Polynomial Roots and Approximate Greatest Common Divisors

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1. NOTATION

A polynomial is written as

$$f(x) = \sum_{i=0}^{m} a_i \phi_i(x)$$

where $\phi_i(x), i = 0, \ldots, m$, is a set of linearly independent polynomial basis functions and $a_i, i = 0, \ldots, m$ is a set of coefficients.

The power basis:	$\phi_i(x) = x^i, \quad i = 0, \ldots, m$
The Bernstein basis:	$\phi_i(x) = \binom{m}{i}(1-x)^{m-i}x^i, \quad i = 0, \ldots, m$
The scaled Bernstein basis:	$\phi_i(x) = (1-x)^{m-i}x^i, \quad i = 0, \ldots, m$

Example 1.1 A third order polynomial is written as:

The power basis:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

The Bernstein basis:

$$\begin{aligned} f(x) &= b_0 \left(\binom{3}{0} (1-x)^3 \right) + b_1 \left(\binom{3}{1} (1-x)^2 x \right) \\ &\quad + b_2 \left(\binom{3}{2} (1-x) x^2 \right) + b_3 \left(\binom{3}{3} x^3 \right) \end{aligned}$$

The scaled Bernstein basis:

$$f(x) = c_0 (1-x)^3 + c_1 (1-x)^2 x + c_2 (1-x) x^2 + c_3 x^3$$

□

The power basis: The most frequently used polynomial basis

The Bernstein basis: Used extensively in geometric modelling because of its elegant geometric properties and numerical stability:

$$((1-x)+x)^m = \sum_{i=0}^{m} \binom{m}{i} (1-x)^{m-i} x^i = 1$$

The scaled Bernstein basis: Simpler to use than the Bernstein basis, and easy to convert between the scaled Bernstein basis and the Bernstein basis. From Example 1.1

$$c_i = \binom{m}{i} b_i, \qquad i = 0, \ldots, m$$

The norm of a polynomial

It is assumed that:

- The coefficients of $f(x)$ are real
- A polynomial of degree m can be identified with a vector space of dimension $m+1$
- The elements of vectors that lie in this space are the coefficients of $f(x)$ with respect to the basis functions $\phi_i(x), i = 0, \ldots, m$

The 2-norm of $f(x)$ is equal to the 2-norm of the vector $\boldsymbol{a}$ of its coefficients

$$\|f(x)\| = \|\boldsymbol{a}\| = \left(\sum_{i=0}^{m} a_i^2 \right)^{\frac{1}{2}}$$

Unless stated otherwise, the 2-norm is used exclusively, $\|\cdot\| = \|\cdot\|_2$.

2. INTRODUCTION

Four problems:

1. Computation of an approximate greatest common divisor (AGCD)

   Calculate an AGCD of the polynomials

$$
\begin{aligned}
f(x) &= (x-1)^8(x-2)^{16}(x-3)^{24}+\epsilon_1 \\
g(x) &= (x-1)^{12}(x+2)^4(x-3)^8(x+4)^2+\epsilon_2
\end{aligned}
$$

   Some applications: Pole-zero cancellation in control systems, error control coding and a novel algorithm to compute the roots of a polynomial

   Extend to polynomials expressed in the Bernstein basis

2. Structured low rank approximation of the Sylvester resultant matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -5 & 1 & 0 & -5.99 & 1 & 0 \\ 8 & -5 & 1 & 10.96 & -5.99 & 1 \\ -4 & 8 & -5 & -5.97 & 10.96 & -5.99 \\ 0 & -4 & 8 & 0 & -5.97 & 10.96 \\ 0 & 0 & -4 & 0 & 0 & -5.97 \end{bmatrix}$$

The singular values are

$$\begin{bmatrix} 25.314 & 14.959 & 5.5837 & 0.42448 & 2.6536e-06 & 6.6615e-16 \end{bmatrix}$$

Application: Intersection problems in geometric modelling

Extend to polynomials expressed in the Bernstein basis

3. A robust algorithm for computing the roots of a polynomial

Calculate the roots of the polynomial

$$f(x) = (x-1)^{20}(x-2)^{30}(x-3)^{40} + \epsilon_1$$

Extend to polynomials expressed in the Bernstein basis

4. The principle of maximum likelihood for the estimation of the rank of a matrix

Estimate the rank of an arbitrary matrix that is corrupted by noise of unknown magnitude

Applications: Signal processing, AGCD computations, regularisation

2.1 Historical background and difficulties of computing polynomial roots

There exist many algorithms for computing the roots of a polynomial:

- Bairstow, Graeffe, Jenkins-Traub, Laguerre, Müller, Newton ...

These methods yield satisfactory results if:

- The polynomial is of moderate degree
- The roots are simple and well-separated
- A good starting point in the iterative scheme is used

This heuristic has exceptions:

$$f(x) = \prod_{i=1}^{20}(x-i) = (x-1)(x-2)\cdots(x-20)$$

Example 2.1 Consider the polynomial

$$x^4 - 4x^3 + 6x^2 - 4x + 1 = (x-1)^4$$

whose root is $x = 1$ with multiplicity 4. MATLAB returns the roots

1.0002, 1.0000 + 0.0002i, 1.0000 - 0.0002i, 0.9998 □

Example 2.2 The roots of the polynomial $(x-1)^{100}$ were computed by MATLAB.

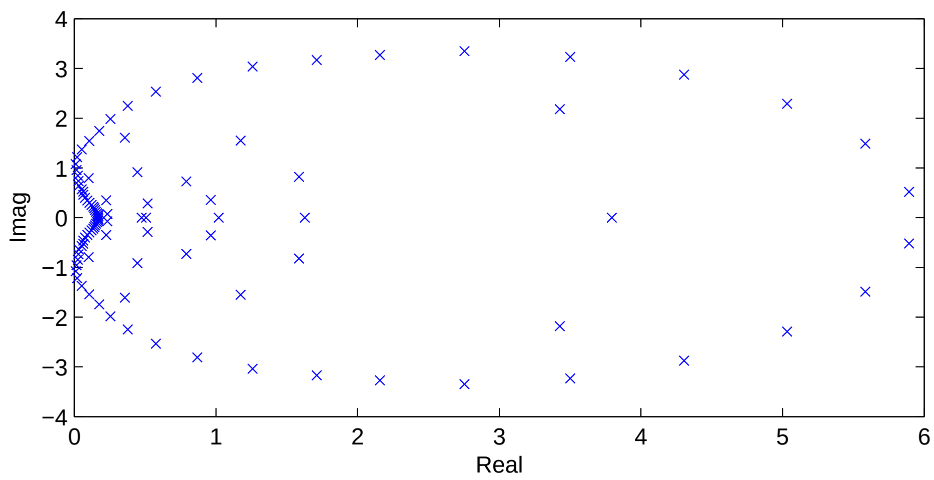

Figure 2.1: The computed roots of $(x-1)^{100}$.

□

Example 2.3 Consider the roots $(x-1)^m$ due to a perturbation of -2^{-10} in the constant coefficient.

$$x = 1 + \frac{1}{2^{\frac{10}{m}}}\left(\cos\frac{2\pi k}{m} + i\sin\frac{2\pi k}{m}\right), \qquad k = 0, \ldots, m-1.$$

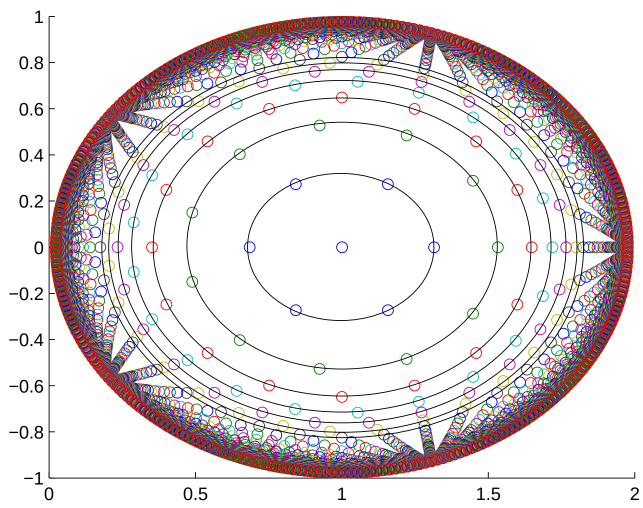

Figure 2.2: Perturbation regions of $(x-1)^m, \quad m = 1, 6, 11, \ldots,$ when the constant term is perturbed by -2^{-10}. □

A robust algorithm to compute the multiple roots of a polynomial includes:

- AGCD computations
- Structured low rank approximations of the Sylvester resultant matrix
- The estimation of the rank of a matrix

3. CONDITION NUMBERS AND ERRORS

All computations must consider the effects of errors. Common sources of error in computation include:

- Modelling errors
  - A low dimensional approximation of a high dimensional system
  - The restriction to low order terms
- Roundoff errors – Due to floating point arithmetic
- Data errors – Experimental data is inexact
- Discretisation errors – The use of the finite element or finite difference methods to solve a partial differential equation

3.1 Backward errors and the forward error

The forward error is a measure of the error δx_0 in the solution x_0

$$\text{forward error } = \frac{|\delta x_0|}{|x_0|}$$

- The exact solution x_0 is not known *a priori*, and so the forward error is not a practical measure of the error.
- The backward error is a more practical error measure because it does not require knowledge of the exact solution.

Note the difference:

The forward error is measured in the solution space

The backward error is measured in the data space

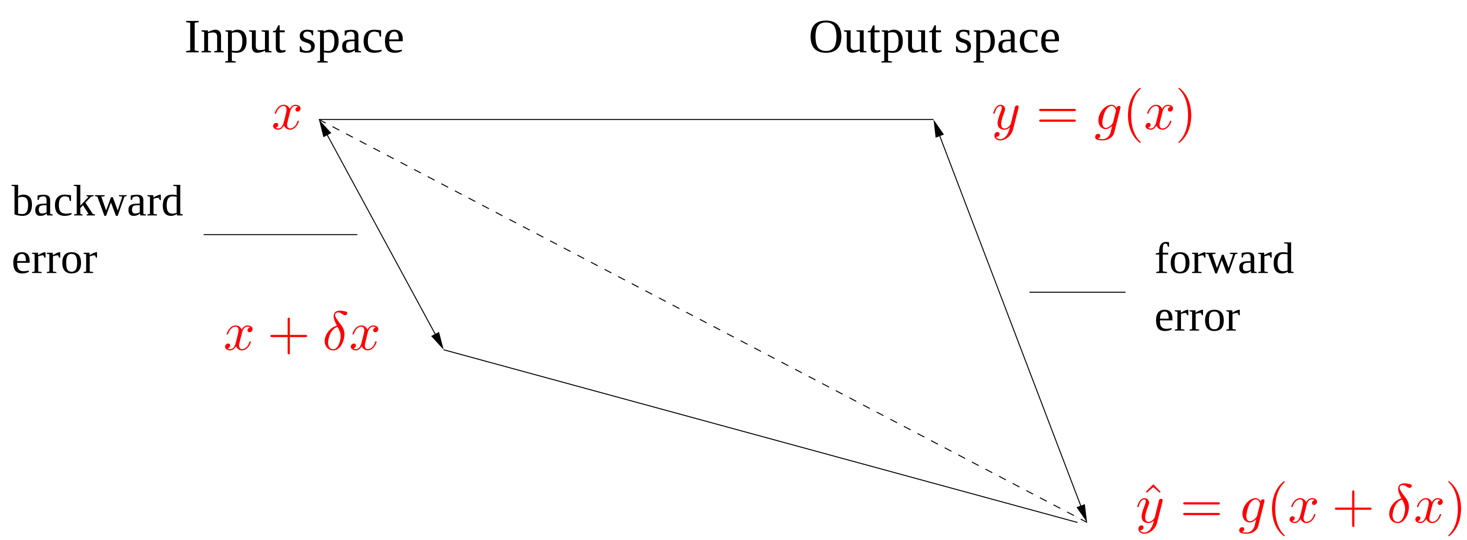

Figure 3.1: Backward and forward errors for $y = g(x)$. The solid lines represent exact computations, and the dashed line represents the computed approximation.

- A small forward error means that the error in the computed solution is small.
- A small backward error means that the approximate solution is the theoretically exact solution of a problem that is near the problem whose solution is desired.

How are the forward and backward errors related?

forward error = condition number $\times$ backward error

Note: This equation takes on a more complex form for multiple roots of a polynomial.

- The forward error is the error in the solution: $\frac{|\delta x_0|}{|x_0|}$
- The backward error is the 'distance' between two problems. How is it defined?
- What is the condition number and how is it defined?

3.1.1 Backward errors

The backward error is the 'distance' between two problems.

It is required to compute the roots of the polynomial

$$f_1(x) = \sum_{i=0}^{m} a_i \phi_i(x)$$

but roundoff errors and the iterative nature of most root finding algorithms cause errors in the solution, such that the roots of the neighbouring polynomial

$$f_2(x) = \sum_{i=0}^{m} (a_i + \delta a_i)\, \phi_i(x)$$

are computed.

What is the backward error of the computed root $\tilde{x}_0$ of $f_2(x)$?

The backward error of a computed root $\tilde{x}_0$ of $f_2(x)$ is the minimum amount that the coefficients of $f_1(x)$ must move such that it has a root $\tilde{x}_0 = x_0 + \delta x_0$:

$$f_1(x_0) = \sum_{i=0}^{m} a_i \phi_i(x_0) = 0 \qquad \text{and} \qquad f_1(\tilde{x}_0) = \sum_{i=0}^{m} a_i \phi_i(\tilde{x}_0) \neq 0$$

but

$$f_2(\tilde{x}_0) = \sum_{i=0}^{m} (a_i + \delta a_i)\, \phi_i(\tilde{x}_0) = 0$$

There are two types of backward error:

- Componentwise backward error

  The relative error in each coefficient has an upper bound:

  $$|\delta a_i| \le \varepsilon_c \, |a_i| \, , \qquad i = 0, \ldots, m$$

  The componentwise backward error is the minimum value of ε_c that satisfies this inequality, such that $f_2(\tilde{x}_0) = 0$.

- Normwise backward error

  The relative error in the coefficients has an upper bound:

  $$\|\delta a\| \le \varepsilon_n \, \|a\|$$

  The normwise backward error is the minimum value of ε_n that satisfies this inequality, such that $f_2(\tilde{x}_0) = 0$.

Why is backward error analysis important?

- Backward error analysis interprets *all* the errors (data errors, roundoff errors, etc.) as being equivalent to perturbations in the data.
- If the backward error is less than the error in the data, then the computed solution is acceptable.
- An expression for the backward error of an approximate solution is independent of the method used for its computation.

Definition 3.1 Componentwise backward error

The componentwise backward error of the approximation $\tilde{x}_0$, which may be complex, of the root x_0 of $f(x)$ is defined as

$$
\begin{aligned}
\eta_c(\tilde{x}_0) &= \min\left\{\varepsilon_c : \sum_{i=0}^{m} \tilde{a}_i \phi_i(\tilde{x}_0) = 0 \text{ and } |\delta a_i| \le \varepsilon_c |a_i| ; \tilde{a} = a + \delta a\right\} \\
&= \frac{|f(\tilde{x}_0)|}{\sum_{i=0}^{m} |a_i \phi_i(\tilde{x}_0)|}
\end{aligned}
$$

The perturbations in the coefficients that achieve this backward error are

$$
\delta a_k = -\left(\frac{a_k f(\tilde{x}_0)}{\sum_{i=0}^{m} |a_i \phi_i(\tilde{x}_0)|}\right)\left(\frac{a_k \overline{\phi_k(\tilde{x}_0)}}{|a_k \phi_k(\tilde{x}_0)|}\right), \qquad k = 0, \ldots, m
$$

Definition 3.2 Normwise backward error

The normwise backward error of the approximation $\tilde{x}_0$, which may be complex, of the root x_0 of $f(x)$ is defined as

$$\eta_n(\tilde{x}_0) = \min\left\{\varepsilon_n : \sum_{i=0}^{m} \tilde{a}_i \phi_i(\tilde{x}_0) = 0 \text{ and } \|\delta a\| \le \varepsilon_n \|a\| ; \tilde{a} = a + \delta a\right\}$$
$$= \frac{|f(\tilde{x}_0)|}{\|\phi(\tilde{x}_0)\| \, \|a\|}$$

The perturbations in the coefficients that achieve this backward error are

$$\delta a_i = \frac{f(\tilde{x}_0)\, v_i(\tilde{x}_0)}{\|\phi(\tilde{x}_0)\|}, \qquad v(\tilde{x}_0)^T \phi(\tilde{x}_0) = -\|\phi(\tilde{x}_0)\|, \qquad \|v(\tilde{x}_0)\| = 1$$

3.2 Condition numbers

The forward and backward errors are related by the condition number of the problem, which is a measure of the sensitivity of the solution to perturbations in the data.

Example 3.1 Consider the evaluation of the function $y = g(x)$ at the point $x = x_0$, and assume that the computed solution is in error.

$$y + \delta y = g(x_0 + \delta x_0) = g(x_0) + g^{(1)}(x_0)\delta x_0 + \frac{g^{(2)}(x_0)}{2}(\delta x_0)^2$$

Thus, to first order

$$\frac{\delta y}{y} = \left(\frac{x_0 g^{(1)}(x_0)}{g(x_0)}\right)\frac{\delta x_0}{x_0}$$

The quantity

$$\left|\frac{x_0 g^{(1)}(x_0)}{g(x_0)}\right|$$

is the condition number of x_0. □

The condition number of x_0 is a measure of the relative change in the output y for a given relative change in the input x:

- A root of a polynomial is ill-conditioned if its condition number is large
- A root of a polynomial is well-conditioned if its condition number is small

Extend this concept to polynomials:

- If the coefficients of the polynomial change by a small amount, how do the roots change?
- Use componentwise and normwise error models to define the perturbations in the coefficients, and thus obtain a componentwise condition number and a normwise condition number.

3.2.1 The componentwise condition number of a root of a polynomial

Theorem 3.1 Let the coefficients a_i of $f(x)$ be perturbed to $a_i + \delta a_i$ where $|\delta a_i| \leq \varepsilon_c |a_i|, i = 0 \ldots m$. Let the real root x_0 of $f(x)$ have multiplicity r, and let one of these r roots be perturbed to $x_0 + \delta x_0$ due to the perturbations in the coefficients. Then the componentwise condition number of x_0 is

$$\kappa_c(x_0) = \max_{|\delta a_i| \leq \varepsilon_c |a_i|} \frac{|\delta x_0|}{|x_0|} \frac{1}{\varepsilon_c} = \frac{1}{\varepsilon_c^{1-\frac{1}{r}}} \frac{1}{|x_0|} \left(\frac{r!}{\left| f^{(r)}(x_0) \right|} \sum_{i=0}^{m} |a_i \phi_i(x_0)| \right)^{\frac{1}{r}}$$

If $r \gg 1$

$$\kappa_c(x_0) \approx \frac{1}{\varepsilon_c} = \text{upper bound of componentwise signal-to-noise ratio}$$

$\kappa_c(x_0)$ defines a circular region in the complex plane in which $x_0 + \delta x_0$ lies.

3.2.2 The normwise condition number of a root of a polynomial

Theorem 3.2 Let the coefficients a_i of $f(x)$ be perturbed to $a_i + \delta a_i$ where $\|\delta a\| \leq \varepsilon_n \|a\|$. Let the real root x_0 of $f(x)$ have multiplicity r, and let one of these r roots be perturbed to $x_0 + \delta x_0$ due to the perturbations in the coefficients. Then the normwise condition number of x_0 is

$$\kappa_n(x_0) = \max_{\|\delta a\| \leq \varepsilon_n \|a\|} \frac{|\delta x_0|}{|x_0|} \frac{1}{\varepsilon_n} = \frac{1}{\varepsilon_n^{1-\frac{1}{r}}} \frac{1}{|x_0|} \left(\frac{r!}{\left| f^{(r)}(x_0) \right|} \|a\| \, \|\phi(x_0)\| \right)^{\frac{1}{r}}$$

If $r \gg 1$

$$\kappa_n(x_0) \approx \frac{1}{\varepsilon_n} = \text{upper bound of normwise signal-to-noise ratio}$$

$\kappa_n(x_0)$ defines a circular region in the complex plane in which $x_0 + \delta x_0$ lies.

3.3 Limitations of the condition numbers

The formulae for $\kappa_c(x_0)$ and $\kappa_n(x_0)$ assume that the lowest order term is used in the Taylor series of the perturbed polynomials:

- What are the implications of this assumption?
- When is the assumption valid, and when is it not valid?

Analogy:

- When two magnets are well separated, their magnetic fields do not not interact.
  - As the distance between the magnets decreases, the magnetic fields interact.
- When two polynomial roots are well separated, the roots can be considered in isolation.
  - As the distance between the roots decreases, they interact and the condition number of a given root is a function of its neighbouring close roots.

Since x_0 is a root of multiplicity r of $f(x)$

$$f\left(x_0+\delta x_0\right) = \sum_{k=0}^{n} \frac{\delta x_0^k}{k!} f^{(k)}\left(x_0\right)$$
$$= \frac{\delta x_0^r}{r!} f^{(r)}\left(x_0\right)+\frac{\delta x_0^{r+1}}{(r+1)!} f^{(r+1)}\left(x_0\right)+\text{higher order terms}$$
$$\approx \frac{\delta x_0^r}{r!} f^{(r)}\left(x_0\right)$$

if

$$\left|\frac{\delta x_0^r}{r!} f^{(r)}\left(x_0\right)\right| \gg\left|\frac{\delta x_0^{r+1}}{(r+1)!} f^{(r+1)}\left(x_0\right)+\text{higher order terms}\right|$$

- The validity of this inequality decreases as two or more roots merge

Example 3.2 Consider the polynomial

$$f(x) = (x-1)^2(x-1-\alpha) = x^3 - (3+\alpha)x^2 + (3+2\alpha)x - (1+\alpha)$$

and let $\varepsilon_c = 10^{-7}$.

Use the complete Taylor expansion, up to and including third order terms, and consider the perturbation regions of

$$x_0 = 1 \text{ (twice)} \quad \text{and} \quad x_1 = 1 + \alpha$$

Note: If only the lowest order term is retained, the perturbation region of each root is a circle.

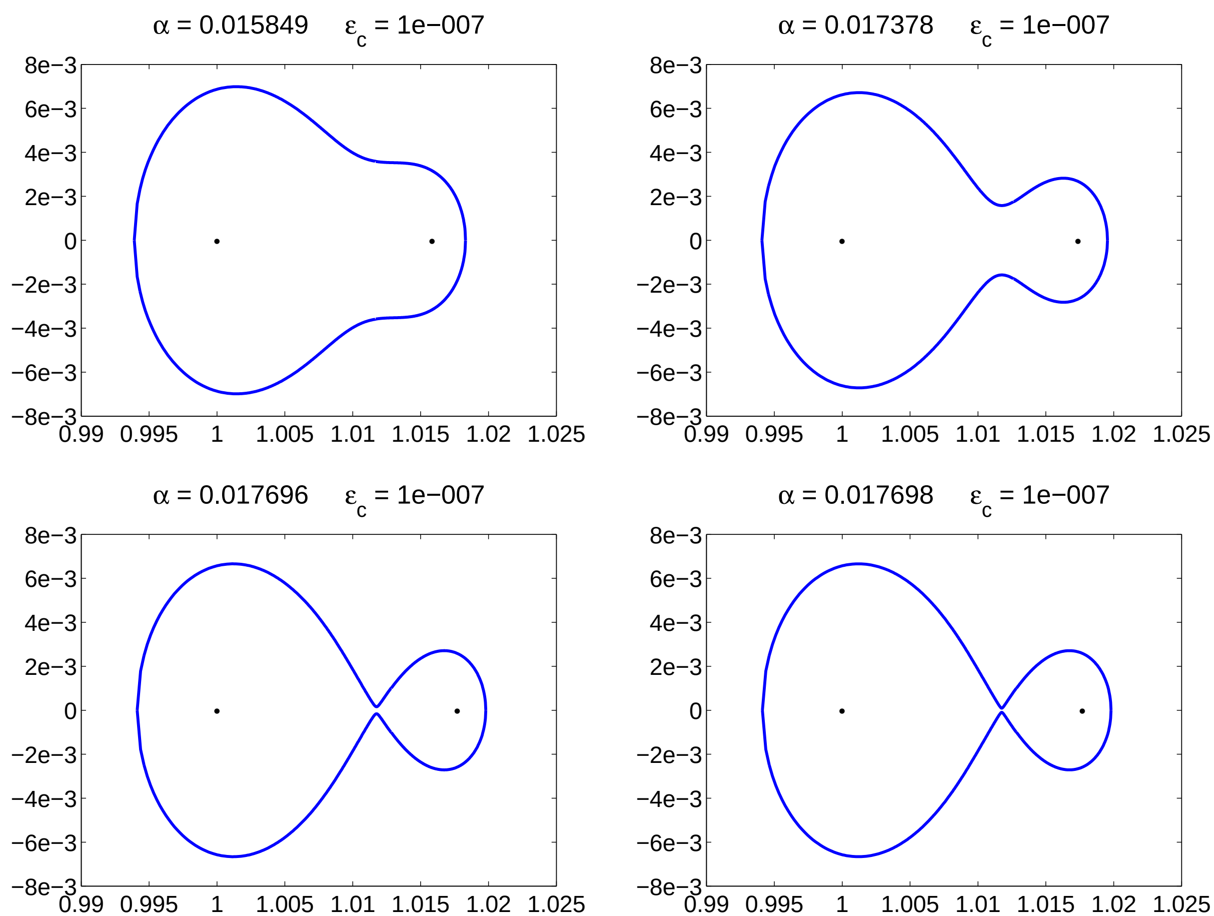

Figure 3.2: The perturbation regions, in the complex plane, for the double root $x_0 = 1$ and the simple root $x_1 = (1 + \alpha)$, for several values of α.

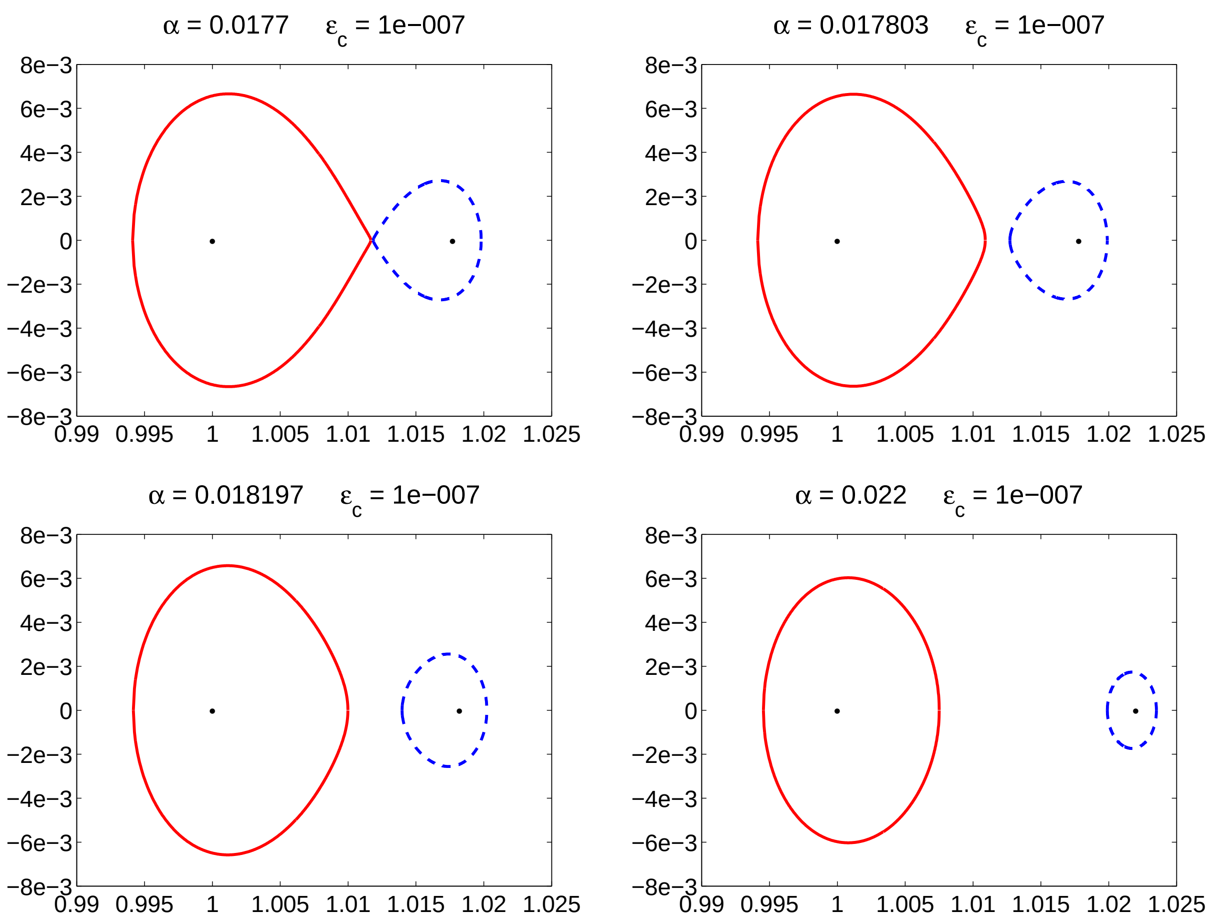

Figure 3.3: The perturbation regions, in the complex plane, for the double root $x_0 = 1$ and the simple root $x_1 = (1+\alpha)$, for several values of α.

$$
\begin{aligned}
f(x) &= (x-1)^2(x-1-\alpha) \\
&= x^3-(3+\alpha)x^2+(3+2\alpha)x-(1+\alpha)
\end{aligned}
$$

Roots of $f(x)$ and their condition numbers using the lowest order term only:

$$x_0 = 1 \text{ (twice)} \qquad \kappa_c\left(x_0\right) = \frac{2}{\sqrt{\varepsilon_c}}\left(\frac{2+\alpha}{\alpha}\right)^{\frac{1}{2}}$$

$$x_1 = 1+\alpha \qquad \kappa_c\left(x_1\right) = 2\left(\frac{2+\alpha}{\alpha}\right)^2$$

When $\alpha = 0$:

$$g(x) = \left. f(x)\right|_{\alpha=0} = (x-1)^3$$

The root of $g(x)$ and its condition number using the lowest order term only:

$$x_2 = 1 \text{ (three times)} \qquad \kappa_c(x_2) = \frac{2}{\varepsilon_c^{\frac{2}{3}}}$$

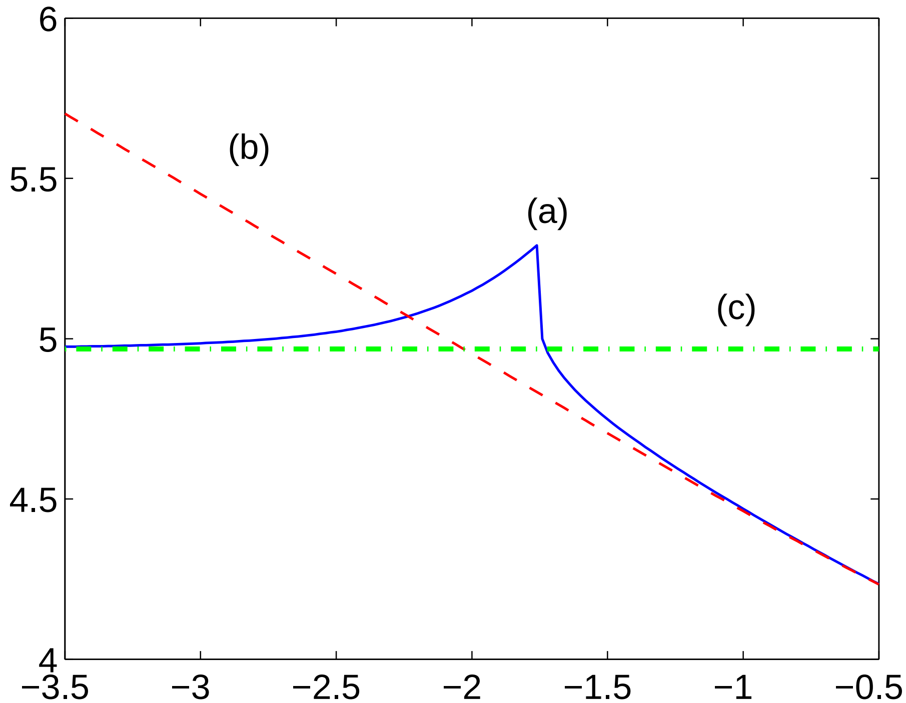

Figure 3.4: The variation of the logarithm, to base 10, of (a) the condition number of the double root $x_0 = 1$ using all terms in the Taylor series, (b) $\kappa_c(x_0)$ using the lowest order term only, and (c) $\kappa_c(x_2)$ of the treble root $x_2 = 1$ of the polynomial $(x-1)^3 = 0$, against $\log_{10}\alpha$.

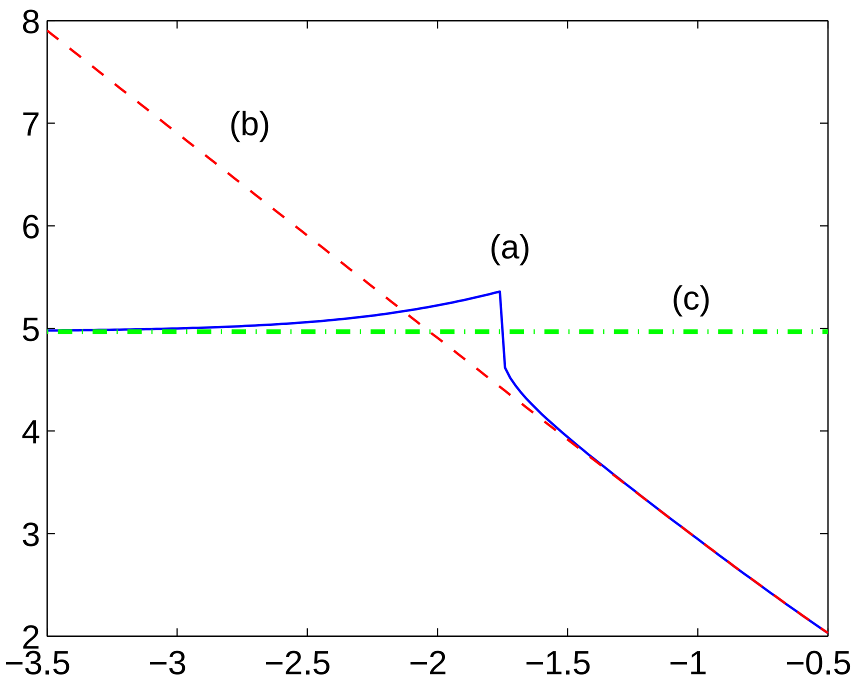

Figure 3.5: The variation of the logarithm, to base 10, of (a) the condition number of the simple root $x_1 = 1 + \alpha$ using all terms in the Taylor series, (b) $\kappa_c(x_1)$ using the lowest order term only, and (c) $\kappa_c(x_2)$ of the treble root $x_2 = 1$ of the polynomial $(x-1)^3 = 0$, against $\log_{10} \alpha$. □

3.4 Principal axes of perturbed roots

Figures 3.2 and 3.3 define the regions in which the perturbed roots may lie.

- Are the perturbed roots randomly scattered in this region, or do they only lie in some portions of the regions, and not other portions?

Example 3.3 Consider the polynomial in Example 3.2

$$f(x) = (x-1)^2(x-1-\alpha) = x^3 - (3+\alpha)x^2 + (3+2\alpha)x - (1+\alpha)$$

and let $\varepsilon_c = 10^{-7}$. Compute the roots of this polynomial 1000 times and consider the distribution of the roots.

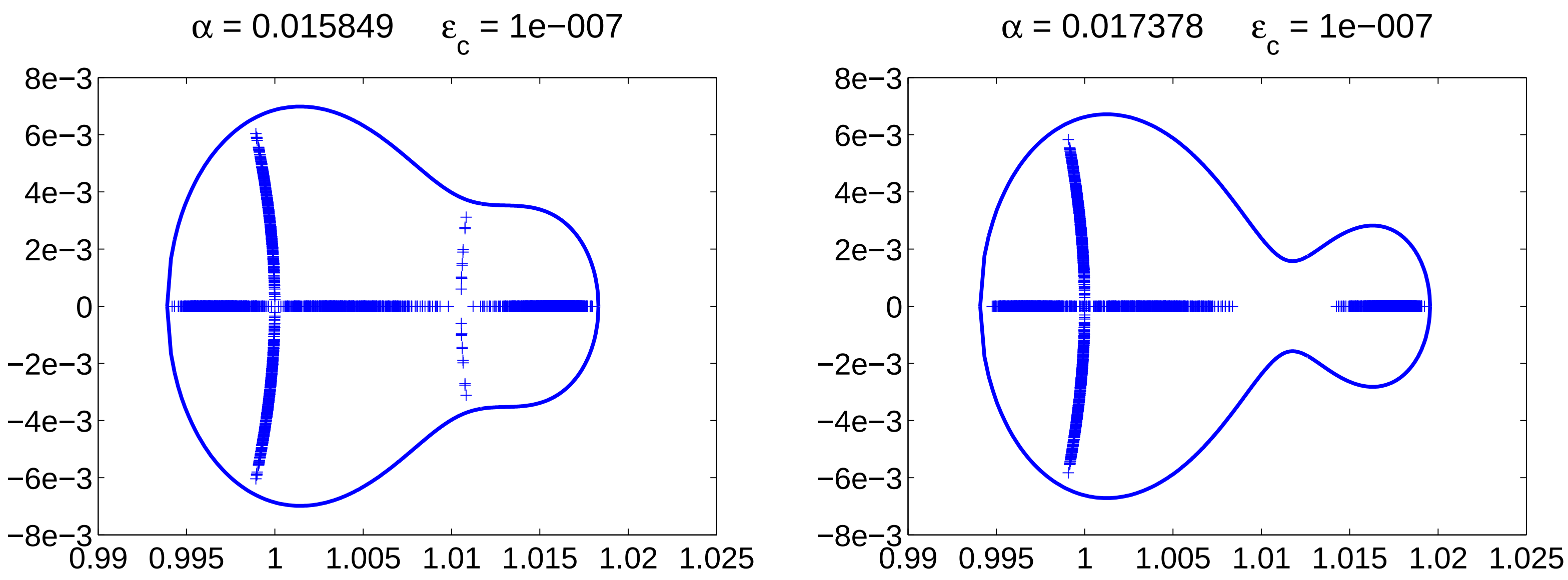

Figure 3.6a: The regions within which the roots of the perturbed polynomial $f(x)$ lie, and their principal axes, for two values of α.

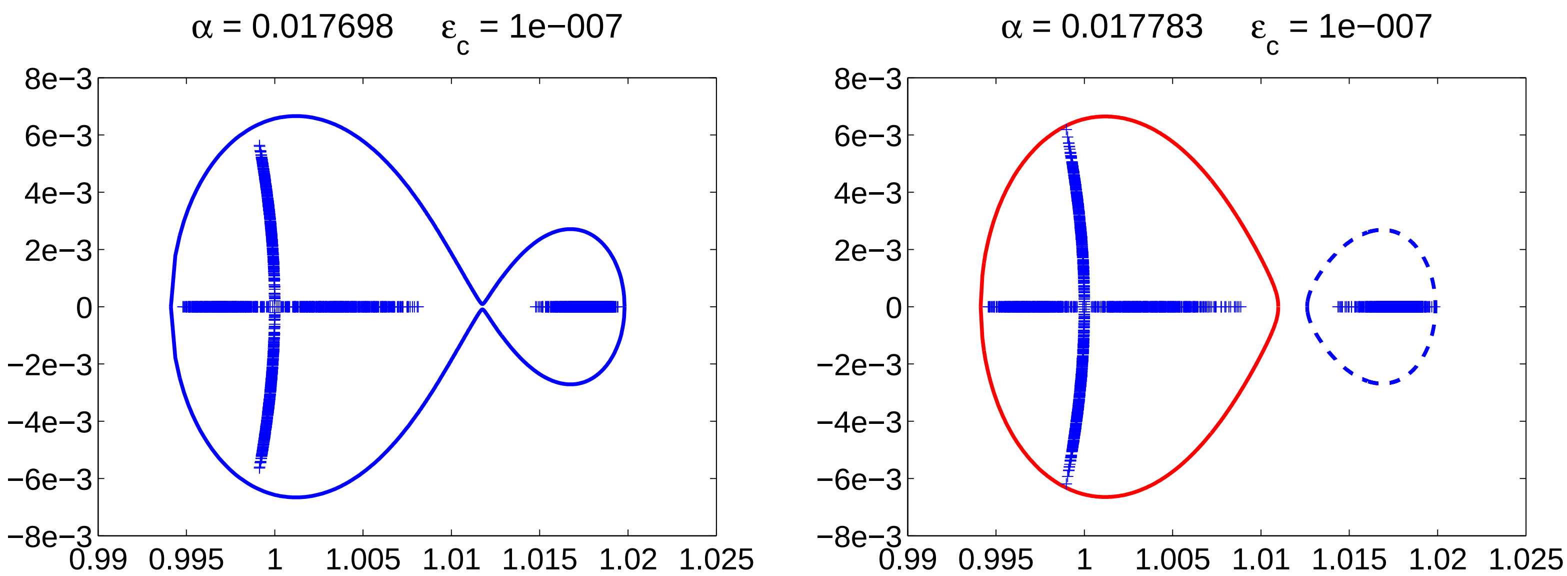

Figure 3.6b: The regions within which the roots of the perturbed polynomial $f(x)$ lie, and their principal axes, for two values of α.

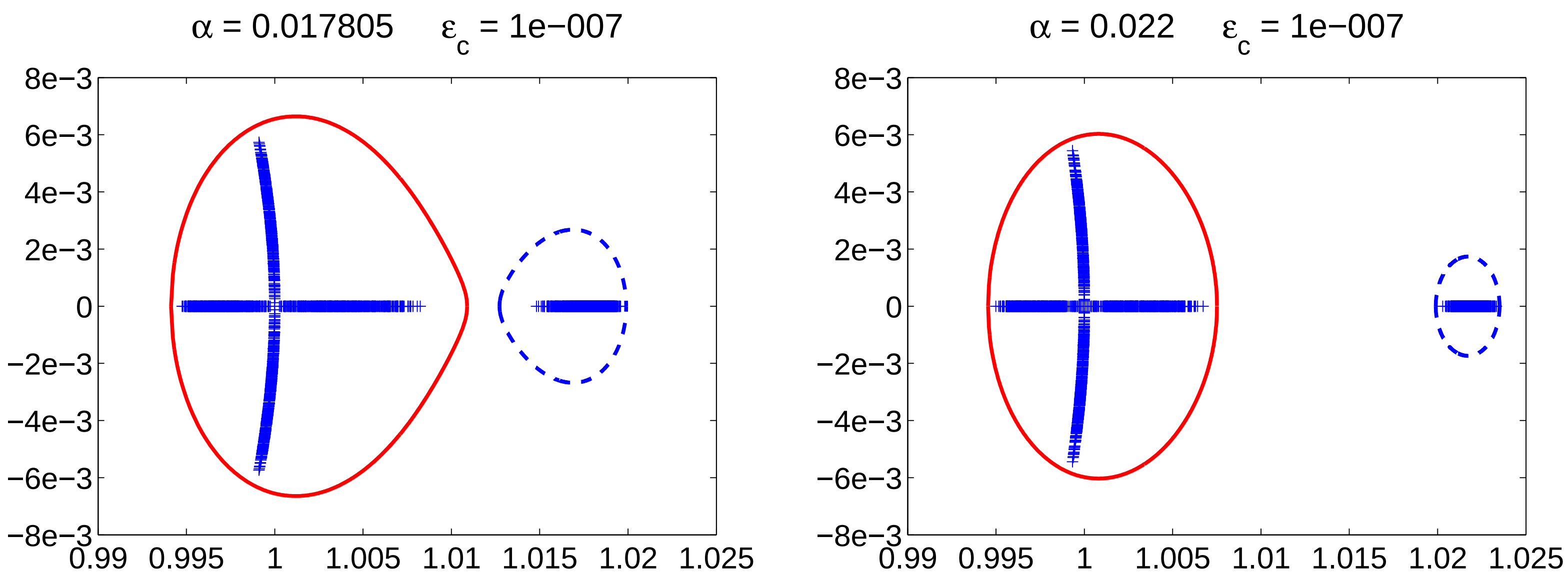

Figure 3.6c: The regions within which the roots of the perturbed polynomial $f(x)$ lie, and their principal axes, for two values of α. □

3.5 Condition numbers, backward errors and the forward error

The forward error, backward error and condition number are related

$$
\begin{aligned}
\frac{|\delta x_0|}{|x_0|} &= \kappa_c(x_0)\left(\frac{\eta_c(\tilde{x}_0)}{\varepsilon_c}\right)^{\frac{1}{r}}\varepsilon_c \\
\frac{|\delta x_0|}{|x_0|} &= \kappa_n(x_0)\left(\frac{\eta_n(\tilde{x}_0)}{\varepsilon_n}\right)^{\frac{1}{r}}\varepsilon_n
\end{aligned}
$$

If x_0 is a simple root ($r = 1$)

forward error of $x_0 =$ condition number of $x_0 \times$ backward error of $\tilde{x}_0$

If x_0 is a root of high multiplicity ($r \gg 1$)

relative error of x_0 $=$ condition number of $x_0 \times$ upper bound of signal-to-noise ratio

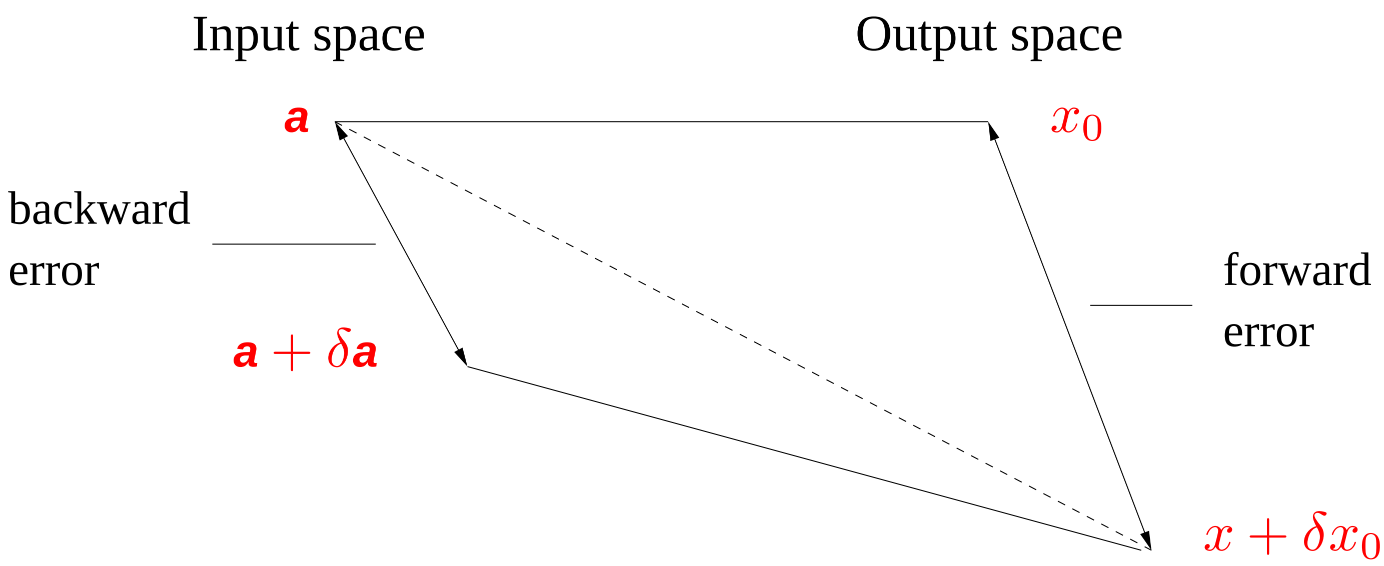

Figure 3.7: The condition number of a simple root x_0 of a polynomial with coefficients ***a*** is the ratio of the forward error to the backward error.

A multiple root breaks up into a cluster of closely spaced roots due to roundoff errors, or when one or more of its coefficients is perturbed.

- Can simple averaging be used to regroup closely spaced roots?
  - If the radius of the cluster is small, and it is sufficiently far from the nearest neighbouring cluster or isolated root, then a simple interpretation of the computed solution is the approximation of the cluster of roots by a multiple root at the arithmetic mean of the cluster.

Does this simple method always work?

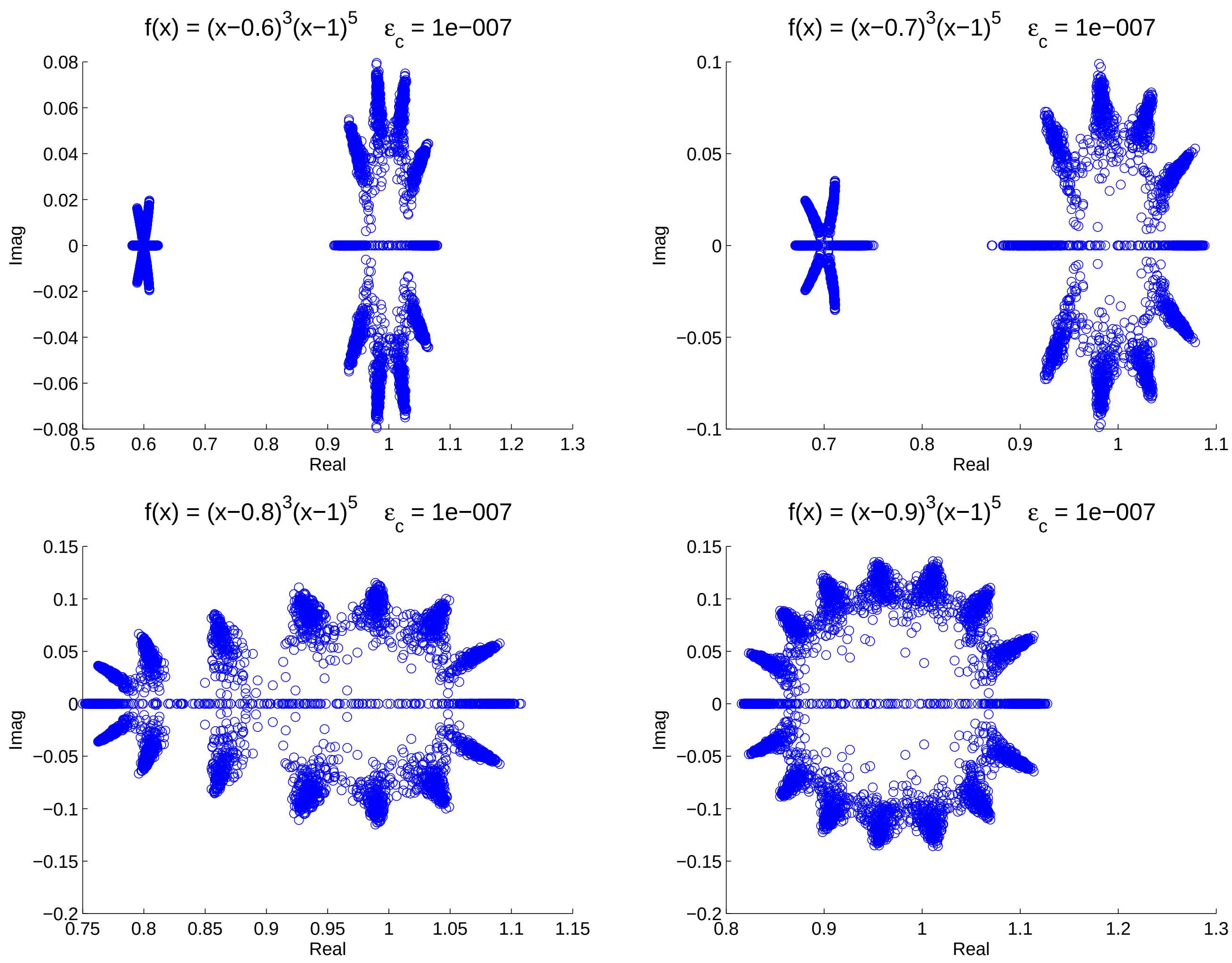

Figure 3.8: The root distribution of four perturbed polynomials.

4. THE GEOMETRY OF ILL-CONDITIONED POLYNOMIALS

- A root x_0 of multiplicity r introduces $(r-1)$ constraints on the coefficients.
- A monic polynomial of degree m has m degrees of freedom.
- The root x_0 lies on a manifold of dimension $(m-r+1)$ in a space of dimension m.
- This manifold is called a pejorative manifold because polynomials near this manifold are ill-conditioned.
- A polynomial that lies on a pejorative manifold is well-conditioned with respect to (the structured) perturbations that keep it on the manifold, which corresponds to the situation in which the multiplicity of the roots is preserved.
- A polynomial is ill-conditioned with respect to perturbations that move it off the manifold, which corresponds to the situation in which a multiple root breaks up into a cluster of simple roots.

Example 4.1 Consider the problem of determining the roots of the polynomial

$$f(x) = (x + 1 + \sqrt{\epsilon})(x + 1 - \sqrt{\epsilon})$$

$$x_0 = -1 - \sqrt{\epsilon} \quad \text{and} \quad x_1 = -1 + \sqrt{\epsilon}$$

and thus

$$\frac{\mathrm{d}x_0}{\mathrm{d}\epsilon} = -\frac{1}{2\sqrt{\epsilon}} \quad \text{and} \quad \frac{\mathrm{d}x_1}{\mathrm{d}\epsilon} = \frac{1}{2\sqrt{\epsilon}}$$

A monic quadratic polynomial is of the form $x^2 + bx + c$, and a double root exists if

$$b^2 = 4c$$

All real quadratic monic polynomials whose coefficients lie on the curve in $(b, c) \in \mathbb{R}^2$ have a double root, and this curve is therefore the pejorative manifold for this class of polynomial.

The numerical condition

$$\frac{\mathrm{d}x_0}{\mathrm{d}\epsilon} = -\frac{1}{2\sqrt{\epsilon}} \qquad \text{and} \qquad \frac{\mathrm{d}x_1}{\mathrm{d}\epsilon} = \frac{1}{2\sqrt{\epsilon}}$$

of the roots x_0, x_1 is inversely proportional to the square root of its distance ϵ from the quadratic polynomial that has a double root.

- This is a particular example of the more general fact that a polynomial is ill-conditioned if it is near a polynomial that has a multiple root.
- The condition $b^2 = 4ac$ is satisfied by $\epsilon = 0$, and the polynomial $f(x)$ lies near this manifold. This proximity to the manifold is the cause of the ill-conditioning of this polynomial. □

Example 4.2 Consider a cubic polynomial $f(x)$ with real roots x_0, x_1 and x_2

$$\begin{aligned}(x-x_0)(x-x_1)(x-x_2) &= x^3-(x_0+x_1+x_2)x^2+\\ &\quad (x_0x_1+x_1x_2+x_2x_0)x-x_0x_1x_2\end{aligned}$$

- If $f(x)$ has one double root and one simple root, then $x_0 = x_1 \neq x_2$ and thus $f(x)$ can be written as

$$x^3-(2x_1+x_2)x^2+(x_1^2+2x_1x_2)x-x_1^2x_2$$

The pejorative manifold of a cubic polynomial that has a double root is the surface defined by

$$\begin{pmatrix} -(2x_1+x_2) & (x_1^2+2x_1x_2) & -x_1^2x_2 \end{pmatrix} \qquad x_1 \neq x_2$$

- If $f(x)$ has a triple root, then $x_0 = x_1 = x_2$, and thus $f(x)$ can be written as

$$x^3 - 3x_0 x^2 + 3x_0^2 x - x_0^3$$

The pejorative manifold of a cubic polynomial that has a triple root is the curve defined by

$$\begin{pmatrix} -3x_0 & 3x_0^2 & -x_0^3 \end{pmatrix}$$

□

Theorem 4.1 The condition number of the real root x_0 of multiplicity r of the polynomial $f(x) = (x - x_0)^r$, such that the perturbed polynomial also has a root of multiplicity r, is

$$\rho(x_0) := \frac{\Delta x_0}{\Delta f} = \frac{1}{r\,|x_0|}\frac{\|(x-x_0)^r\|}{\|(x-x_0)^{r-1}\|} = \frac{1}{r\,|x_0|}\left(\frac{\sum_{i=0}^{r}\binom{r}{i}^2 (x_0)^{2i}}{\sum_{i=0}^{r-1}\binom{r-1}{i}^2 (x_0)^{2i}}\right)^{\frac{1}{2}}$$

where

$$\Delta f = \frac{\|\delta f\|}{\|f\|} \quad \text{and} \quad \Delta x_0 = \frac{|\delta x_0|}{|x_0|}$$

Example 4.3 The condition number $\rho(1)$ of the root $x_0 = 1$ of $(x-1)^r$ is

$$\rho(1) = \frac{1}{r}\left(\frac{\sum_{i=0}^{r}\binom{r}{i}^2}{\sum_{i=0}^{r-1}\binom{r-1}{i}^2}\right)^{\frac{1}{2}}$$

This expression reduces to

$$\rho(1) = \frac{1}{r}\sqrt{\frac{\binom{2r}{r}}{\binom{2(r-1)}{r-1}}} = \frac{1}{r}\sqrt{\frac{2(2r-1)}{r}} \approx \frac{2}{r} \quad \text{for large } r$$

Compare with the componentwise and normwise condition numbers

$$\kappa_c(1) \approx \frac{1}{\varepsilon_c} \qquad \text{and} \qquad \kappa_n(1) \approx \frac{1}{\varepsilon_n}$$

- $\rho(1)$ is independent of the the noise level (assumed to be small)
- $\rho(1)$ decreases as the multiplicity r of the root $x_0 = 1$ increases □

5. A SIMPLE POLYNOMIAL ROOT FINDER

Guass developed a polynomial root finder that is fundamentally different to the root finders mentioned earlier:

Let $w_1(x)$ be the product of all linear factors of $f(x)$

Let $w_2(x)$ be the product of all quadratic factors of $f(x)$

$$\cdots$$

Let $w_i(x)$ be the product of all factors of degree i of $f(x)$

Thus $f(x)$ can be written as

$$f(x) = w_1(x)w_2^2(x)w_3^3(x) \cdots w_{r_{\max}}^{r_{\max}}(x)$$

Perform a sequence of GCD computations

$$
\begin{aligned}
q_1(x) &= \text{GCD}\left(f(x), f^{(1)}(x)\right) &&= w_2(x) w_3^2(x) w_4^3(x) \cdots w_{r_{\max}}^{r_{\max}-1}(x) \\
q_2(x) &= \text{GCD}\left(q_1(x), q_1^{(1)}(x)\right) &&= w_3(x) w_4^2(x) w_5^3(x) \cdots w_{r_{\max}}^{r_{\max}-2}(x) \\
q_3(x) &= \text{GCD}\left(q_2(x), q_2^{(1)}(x)\right) &&= w_4(x) w_5^2(x) w_6^3(x) \cdots w_{r_{\max}}^{r_{\max}-3}(x) \\
q_4(x) &= \text{GCD}\left(q_3(x), q_3^{(1)}(x)\right) &&= w_5(x) w_6^2(x) \cdots w_{r_{\max}}^{r_{\max}-4}(x) \\
& \vdots
\end{aligned}
$$

The sequence terminates at $q_{r_{\max}}(x)$, which is a constant.

A set of polynomials $h_i(x), i = 1, \ldots, r_{\max}$, is defined such that

$$
\begin{aligned}
h_1(x) &= \frac{f(x)}{q_1(x)} &&= w_1(x)w_2(x)w_3(x)\cdots \\
h_2(x) &= \frac{q_1(x)}{q_2(x)} &&= w_2(x)w_3(x)\cdots \\
h_3(x) &= \frac{q_2(x)}{q_3(x)} &&= w_3(x)\cdots \\
&\vdots \\
h_{r_{\max}}(x) &= \frac{q_{r_{\max}-2}}{q_{r_{\max}-1}} &&= w_{r_{\max}}(x)
\end{aligned}
$$

The functions, $w_1(x), w_2(x), \cdots, w_{r_{\max}}(x)$, are determined from

$$w_1(x) = \frac{h_1(x)}{h_2(x)}, \quad w_2(x) = \frac{h_2(x)}{h_3(x)}, \quad \cdots \quad , w_{r_{\max}-1}(x) = \frac{h_{r_{\max}-1}(x)}{h_{r_{\max}}(x)}$$

until

$$w_{r_{\max}}(x) = h_{r_{\max}}(x)$$

The equations

$$w_1(x) = 0, \quad w_2(x) = 0, \quad \cdots \quad , w_{r_{\max}}(x) = 0$$

contain only simple roots, and they yield the simple, double, triple, etc., roots of $f(x)$.

- If x_0 is a root of $w_i(x)$, then it is a root of multiplicity i of $f(x)$.

Example 5.1 Calculate the roots of the polynomial

$$f(x) = x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2$$

whose derivative is

$$f^{(1)}(x) = 6x^5 - 15x^4 + 18x^2 - 6x - 3$$

Perform a sequence of GCD computations:

$$\begin{aligned} q_1(x) &= \text{GCD}\left(f(x), f^{(1)}(x)\right) &= x^3 - x^2 - x + 1 \\ q_2(x) &= \text{GCD}\left(q_1(x), q_1^{(1)}(x)\right) &= x - 1 \\ q_3(x) &= \text{GCD}\left(q_2(x), q_2^{(1)}(x)\right) &= 1 \end{aligned}$$

The maximum degree of a divisor of $f(x)$ is 3 because the sequence terminates at $q_3(x)$.

The polynomials $h_i(x)$ are:

$$\begin{aligned} h_1(x) &= \frac{f(x)}{q_1(x)} &= x^3 - 2x^2 - x + 2 \\ h_2(x) &= \frac{q_1(x)}{q_2(x)} &= x^2 - 1 \\ h_3(x) &= \frac{q_2(x)}{q_3(x)} &= x - 1 \end{aligned}$$

The polynomials $w_i(x)$ are

$$\begin{aligned} w_1(x) &= \frac{h_1(x)}{h_2(x)} &= x - 2 \\ w_2(x) &= \frac{h_2(x)}{h_3(x)} &= x + 1 \\ w_3(x) &= h_3(x) &= x - 1 \end{aligned}$$

and thus the factors of $f(x)$ are

$$f(x) = (x-2)(x+1)^2(x-1)^3$$

□

5.1 Discussion of the method

- The computation of the GCD of two polynomials is an ill-posed problem because it is not a continuous function of their coefficients:
  - The polynomials $f(x)$ and $g(x)$ may have a non-constant GCD, but the perturbed polynomials $f(x) + \delta f(x)$ and $g(x) + \delta g(x)$ may be coprime.
- The determination of the degree of the GCD of two polynomials reduces to the determination of the rank of a resultant matrix, but the rank of a matrix is not defined in a floating point environment. The determination of the rank of a noisy matrix is a challenging problem that arises in many applications.
- Polynomial division, which reduces to the deconvolution of the coefficients of the polynomials, is an ill-conditioned problem that must be implemented with care in order to obtain a computationally reliable solution.

- The data in many practical examples is inexact, and thus the polynomials are only specified within a tolerance. The given inexact polynomials may be coprime, and it may therefore be necessary to perturb each polynomial slightly, such that they have a non-constant GCD. This GCD is called an approximate greatest common divisor of the given inexact polynomials, and it is necessary to compute the smallest perturbations such that the perturbed polynomials have a non-constant exact GCD.
- The amplitude of the noise may or may not be known in practical examples, and it may only be known approximately. It is desirable that a polynomial root finder not require an estimate of the noise level, and that all parameters and thresholds be calculated from the polynomial coefficients.

6. SUMMARY

- Multiple roots break up into a cluster of simple roots when they are computed by conventional algorithms. They cannot, in general, be grouped by clustering.
- A multiple root is ill-conditioned with respect to random (unstructured) perturbations, but it is well-conditioned with respect to (structured) perturbations that preserve its multiplicity.
- A multiple root is ill-conditioned if the perturbations move it off its pejorative manifold, but it is well-conditioned if the perturbations keep it on its pejorative manifold.
- The backward error of an approximate root is measured in the data space and the forward error is measured in the solution space.
- The condition number of a simple root is the ratio of its forward error to its backward error.