Nonlinear models of suspension bridges

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Tacoma bridge

Tacoma bridge

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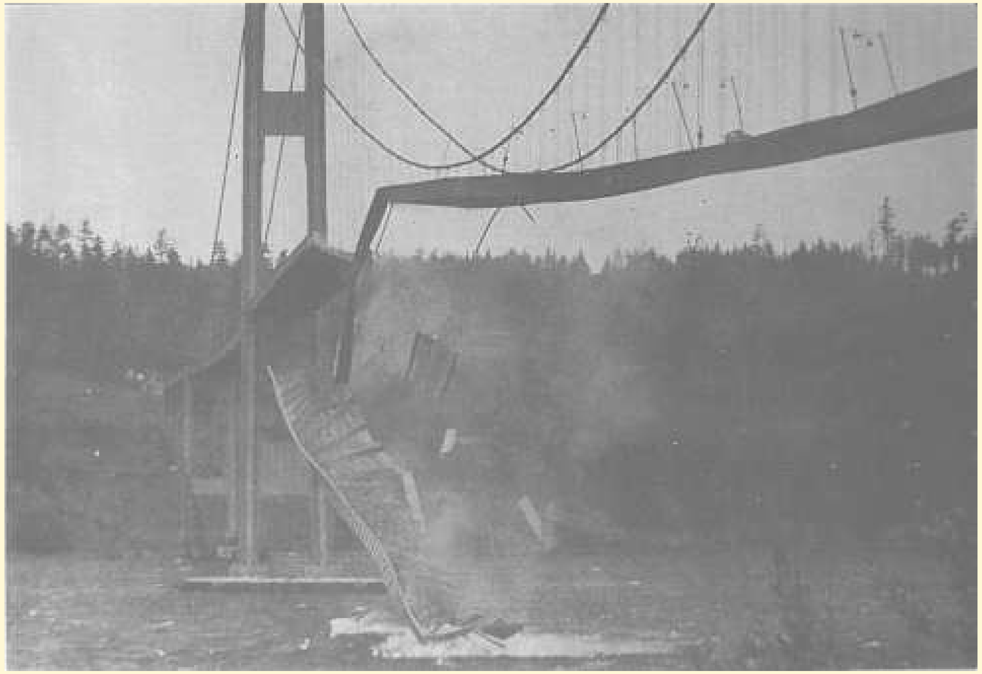
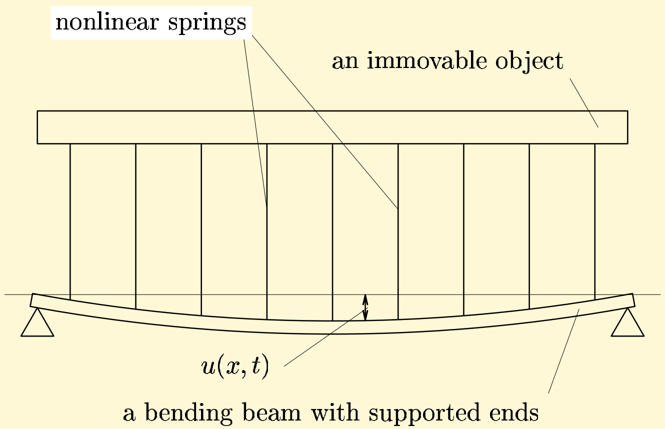
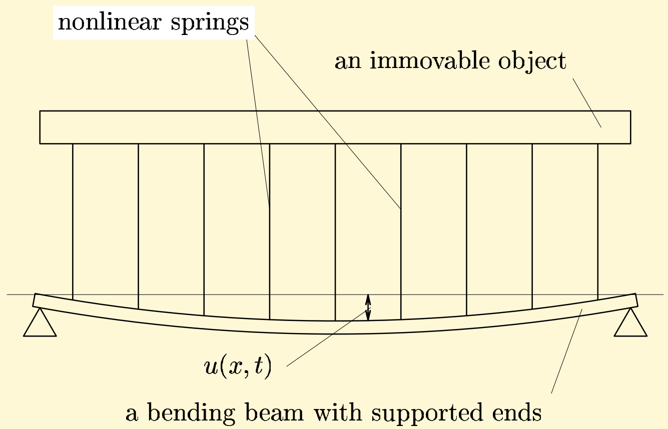


Introduction - model 1

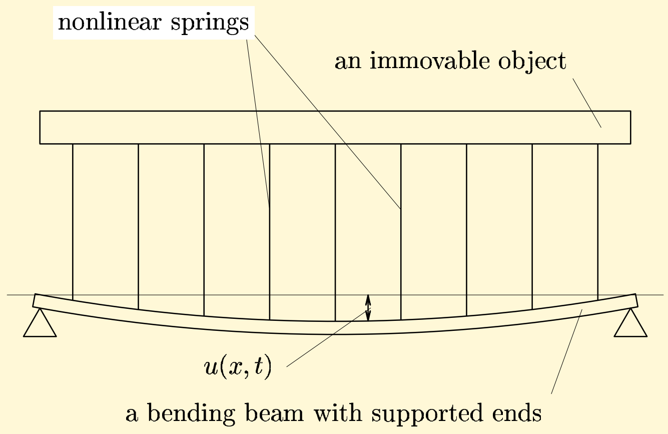


Introduction - model 1



Differential equation

$$m\frac{\partial^2 u(x,t)}{\partial t^2} + EI\frac{\partial^4 u(x,t)}{\partial x^4} + b\frac{\partial u(x,t)}{\partial t} = -ku^+(x,t) + W(x) + f(x,t)$$



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Boundary conditions

$$u(0,t)=u(L,t)=u_{xx}(0,t)=u_{xx}(L,t)=0,$$

$$u(x,t+2\pi)=u(x,t),\ -\infty<t<\infty, x\in(0,L)$$

Introduction - references 1

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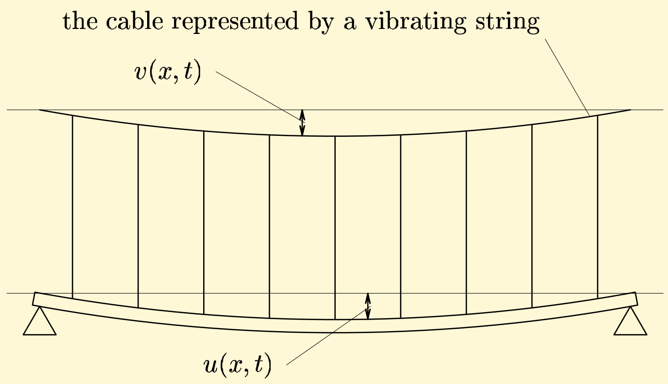
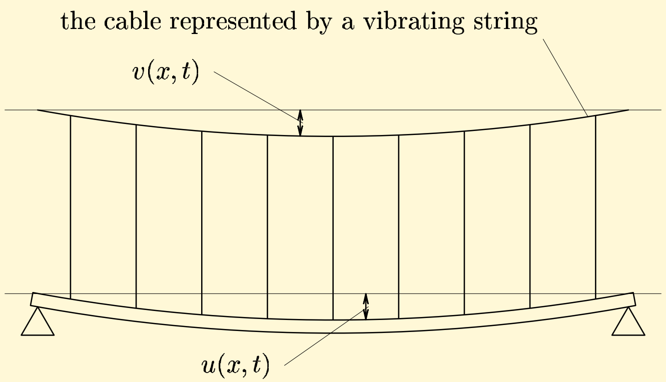
Introduction - model 2





Differential equation and boundary conditions

$$m_1 \frac{\partial^2 v.}{\partial t^2} + T \frac{\partial^2 v}{\partial x^2} + b_1 \frac{\partial v}{\partial t} = k(u - v)^+ + W_1 + f_1(x, t),$$

$$m_2 \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} + b_2 \frac{\partial u}{\partial t} = -k(u - v)^+ + W_2 + f_2(x, t),$$

$$u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = v(0, t) = v(l, t) = 0,$$

$$u(x, t + 2\pi) = u(x, t), v(x, t + 2\pi) = v(x, t), \ -\infty < t < \infty, x \in (0, L)$$

Introduction - references 2

J. Berkovits, P. Drábek, H. Leinfelder, V. Mustonen, G. Tajčová, *Time–periodic oscillations in suspension bridges: existence of unique solution*, Nonlinear Analysis: Real World Applications 1(2000), pp. 345-362.

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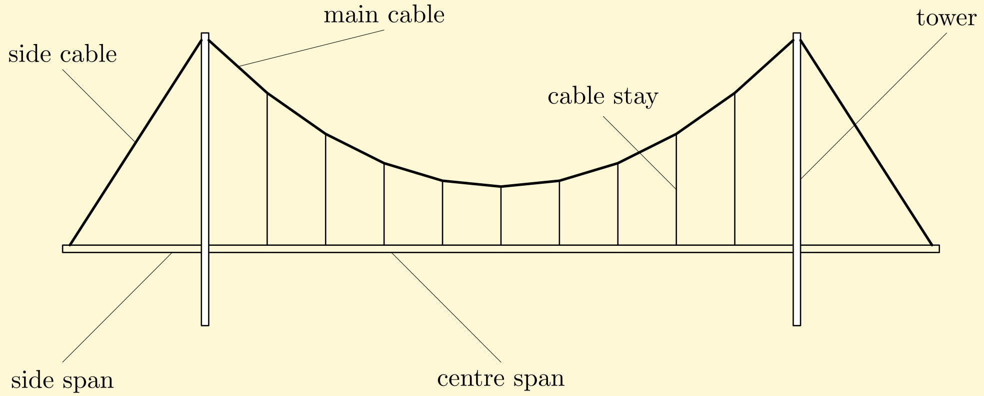
Suspension bridge - steady state problem



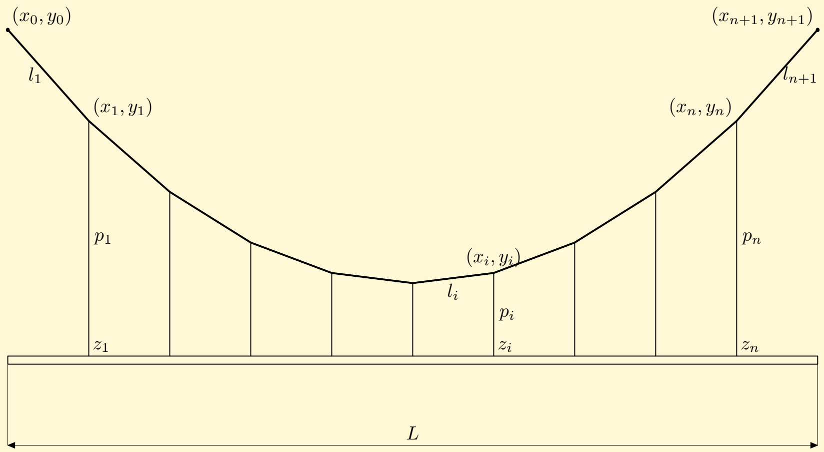
Assumptions:

The main cable is perfectly flexible and inextensible.

The behaviour of the cable stays is nonlinear: they resist tension and do not resist compression.

Two different models

The road bed is divided into three separate parts – centre span and two side spans



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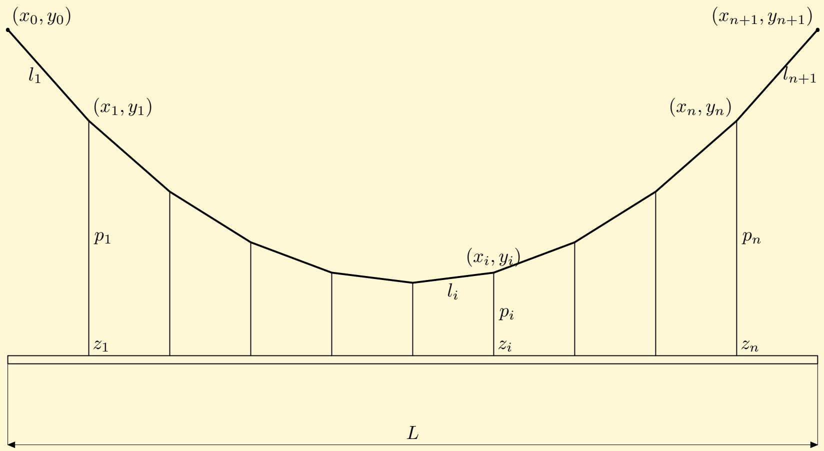
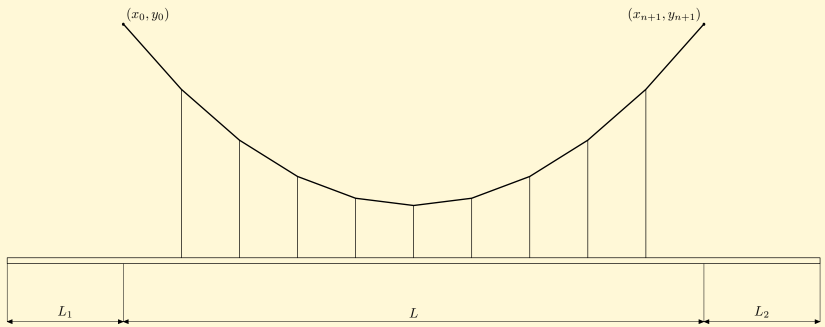


The road bed is undivided



Deformation energy of the centre span

The basic bilinear form

$$b_V(u,v) = \int_0^L K_V u'' v'' dz.$$

$$K_V(z) > \varepsilon > 0, \quad z \in (0,L).$$

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The bilinear form $b_V(.,.)$ is defined on the space

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The deformation energy of the centre span connected with u and supported at the ends is

$$\frac{1}{2} b_V(u,u).$$

Main cable and cable stays

The main cable is a polygon with the points (x_i, y_i), $i = 1, \dots n+1$.
(x_0, y_0), (x_{n+1}, y_{n+1}) are fixed.
$x = (x_1, \dots x_n)$, $y = (y_1, \dots y_n)$.

(A) $$f_i(x,y) = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 = l_i^2, \quad i = 1, \dots n+1,$$

where $l_i, i = 1, \dots n+1$ are the given positive numbers.

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where $l_i, i = 1, \ldots n+1$ are the given positive numbers.

The deformation energy of the cable stays $\phi(y,u)$ is

$$\phi(y,u) = \frac{1}{2} \sum_{i=1}^{n} k_i \left\{ (y_i - d_i - u(z_i))^+ \right\}^2,$$

where $x^+ = \max\{0, x\}$.

Outer forces

The gravitation forces acting on the centre span $F \in L^2(0, L)$.

$$L_F(u) = \int_0^L F u dz$$

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The forces acting on the main cable.

$$F_i = -\rho_i g, \ i = 1, \dots n,$$

$$\rho_i = \rho_c(l_i + l_{i+1})/2 + \rho_s d_i, \ i = 1, \dots n,$$

The forces F_i, $i = 1, \dots n$ define the linear form

$$L_c(y) = \sum_{i=1}^{n} F_i y_i.$$

The functional of potential energy

The solution to the problem is a minimum of

$$J(x,y,u) = \frac{1}{2} b_V(u,u) + 2\Phi(y,u) - L_F(u) - 2L_c(y)$$

defined on $R^n \times R^n \times W_V$, where $x, y \in R^n$ satisfy the restrictions (A).

Denote

$$\text{grad}_x f_i = \left(\frac{\partial f_i}{\partial x_1}, \dots \frac{\partial f_i}{\partial x_n} \right),$$

$$\text{grad}_y f_i = \left(\frac{\partial f_i}{\partial y_1}, \dots \frac{\partial f_i}{\partial y_n} \right), \; i = 1 \dots n+1.$$

Variational equation 1

Denote

$$\text{grad}_x f_i = \left(\frac{\partial f_i}{\partial x_1}, \dots \frac{\partial f_i}{\partial x_n}\right),$$

$$\text{grad}_y f_i = \left(\frac{\partial f_i}{\partial y_1}, \dots \frac{\partial f_i}{\partial y_n}\right), \ i = 1 \dots n+1.$$

We say that $\tilde{y}$ fulfills the relation (A) in differential sense if there exists $\tilde{x} \in R^n$ such that the equations

$$(\text{grad}_x f_i(x,y), \tilde{x}) + (\text{grad}_y f_i(x,y), \tilde{y}) = 0, \ i = 1, \dots n+1$$

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hold.

If x, y, u is a minimum of the functional $J(.,.,.)$ and x, y satisfy (A), then y satisfies the variational equation

$$\sum_{i=1}^{n} \left(k_i(y_i - p_i - u(z_i))^+ - F_i\right) \tilde{y}_i = 0,$$

where $\tilde{y}$ satisfies (A) in differential sense.

Variational equation 2

The function $u \in W_V$ satisfies the variational equality

$$b_V(u, v) - 2\sum_{i=1}^{n} k_i(y_i - p_i - u(z_i))^+ v(z_i) = L_F(v)$$

for all $v \in W_V$.

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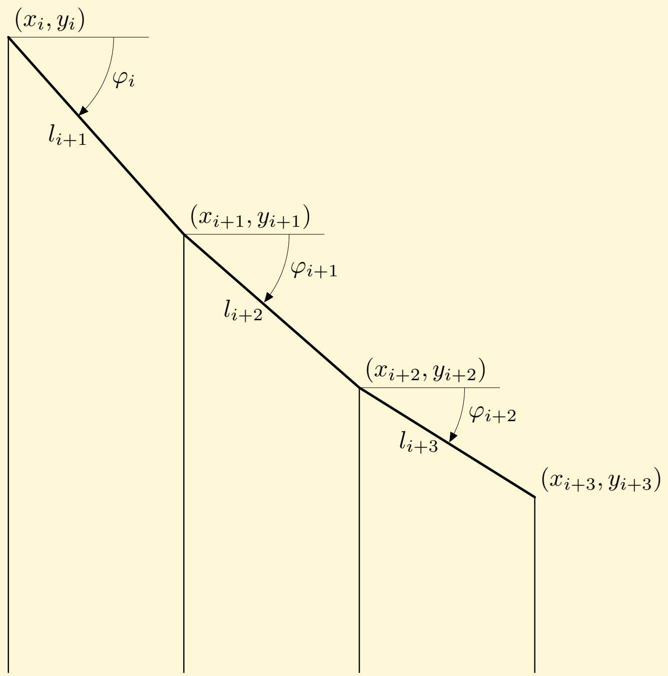
Denote

$$s_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}},\ i = 1, \ldots n+1.$$

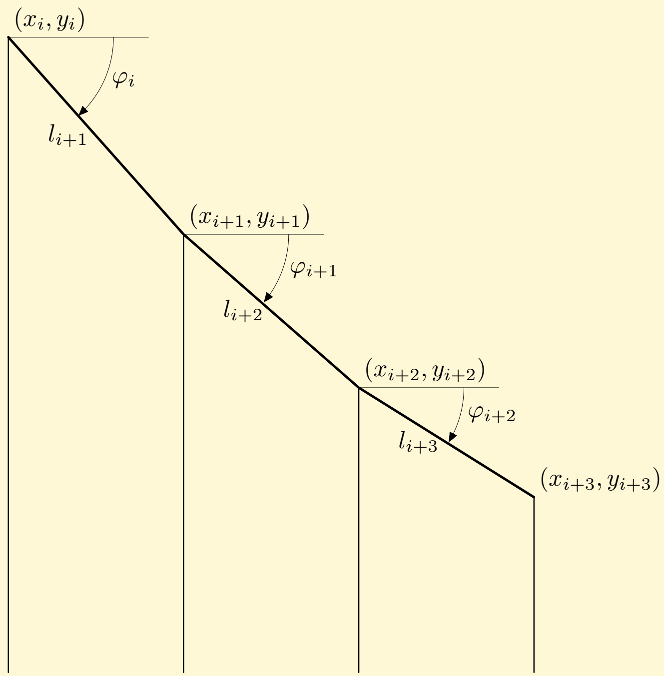
Consider that $x, y \in R^n$, which solves variational equation above, satisfy the relations

$$\text{(B)} \qquad x_{i-1} < x_i,\ s_i < s_{i+1},\ i = 1, \ldots n+1.$$

Variational equation 3



Variational equation 3



Let (x_0, y_0), (x_{n+1}, y_{n+1}), $l_i > 0$, $F_j > 0$, $i = 1 \ldots n+1$, $j = 1 \ldots n$, $F \in L^2(0, L)$ be given, then x, y, u from R^n, R^n, W_V are a solution to the problem $\mathcal{P}$ if they satisfy the variational equalities above and x, y fulfill the relations (A), (B).

Auxiliary problem 1

Let $a = (a_1, \dots a_n) \in R^n$ satisfy

$$a_i > 0, \quad i = 1 \dots n\,,$$

$x, y \in R$ satisfy the conditions (A), (B). Then x, y is a solution to the auxiliary problem $\mathcal{A}$, if the equation

$$\sum_{i=1}^{n} a_i \tilde{y}_1 = 0$$

holds for all $\tilde{y} = (\bar{y}_1, \dots \tilde{y}_n)$ fulfilling the restrictions (A) at x, y in differential sense.

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Lemma Let (x_0, y_0), (x_{n+1}, y_{n+1}), l_i, $i = 1, \dots n+1$, $a \in R^n$ be given. Then $x, y \in R^n$ satisfying (A), (B) are a solution to $\mathcal{A}$ if and only if the equations

$$\frac{a_{j+1}}{a_j} = \frac{s_{j+2} - s_{j+1}}{s_{j+1} - s_j} \,, \; j = 1, \dots, n-1$$

hold.

Auxiliary problem 2

Let (x_0, y_0), (x_{n+1}, y_{n+1}), $l_i > 0$, $i = 1 \ldots n+1$ be given, we stay that the parameters satisfy the assumption (C) if the inequalities

$$x_{n+1} - x_0 > l_i,\ i = 1, \ldots n+1$$

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hold.

We say that the parameters (x_0, y_0), (x_{n+1}, y_{n+1}), $l_i > 0$, $i = 1 \dots n+1$ satisfy the assumption (D) if there exist $x = (x_1, \dots x_n)$, $y = (y_1, \dots y_n)$ which satisfy (A), (B).

Solution to $\mathcal{A}$

Theorem Let (x_0, y_0), (x_{n+1}, y_n), $l_i > 0$, $i = 1, \dots n+1$ satisfy (C), (D). Then there exist the smooth functions

$$\Phi : K \to R^n, \quad \Psi : K \to R^n$$

which are bounded and $\Psi = (\Psi_1, \dots \Psi_n)$ satisfies the inequality

$$x_0 \; < \; \Psi_i(b) \; < \; x_{n+1}, \, i = 1, \dots n \, .$$

If $a = (a_1, \dots a_n) \in R^n$ fulfill $a_i > 0$, $i = 1, \dots n$, then

$$x \; = \; \Phi\left(\frac{a_2}{a_1}, \dots \frac{a_n}{a_{n-1}}\right) \, , \; y \; = \; \Psi\left(\frac{a_2}{a_1}, \dots \frac{a_n}{a_{n-1}}\right)$$

are the unique solution to $\mathcal{A}$ for a.

Another auxiliary problem

$$b_V(u,v) - 2\sum_{i=1}^{n} k_i(y_i - p_i - u(z_i))^+ v(z_i) = L_F(v).$$

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Lemma Let the sequence y_i^k converge to y_i^0, $i = 1, \dots n$ if $k \to \infty$. Let $F^k \in L^2(0,L)$ converge to $F^0 \in L^2(0,L)$ if $k \to \infty$. Let u^k be the solutions to equation above corresponding to y_i^k, F^k, then u^k converges in W_V to $u^0 \in W_V$ if $k \to \infty$ and u^0 is the solution to the equation above corresponding to y_i^0, F^0.

Some useful functions

Define the functions:

$\Theta : M \to R^{n-1}$, where

$$\Theta(a) = (\Theta_1(a), \ldots \Theta_{n-1}(a)),$$

$$\Theta_i(a) = \frac{a_{i+1}}{a_i},\ i = 1 \ldots n-1.$$

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$\kappa : R^n \times L^2(0,L) \to M$

$$\kappa(y,F) = (\kappa_1(y,F), \dots \kappa_n(y,F)),$$

$$\kappa_i(y,F) = k_i(y_i - p_i - u(z_i))^+ - F_i,$$

where u is the solution to the variational equation above for y, F.

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where u is the solution to the variational equation above for y, F.

$R : R^n \times L^2(0,L) \to R^n$

$$R(y,F) = \Psi \circ \Theta \circ \kappa(y,F).$$

Theorem Let (x_0, y_0), (x_{n+1}, y_{n+1}), l_i, $i = 1 \dots n+1$ satisfy the assumptions (C), (D) and $F \in L^2(0, L)$. Then there exist solutions to $\mathcal{P}$.

Local properties of solution

Let x, y, u from N, R^n, W_V be a solution to $\mathcal{P}$, then we say that the solution satisfies the condition (E) if the inequalities

$$y_i - p_i - u(z_i) > 0, \quad i = 1, \ldots n$$

hold.

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$$y_i - p_i - u(z_i) > 0, \quad i = 1, \dots n$$

hold.

Let us define the function $S : R^n \times L^2(0, L) \to R^{n \times n}$ in the following way:

$$S(y, F) = \frac{\partial}{\partial y} \Psi \circ \Theta \circ \kappa(y, F)\,,$$

where the functions Ψ, Θ are defined above and the values of S are matrices $n \times n$.

Local uniqueness

Theorem Let $\tilde{x}$, $\tilde{y}$, $\tilde{u}$ from N, R^n, W_V is a solution to the problem $\mathcal{P}$ corresponding to $\tilde{F} \in L^2(0, L)$. Let the solution satisfy the condition (E) and the matrix $S(\tilde{y}, \tilde{F})$ not have the eigenvalue equal to one. Then there exist $\alpha, \beta, > 0$ and the functions $\xi_1 : B \to N$, $\xi_2 B \to R^n$, $\xi_3 : B \to W_V$, where

$$B = \{F \in L^2(0, L) \mid \| F - \tilde{F} \|_{L^2(\alpha, L)} < \alpha\} .$$

The functions are continuously differentiable on B and satisfy the relations

$$\xi_1(\tilde{F}) = \tilde{x}, \; \xi_2(\tilde{F}) = \tilde{y}, \; \xi_3(\tilde{F}) = \tilde{u},$$

together with the inequalities

$$| \tilde{x} - \xi_1(\tilde{F}) |_{R^n} < \beta, \; | \tilde{y} - \xi_2(\tilde{F}) |_{R^n} < \beta, \; \| \tilde{u} - \xi_3(\tilde{F}) \|_{W_V} < \beta$$

on the set B. Moreover, $\xi_1(F)$, $\xi_2(F)$, $\xi_3(F)$ is a solution to $\mathcal{P}$ for any $F \in B$, where F corresponds to $L_F(.)$, and the solution is unique if the inequalities above are satisfied.

Remarks

Remark 1. The matrix $S(y, F)$ can be expressed explicitly and the Newton method can be applied for looking for a solution to $\mathcal{P}$ satisfying the condition (E).

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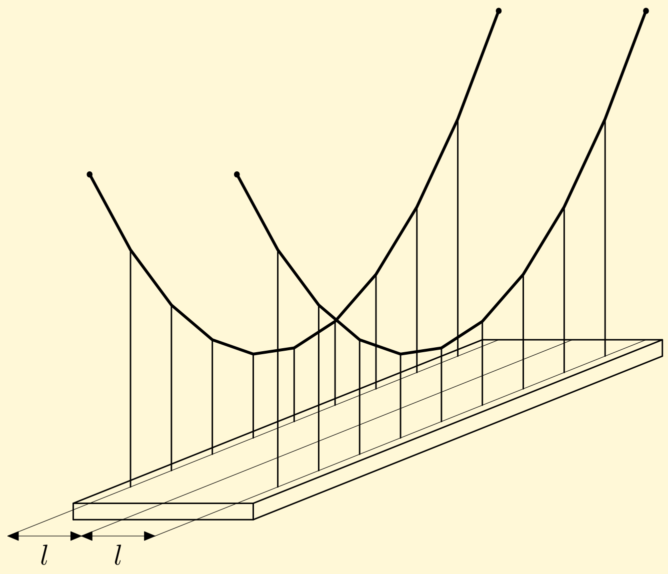
Remark 2. The stability can be disturbed if one is among eigenvalues of S or S is near to the state, which can be expressed by the number

$$\inf_{x \in R^n} \frac{\mid x - Sx \mid_c}{\mid x \mid_c},$$

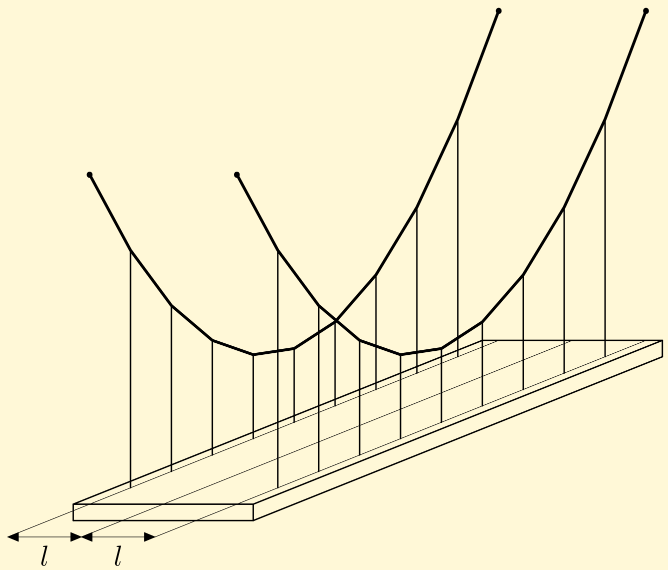
where $\mid x \mid_c$ is the norm on R^ngiven by the formula

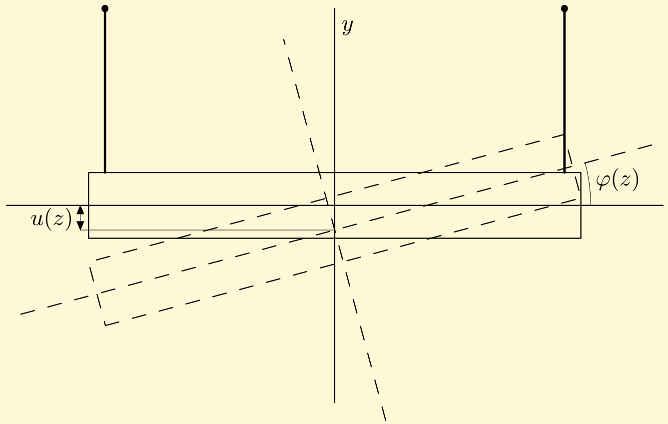
$$\mid x \mid_c = \sqrt{\sum_{i=1}^{n} k_i x_i^2}.$$

Generalized model - torsion



Generalized model - torsion





The generalized functional of potential energy

The solution to the generalized problem is a minimum of

$$K(x^1, y^1, x^2, y^2, u, \varphi) = \frac{1}{2} b_V(u,u) + \frac{1}{2} b_T(\varphi,\varphi) + \phi(y^1, u + l\varphi) + \phi(y^2, u - l\varphi) - L_F(u) - L_G(u) - L_c(y^1) - L_c(y^2),$$

defined on a subset of $R^n \times R^n \times R^n \times R^n \times W_V \times W_T$, where x^1, y^1 and x^2, y^2 satisfy the conditions (A).

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defined on a subset of $R^n \times R^n \times R^n \times R^n \times W_V \times W_T$, where x^1, y^1 and x^2, y^2 satisfy the conditions (A).

$$b_T(\varphi, \psi) = \int_0^L K_T \varphi' \psi' dz,$$

The generalized functional of potential energy

The solution to the generalized problem is a minimum of

$$K(x^1, y^1, x^2, y^2, u, \varphi) = \frac{1}{2} b_V(u,u) + \frac{1}{2} b_T(\varphi,\varphi) + \phi(y^1, u + l\varphi) + \phi(y^2, u - l\varphi) - L_F(u) - L_G(u) - L_c(y^1) - L_c(y^2),$$

defined on a subset of $R^n \times R^n \times R^n \times R^n \times W_V \times W_T$, where x^1, y^1 and x^2, y^2 satisfy the conditions (A).

$$b_T(\varphi,\psi) = \int_0^L K_T \varphi' \psi' dz,$$

$$W_T = \{\varphi \in H^1(0,L) \mid \varphi(0) = \varphi(L) = 0\},$$

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