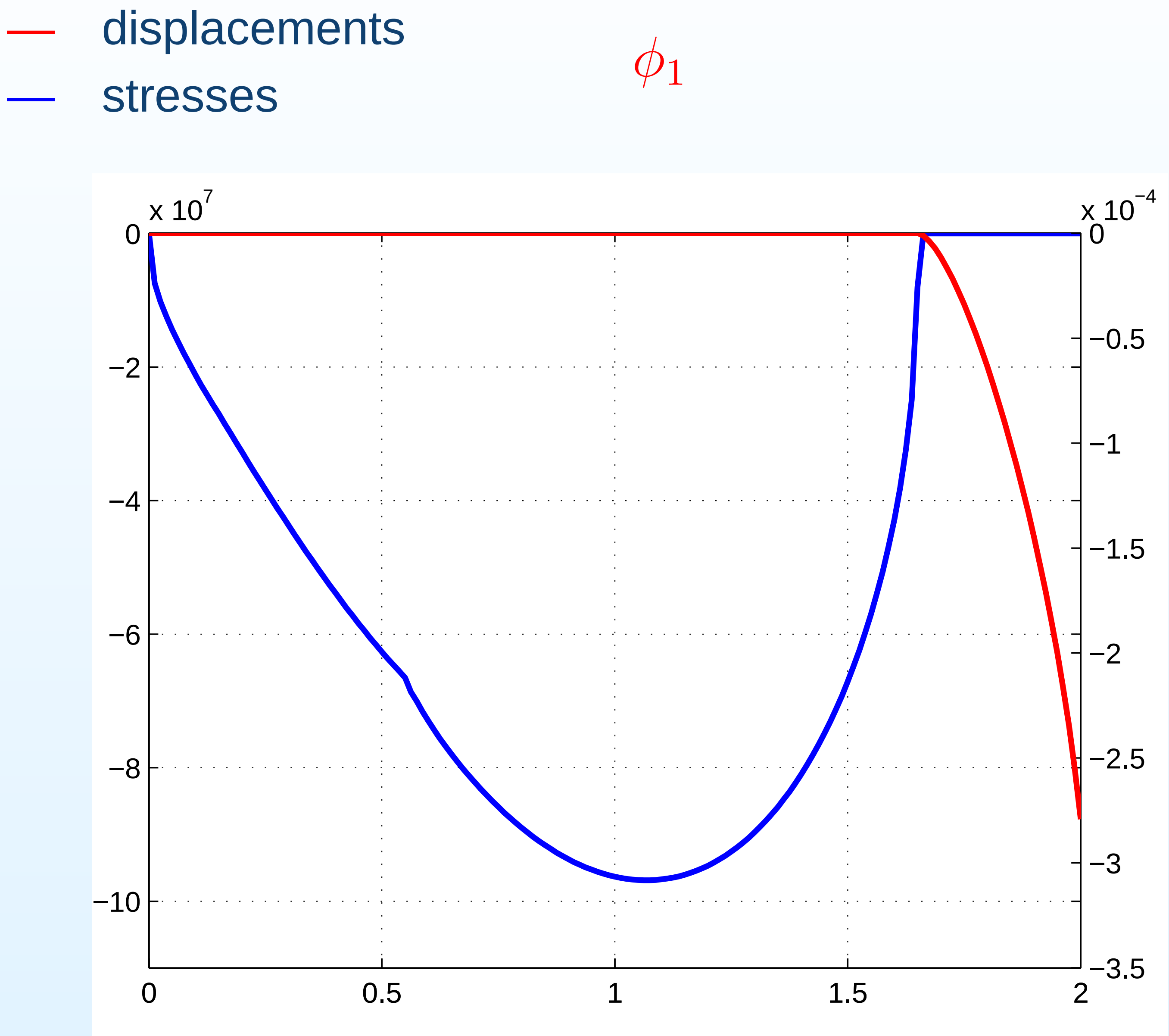


Deformation of Ω at $t = 1$ enlarged $300\times$

ϕ_2



Normal stress and displacement on Γ_c at $t = 1$

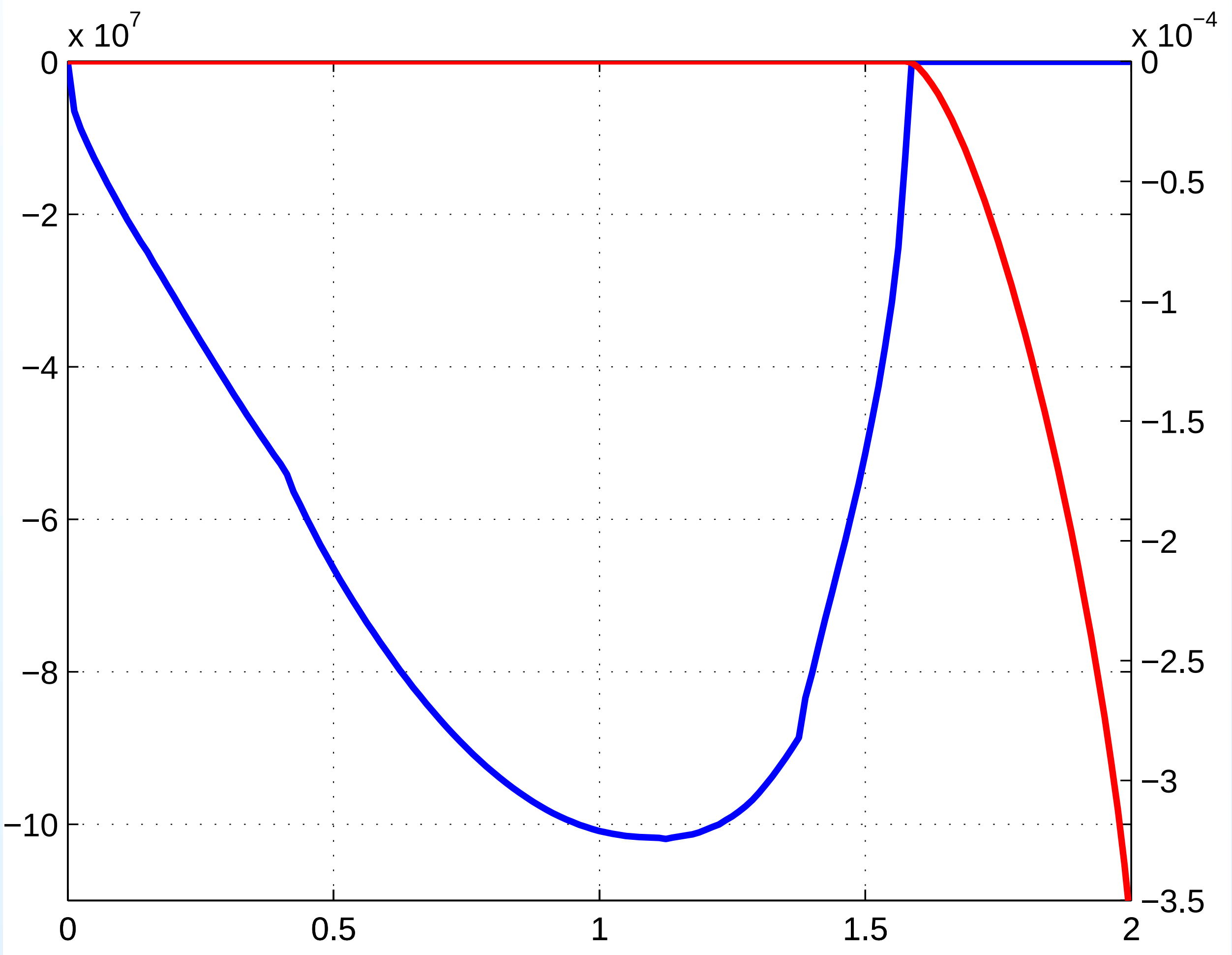


Normal stress and displacement on Γ_c at $t = 1$

— displacements

— stresses

ϕ_2

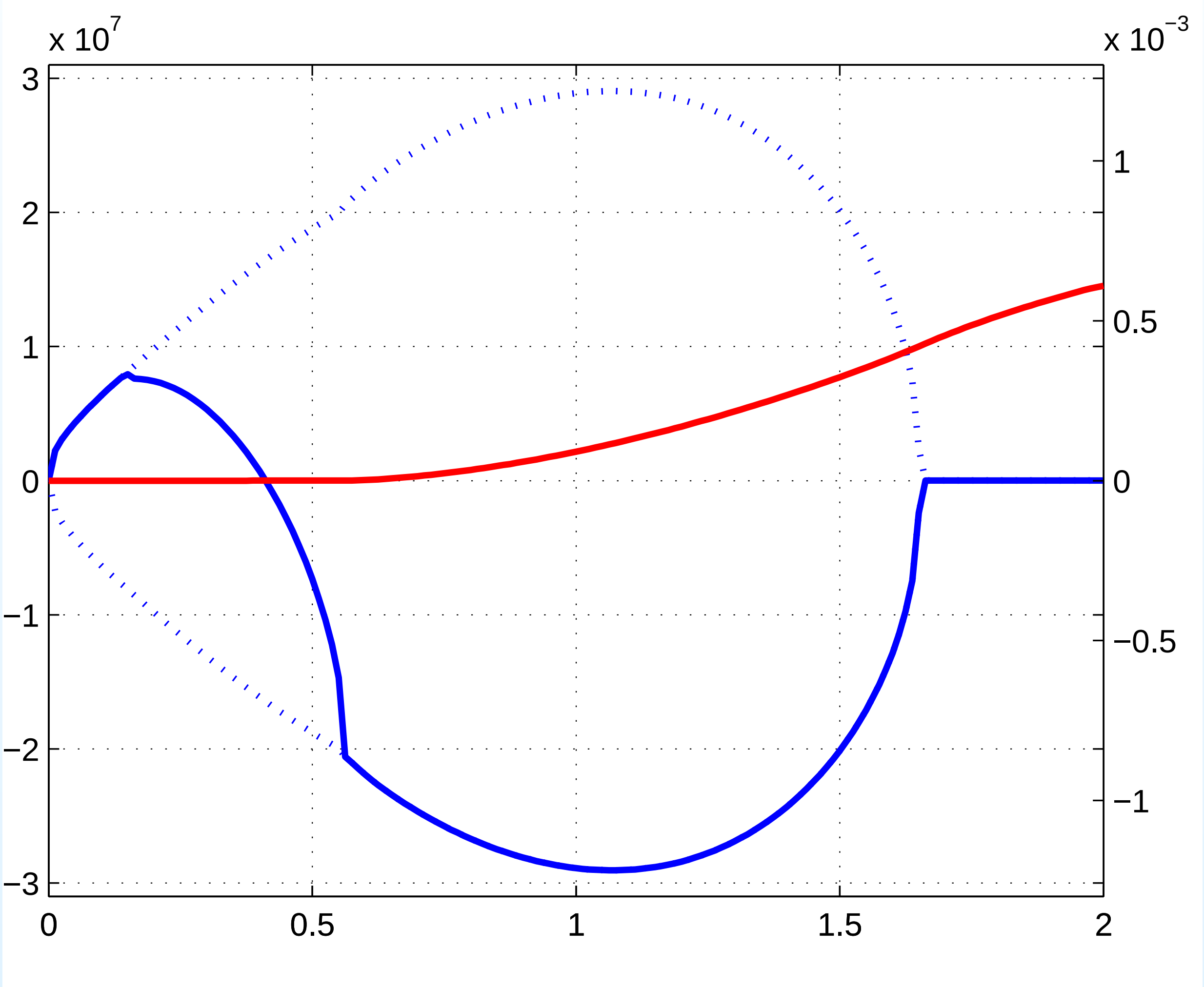


Tangential stress and displacement on Γ_c at $t = 1$

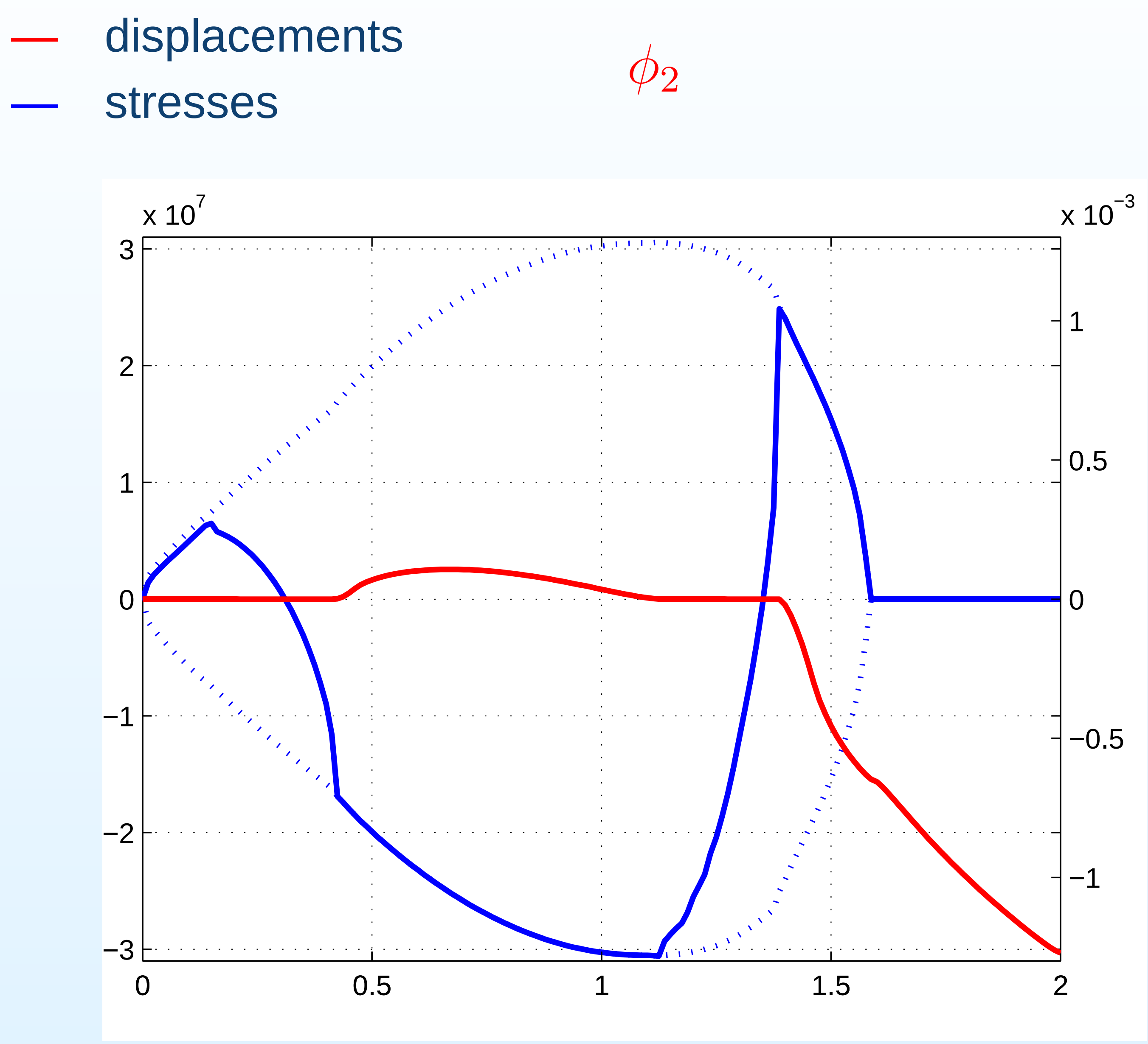
— displacements

— stresses

ϕ_1



Tangential stress and displacement on Γ_c at $t = 1$



Reference

Contact problems with friction (static case)

Theoretical analysis: Duvaut, Lions, Nečas, Jarušek, Eck, Oden, Kikuchi, Frémond, . . .

Discretization, numerical realization: Panagiotopoulos, Raous, Cocou, Hild, Laborde, Renard, Bisegna, Lebon, Hlaváček, Kikuchi, Wohlmuth, Krause, . . .

Qualitate analysis: Hild, Renard, Ionescu, Balard, . . .